NATURAL SCIENCES TRIPOS Part II

Wednesday 14 January 2009 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A bead of mass m slides freely on a light wire of parabolic shape, which is forced to rotate with angular velocity ω about a vertical axis. The equation of the parabola is

$$z = \frac{1}{2} a r^2$$

where z is the height and r is the distance from the axis of rotation.

(a) Show that the Lagrangian for this system is [5]

$$L = \frac{1}{2} m \left[\left(1 + a^2 r^2\right) \dot{r}^2 + \left(\omega^2 - ag\right) r^2 \right]$$

(b) Find a constant of the motion. [6]

(c) The bead is released at $r = 1/a$ with $\dot{r} = v$. Show that if $\omega^2 \geq ag$ the bead escapes to infinity. Show that if $\omega^2 < ag$ it oscillates about $r = 0$, and find the maximum value of r. [9]

(d) Now suppose the wire is not forced but rotates freely about the vertical axis with angular velocity $\dot{\phi}$. Find the new Lagrangian and constants of the motion. [7]

(e) If the bead is released with the same initial conditions as before, i.e. $r = 1/a$, $\dot{r} = v$, $\dot{\phi} = \omega$, show that in this case it cannot escape to infinity for any value of ω, and find the maximum and minimum values of r. [6]

2 Define the Hamiltonian of a dynamical system with a finite number of degrees of freedom. [3]

Explain briefly the concept of a canonical transformation. [3]

Show by means of a canonical transformation that the Hamiltonian

$$H = \frac{p^2}{2m} + p f(q) + \frac{1}{2} k q^2$$

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describes the motion of a particle of mass m in some potential $U(q)$, and express $U(q)$ in terms of $f(q)$. [8]

Show that the following transformation is canonical:

$$Q = \tan^{-1}\frac{\lambda q}{p}\ , \quad P = \frac{p^2+\lambda^2 q^2}{2\lambda} + g\left(\frac{p}{q}\right)\ ,$$

where λ is an arbitrary constant and $g(x)$ is an arbitrary function. [8]

Hence, or otherwise, calculate the motion of the particle in the potential

$$U(q) = \frac{1}{2}kq^2 - \frac{1}{2}\frac{A^2}{q^2}$$

where $k > 0$. [8]

Discuss your answer. [3]

3 Show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}$$

leads to Maxwell's equations for a free electromagnetic field. [6]

Given that the electromagnetic stress-energy tensor is

$$T^{\mu\nu} = -\frac{1}{\mu_0}F^{\mu}{}_{\lambda}F^{\nu\lambda} - g^{\mu\nu}\mathcal{L}\ ,$$

show explicitly that this tensor is conserved. [6]

An electromagnetic wave is represented by the 4-vector potential

$$A^{\mu} = (0, A\cos(kz-\omega t), A\sin(kz-\omega t), 0)\ .$$

(a) Evaluate the electric and magnetic fields. [6]

(b) Evaluate the Lagrangian density. [6]

(c) Evaluate the stress-energy tensor and interpret its components. [9]

$$\left[\textit{You may assume that } F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\right]$$

4 A real scalar field $\varphi(x)$ has Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi)$$

(a) Derive the equation of motion, the canonical momentum density and the Hamiltonian density. [6]

(b) Write a Fourier representation of the field and find the dispersion relation between the frequency and wave vector. [5]

(c) Derive the stress-energy tensor and show that it is conserved. [6]

(d) The system has a shift symmetry under $\varphi \to \varphi' = \varphi + c$ where c is a constant. Derive the associated Noether current and show that it is conserved. [6]

(e) Discuss whether you would expect the shift symmetry to be spontaneously broken. [5]

(f) State Goldstone's theorem and discuss its applicability to this case. [5]

5 In the Nambu-Jona-Lasinio model, a Dirac field ψ has Lagrangian density

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi + \frac{\lambda}{4}\left[(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)^2\right]$$

where $\overline{\psi} = \psi^\dagger \gamma^0$ and λ is a real, positive constant.

(a) Derive the equations of motion for ψ and $\overline{\psi}$ and show that they are consistent. [8]

(b) Express $\mathcal{L}$ in terms of the left- and right-handed fields

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi\ , \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi\ ,$$

and derive the equations of motion for ψ_L and ψ_R. [8]

(c) Show that there is a global symmetry with respect to independent phase changes in these fields, i.e.

$$\psi_L \to e^{i\alpha}\psi_L\ , \quad \psi_R \to e^{i\beta}\psi_R$$

where α and β are real constants. [8]

(d) Show that this symmetry is spontaneously broken but there remains a global symmetry with respect to identical phase changes in these fields, i.e. $\alpha = \beta$. [9]

[*You may assume that* $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\ , \quad \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0\ ,$
$\gamma^5\gamma^5 = 1\ , \quad \gamma^{5\dagger} = \gamma^5$ *and* $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.]

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6 The current density $j(t)$ in a conductor due to an applied electric field $\mathcal{E}(t)$ is given by

$$j(t) = \int \sigma(t-t')\mathcal{E}(t')\,dt'$$

where the linear response function $\sigma(t-t')$ vanishes for $t < t'$ and its Fourier transform gives the conductivity as a function of the frequency ω:

$$\sigma(\omega) = \int_0^\infty \sigma(\tau)e^{i\omega\tau}\,d\tau$$

(a) For a real electric field

$$\mathcal{E}(t) = Fe^{-i\omega t} + F^* e^{i\omega t}$$

show that the current density is

$$j(t) = \sigma(\omega)Fe^{-i\omega t} + \sigma(-\omega)F^* e^{i\omega t}$$

and hence that the real and imaginary parts of σ are even and odd functions of ω, respectively. [7]

(b) At high frequencies the conductor can be treated as a free electron gas. By considering the motion of an electron in the above electric field, show that this implies

$$\sigma(\omega) \underset{\omega\to\infty}{\longrightarrow} i\frac{ne^2}{m\omega}$$

where n is the electron number density and e and m are the electron charge and mass. [7]

(c) At low frequencies the conductivity has the form

$$\sigma(\omega) \underset{\omega\to 0}{\longrightarrow} i\frac{A}{\omega}$$

where A is a real constant.

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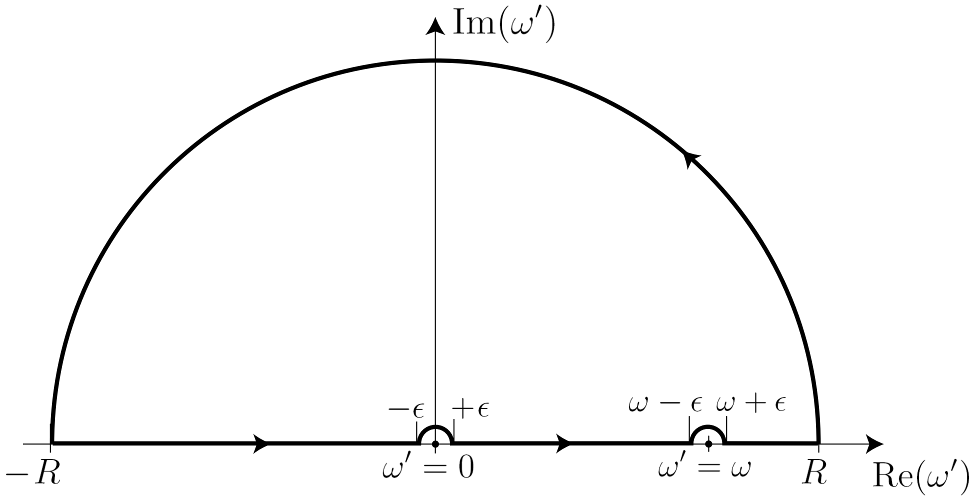

By considering the integral

$$\oint \frac{\sigma(\omega')}{\omega' - \omega}\, d\omega'$$

on the contour shown in the figure and taking the limits $R \to \infty$ and $\epsilon \to 0$, show that the real and imaginary parts of the conductivity, $\sigma_1(\omega)$ and $\sigma_2(\omega)$ respectively, satisfy the Kramers-Kronig relations [8]

$$\begin{aligned} \sigma_1(\omega) &= \frac{1}{\pi} P \int \frac{\sigma_2(\omega')}{\omega' - \omega} d\omega' \\ \sigma_2(\omega) &= -\frac{1}{\pi} P \int \frac{\sigma_1(\omega')}{\omega' - \omega} d\omega' + \frac{A}{\omega} \end{aligned}$$

(d) Show also that [3]

$$A = \frac{ne^2}{m} - \frac{1}{\pi} \int \sigma_1(\omega')\, d\omega'$$

(e) Given that the real part of the conductivity has the form

$$\sigma_1(\omega) = \sum_{\alpha,\beta} |\mathcal{M}_{\alpha\beta}|^2 \frac{f_\alpha - f_\beta}{\omega_{\beta\alpha}} \delta(\omega - \omega_{\beta\alpha})$$

where $\mathcal{M}_{\alpha\beta}$ is a quantum-mechanical matrix element between states α and β with energies E_α and E_β, $f_\alpha = f(E_\alpha)$ where $f(E)$ is the Fermi-Dirac distribution function, and $\omega_{\beta\alpha} = (E_\beta - E_\alpha)/\hbar$, show that [8]

$$\sigma(\omega) = i\frac{ne^2}{m\omega} - \lim_{\epsilon \to 0^+} \frac{i}{\pi\omega} \sum_{\alpha,\beta} |\mathcal{M}_{\alpha\beta}|^2 \frac{f_\alpha - f_\beta}{\omega_{\beta\alpha} - \omega - i\epsilon}$$

[*You may assume that* $\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x)$]

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