Theoretical Physics 1  
Answers to Examination 2006 *2007*

Warning — these answers have been completely retyped...  
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Q1. Bookwork: the canonical momenta are \( p_i \equiv \partial L / \partial \dot{q}_i \). The Hamiltonian is

\[
H \equiv \sum_i p_i \dot{q}_i - L ,
\]

which is a function of \( (q_i, p_i) \) but not \( \dot{q}_i \). Hamilton's equations are

\[
\dot{q}_i = \frac{\partial H}{\partial p_i} ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} ,
\]

i.e. a set of \( 2N \) first-order equations for the coordinates and momenta. For a charged particle we add the scalar \(-q(\phi - A \cdot \mathbf{x})\) to the Lagrangian. The canonical momentum is then \( \mathbf{p} = m \mathbf{x} + q \mathbf{A} \), but the Hamiltonian is still \( H = \frac{1}{2} m \mathbf{x}^2 + q \phi \). Expressed as a function of \( \mathbf{p} \) we have

\[
H = \frac{(\mathbf{p} - q \mathbf{A})^2}{2m} + q \phi \tag{8}
\]

The vector potential \((-By, 0, 0)\) has \( \nabla \times \mathbf{A} = (0, 0, B) \) as required and \( \mathbf{E} = -\nabla \phi \) as required.

The Hamiltonian is

\[
H = \frac{(p_x + qBy)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2} m \omega_0^2 y^2 .
\tag{4}
\]

It does not depend on \( x, z \) or \( t \), so \( p_x, p_z \) and \( H \) are constants of motion. The equations for \( p_y, x \) and \( y \) are

\[
\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{qB}{m} (p_x + qBy) - m \omega_0^2 y ;
\tag{2}
\]

\[
\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x + qBy}{m} ;
\tag{1}
\]

\[
\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} , \quad \dot{z} = \frac{qz}{m} .
\tag{1}
\]

Differentiating the \( \dot{y} \) equation and substituting we get the required result

\[
\ddot{y} + (\omega^2 + \omega_0^2) y = -\frac{\omega p_y}{m} \tag{3}
\]

where \( \omega = qB/m \), the Larmor frequency. This has general solution

\[
y = A \cos(\Omega t + \delta) - \frac{p_x}{m} \frac{\omega}{\Omega^2} \tag{1}
\]

(22 January 2007)
where $A, \delta$ are arbitrary constants and $\Omega^2 = \omega^2 + \omega_0^2$. Using this general solution we have
\[
\dot{x} = \omega \left[ A \cos(\Omega t + \delta) - \frac{p_x}{m} \frac{\omega}{\Omega^2} \right] + \frac{p_x}{m}
\]
so that
\[
x = \frac{\omega}{\Omega} A \sin(\Omega t + \delta) - \frac{p_x}{\Omega^2} \frac{\omega^2}{m} t + \frac{p_x}{m} + \text{const}
\]
If we now apply the boundary condition $y = (v_x, 0, 0)$ at $t = 0$ and remove irrelevant constants we have
\[
x = \nu \frac{\omega^2}{\Omega} t + \beta \sin(\Omega t) \\
y = -\nu \frac{\omega}{\Omega^2} t + \beta \cos(\Omega t)
\]
Where $\nu = \frac{p_x}{m}$ and $\beta = v_x - \nu \omega_0 / \Omega$.

These solutions have a number of special cases. For $\beta = 0$ the trajectory in the $x, y$ plane is a straight line with $y = -\nu \omega / \Omega^2$. For small $\beta$ the trajectory is a sinusoidal oscillation around this value of $y$. Larger values of $\beta$ produce a helical trajectory. For large $B$ the kinetic energy in the $x$ direction is quenched and the trajectory tends towards a closed circle.

Q2. Hamilton’s principle states that $\delta \int dt L(q, \dot{q}, t) = 0$ and leads to (via calculus of variations)
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}
\]
i.e. a collection of $N$ 2nd order equations for the coordinates $q_i$.

The kinetic energy of the masses at B and $B'$ is $2 \times \frac{1}{2} m_1 a^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta)$.
The mass at A' has the velocity $2a \dot{\theta} \sin \theta$ and so contributes the kinetic energy $2m_2 a^2 \sin^2 \theta \dot{\theta}^2$. The potential energy is made of two contributions from the $m_1$ masses and one from $m_2$, giving $V = -ga \cos \theta (2m_1 + 2m_2)$. This gives the Lagrangian in question, $L = T - V$, with the only variable $\theta(t)$.

The corresponding canonical momentum $p = \partial L / \partial \dot{\theta} = a^2 \dot{\theta} (2m_1 + 4m_2 \sin^2 \theta)$.
The equation of motion is (note the partial cancellation of the $\dot{\theta}$ term)
\[
a^2 (2m_1 + 4m_2 \sin^2 \theta) \ddot{\theta} + 4a^2 m_2 \sin \theta \cos \theta \dot{\theta} = 2a \sin \theta (m_1 a \Omega^2 \cos \theta - g[m_1 + m_2])
\]
In equilibrium the l.h.s. is zero and so
\[
\cos \theta = \frac{g[m_1 + m_2]}{m_1 a \Omega^2}
\]

(22 January 2007)
The stable position has to be at $\theta = 0$ unless $\cos \theta_0 \leq 1$, which gives the critical spinning velocity

$$\Omega_c^2 = \frac{g(m_1 + m_2)}{m_1 a}$$

For small oscillations about $\theta = \theta_0$ we ignore the $\dot{\theta}^2$ term and expand the r.h.s. of the dynamic equation, obtaining

$$a(m_1 + 2m_2 \sin^2 \theta_0) \ddot{\theta} \approx \delta \theta (m_1 a \Omega^2 [\cos^2 \theta_0 - \sin^2 \theta_0] - g(m_1 + m_2) \cos \theta_0)$$

(7)

where $\delta \theta = \theta - \theta_0$ is the small deviation from equilibrium. Substituting $g(m_1 + m_2)$ from the expression above leads to cancellation of $\cos^2$ terms and the final equation

$$a(m_1 + 2m_2 \sin^2 \theta_0) \ddot{\theta} + m_1 a \Omega^2 \sin^2 \theta_0 \delta \theta$$

(8)

The frequency of the resulting small oscillations is, therefore, as given in the question.

Q3. First of all, let's write down the Lagrangian in the simplifying case. Now $(dx^0, dx^1) = (cdt, dx)$ and

$$g_{\mu\nu} = \begin{pmatrix} g(x) & 0 \\ 0 & -g(x) \end{pmatrix}$$

which gives, after multiplication under the root,

$$L = -m_0 \sqrt{c^2 g(x) - \dot{x}^2 g(x)} = -m_0 c \sqrt{g} \sqrt{1 - v^2 / c^2}$$

The l.h.s. of the Euler-Lagrange equation will then take the form

$$\frac{d}{dt} \left( m_0 \frac{\Gamma}{\sqrt{c^2 - \dot{x}^2}} \right) = \frac{d}{dt} \left( m_0 v \frac{\sqrt{g}}{\sqrt{c^2 - v^2}} \right)$$

(4)

(the factor following the $m_0 v$ is therefore denoted as $\Gamma$ in the question. The r.h.s. is

$$\frac{\partial L}{\partial x} = -m_0 c \sqrt{1 - v^2 / c^2} \left( \frac{1}{2 \sqrt{g}} \frac{\partial g}{\partial x} \right) = -\frac{m_0}{\Gamma} \frac{\partial}{\partial x} \left[ \frac{1}{2} \dot{g}(x) \right]$$

where $\phi$ is the expression in square brackets.

For the general case of $L = -m_0 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$ we just need to be careful with components and indices. For the three spatial components of the 4-vector variable, we'll have in the l.h.s. of the Euler-Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) = \frac{d}{dt} \left( -\frac{2 g_{\mu \nu} \ddot{x}^\mu}{2 \sqrt{g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu}} \right) = \frac{d}{dt} \left( \gamma g_{\mu \nu} \ddot{x}^\mu \right)$$

(22 January 2007)
Here \( i = (1, 2, 3) \) and \( \mu, \nu = (0, 1, 2, 3) \). Now evaluating the derivatives in the r.h.s. we should group terms together into \( \gamma = \frac{m_0}{\sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \) (or, equivalently, without \( m_0 \) as this cancels on both sides of the linear equation):

\[
\frac{\partial L}{\partial x_i} = -m_0 \left( \frac{\partial g_{\mu\nu}}{\partial x_i} \dot{x}^\mu \dot{x}^\nu \right) \equiv \frac{1}{2} \gamma \left( \frac{\partial g_{\mu\nu}}{\partial x_i} \right) \dot{x}^\mu \dot{x}^\nu.
\]

Q4. The inverse transform is

\[
\rho(r) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k \ \hat{\rho}(k) \exp(-ik \cdot r)
\]

The relation between the Fourier transforms is

\[
|k|^2 \hat{\varphi} = \frac{\hat{\rho}}{\epsilon_0}
\]

so we can (in the absence of noise) find the potential via the relation

\[
\varphi(r) = \frac{1}{(2\pi)^2 \epsilon_0} \int_{-\infty}^{\infty} d^3k \ \frac{\hat{\rho}(k)}{|k|^2} \exp(-ik \cdot r)
\]

For the case \( \rho(r) = A \cos(Qx) \) for the layer \(-t \leq z \leq t\), we have the Fourier transform

\[
\hat{\rho}(x, y, z) = \int_{-t}^{t} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \ A \cos(Qx) \exp(i(k_x x + k_y y + k_z z))
\]

Writing \( \cos(Qx) = \frac{1}{2} (\exp(iQx) + \exp(-iQx)) \), using

\[
\int_{-\infty}^{\infty} dx \ \exp(-ikx) = 2\pi \delta(k)
\]

for the \( x \) and \( y \) integrals and doing the \( z \) integral explicitly, we find

\[
\hat{\rho}(k) = (2\pi)^3 A \delta(k_y) \left( \delta(k_z - Q) + \delta(k_z + Q) \right) \frac{\sin(k_z t)}{k_z}.
\]

The back-transform is only required for \( y = z = 0 \) (the potential is independent of \( y \) anyway, but the variation in \( z \) is quite interesting...), so, after the trivial \( k_y \) integral, we have

\[
\varphi(x, 0, 0) = \frac{A}{2\pi \epsilon_0} \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_x \left( \delta(k_x - Q) + \delta(k_x + Q) \right) \frac{\sin(k_z t)}{k_z} \frac{\exp(-i(k_z x))}{k_z^2 + k_x^2 + Q^2}.
\]

Doing the \( k_x \) integral leaves

\[
\varphi(x, 0, 0) = \frac{A \cos(Qx)}{\pi \epsilon_0} \int_{-\infty}^{\infty} dk_z \frac{\sin(k_z t)}{k_z^2 + Q^2} = \frac{A \cos(Qx)}{\pi \epsilon_0} t^2 \frac{1}{2} \left( \frac{A}{t^2} \right).
\]

(22 January 2007)
using the definition of $I(a)$ given.

To do the integral, you can either write $\sin k = (\exp(ik) - \exp(-ik))/2i$ and close over the top for the first term and underneath for the second one, or express it as $\Im(\exp(ik))$ and just use the pole at $k = ia$, which has residue $\exp(-a)/2a^2$. There is a slight subtlety with the pole at the origin, which has residue $1/a^2$, but only contributes $\pi i \times$ residue because it is exactly on the path of integration.

The final answer is $\epsilon_0 \varphi(x, 0, 0) = A \cos(Qx) (1 - \exp(-Qt))/Q^2$.

Q5. We wish to evaluate

$$I = \int_{-1}^{1} \sqrt{1 - x^2} \, dx$$

using contour integration.

Consider $(z^2 - 1)^{1/2}$ with a branch cut from -1 to 1. For $z = x$ on the real axis, just above the cut we have

$$(z^2 - 1)^{1/2} = i\sqrt{1 - x^2}$$

Consider the contour $C$ in the figure:

\[ \oint (z^2 - 1)^{1/2} \, dz = \int_{-1}^{1} i\sqrt{1 - x^2} \, dx \]

$$= -2i I$$

We can deform the contour $C$ to the contour $C_R$, the circle of radius $R$, as there are no singularities between $C$ and $C_R$. Hence

$$I = \frac{1}{2i} \oint_{C_R} (z^2 - 1)^{1/2} \, dz = \int_{-1}^{1} i\sqrt{1 - x^2} \, dx$$

$$= -2i I$$

(22 January 2007) (TURN OVER)
Let \( \zeta = 1/z \), and let \( C_{1/R} \) be the circle of radius \( 1/R \) traversed clockwise, so that \( C_{1/R} \) is the image of \( C_R \) under the transformation \( z \to \zeta \). Then
\[
z = -\zeta^{-2}d\zeta,
\]
s making the substitution
\[
I = \frac{1}{2i} \oint_{C_{1/R}} (\zeta^{-2} - 1)^{1/2}(-\zeta^{-2})d\zeta
\]

Now,
\[
(\zeta^{-2} - 1)^{1/2}(-\zeta^{-2}) = -\zeta^{-3}(1 - \frac{1}{2}\zeta^2 + \ldots)
\]
so the integrand has a singularity at \( \zeta = 0 \) with residue \( 1/2 \). Therefore (introducing a minus sign because \( C_{1/R} \) is traversed in the negative sense),
\[
I = -\frac{1}{2i} \times 2\pi i \times \frac{1}{2} = \pi^{1/2}
\]

We wish to prove that
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2\pi a}
\]
using the identity
\[
\oint_C \frac{\pi \cot \pi z}{(a+z)^2} dz = 0,
\]
The integrand of the identity has (i) simple poles at \( z = n \) where \( n \) is any integer and (ii) a double pole at \( z = -a \).

To find the residue of \( \cot \pi z \), put \( z = n + \xi \) for small \( \xi \):
\[
\cot \pi z = \frac{\cos(n\pi + \xi\pi)}{\sin(n\pi + \xi\pi)} \approx \frac{\cos n\pi}{(\cos n\pi)\xi} = \frac{1}{\xi}
\]
The residue of the integrand at \( z = n \) is thus \( \pi(a+n)^{-2}\pi^{-1} \).

Putting \( z = -1 + \xi \) for small \( \xi \) and determining the coefficient of \( \xi^{-1} \)
\[
\frac{\pi \cot \pi z}{(a+z)^2} = \frac{\pi}{\xi^2} \cot(-a\pi + \xi\pi)
\]
\[
= \frac{\pi}{\xi^2} \left\{ \cot(-a\pi) + \xi \left[ \frac{d}{dz} \cot \pi z \right]_{z=-a} + \ldots \right\}
\]
so that the residue at the double pole \( z = -a \) is
\[
\pi \left[ -\pi \csc^2 \pi z \right]_{z=-a} = -\pi^2 \csc^2 \pi a.
\]

(22 January 2007)
Collecting together these terms and using the residue theorem gives

\[ I = \oint_C \frac{\pi \cot \pi z}{(a + z)^2} \, dz = 2\pi i \left[ \sum_{n=-N}^{N} \frac{1}{(a + n)^2} - \pi^2 \csc^2 \pi a \right] \]

where \( N \) equals the integer part of \( R \). But as the radius \( R \) of \( C \) tends to \( \infty \), \( \cot \pi z \to \pm i \) (depending on whether \( \Im(z) \) is greater or less than zero respectively. Hence,

\[ I < \int k \frac{\, d}{(a + z)^2} \]

which tends to zero as \( R \to \infty \). Thus \( I \to 0 \) as \( R \) (and hence \( N \to \infty \). We therefore have

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(a + n)^2} = \frac{\pi^2}{\sin^2 \pi a} \]

Q6. Need to describe, for a discrete one-dimensional process with length scale \( a \) and timescale \( \tau \), the idea that the transitions rates into \( P_{N+1}(m) \) are given by \( w(m, m')P_N(m') \).

Principle of detailed balance is then \( w(m, m')P(m') = w(m', m)P(m) \) for each pair \( m, m' \).

The idea of the derivation presented in the notes was to consider the case when transitions are made only from \( m \) to \( m \pm 1 \), so that

\[ P_{N+1}(m) = w(m, m + 1)P_N(m + 1) - w(m + 1, m)P_N(m) + w(m, m - 1)P_N(m - 1) - w(m - 1, m)P_N(m) \]  

If the diffusion is symmetric \( w = 1/2 \), and we get the diffusion equation with coefficient \( D = a^2/\tau \)

If there is a vertical asymmetry due to gravity, then transitions to \( k - 1 \) are preferred over those to \( k + 1 \), giving the first-derivative term in

\[ \frac{\partial P}{\partial t} = \frac{1}{2}D \left( \frac{\partial^2 P}{\partial z^2} + \frac{mg}{k_B T} \frac{\partial P}{\partial z} \right) \]

The argument leading to the coefficient on this term will probably be circular (appeal to Boltzmann factors...), but never mind. An alternative is to use the formal Fokker-Planck equation derivation for a constant force \( F = mg \).

The steady-state solution of this equation is

\[ P(z) \propto \exp(-\frac{mgz}{kT}) \]

The critical size of particle is that for which \( \frac{mga}{kT} \sim 1 \). Evaluating this for the given parameters we find \( a \sim 10^{-6} \) m.

(22 January 2007)  

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