Theoretical Physics 1
Answers to Examination ~~2006~~ 2007

Warning — these answers have been completely retyped...
Please report any typos/errors to `emt1000@cam.ac.uk`

Q1. Bookwork: the canonical momenta are $p_i \equiv \partial L/\partial \dot{q}_i$. The Hamiltonian is

$$H \equiv \sum_i p_i \dot{q}_i - L ,$$

which is a function of (q_i, p_i) but not $\dot{q}_i$. Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} ,$$

i.e. a set of $2N$ first-order equations for the coordinates and momenta. For a charged particle we add the scalar $-q(\phi - \boldsymbol{A}\cdot\dot{\boldsymbol{x}})$ to the Lagrangian. The canonical momentum is then $\boldsymbol{p} = m\dot{\boldsymbol{x}} + q\boldsymbol{A}$, but the Hamiltonian is still $H = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + q\phi$. Expressed as a function of $\boldsymbol{p}$ we have

$$H = \frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m} + q\phi$$ (8)

The vector potential $(-By, 0, 0)$ has $\nabla\times\boldsymbol{A} = (0, 0, B)$ as required and $\boldsymbol{E} = -\nabla\phi$ as required. (4)

The Hamiltonian is

$$H = \frac{(p_x + qBy)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_0^2 y^2 .$$

It does not depend on x, z or t, so p_x, p_z and H are constants of motion. (4)
The equations for p_y, x and y are

$$\begin{aligned} \dot{p}_y &= -\frac{\partial H}{\partial y} = -\frac{qB}{m}(p_x + qBy) - m\omega_0^2 y ; && (2) \\ \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x + qBy}{m} ; && (1) \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m} . \quad \dot{z} = p_z/m && (1) \end{aligned}$$

(1)

Differentiating the $\dot{y}$ equation and substituting we get the required result

$$\ddot{y} + (\omega^2 + \omega_0^2)y = -\frac{\omega p_y}{m}$$ (3)

where $\omega = qB/m$, the Larmor frequency. This has general solution

$$y = A\cos(\Omega t + \delta) - \frac{p_x}{m}\cdot\frac{\omega}{\Omega^2}$$ (1)

(22 January 2007)

where A, δ are arbitrary constants and $\Omega^2 = \omega^2 + \omega_0^2$. Using this general solution we have

$$\dot{x} = \omega \left[A\cos(\Omega t + \delta) - \frac{p_x}{m}\frac{\omega}{\Omega^2} \right] + \frac{p_x}{m}$$

(1)

so that

$$x = \frac{\omega}{\Omega} A\sin(\Omega t + \delta) - \frac{p_x}{m}\frac{\omega^2}{\Omega^2} t + \frac{p_x}{m} t + const$$

If we now apply the boundary condition $\underline{v} = (v_x, 0, 0)$ at $t = 0$ and remove irrelevant constants we have

1 + 2 b.cs correctly applied

$$\begin{aligned} x &= \nu \frac{\omega_0^2}{\Omega^2} t + \frac{\beta}{\Omega}\sin(\Omega t) \\ y &= -\nu \frac{\omega}{\Omega^2} t + \frac{\beta}{\omega}\cos(\Omega t) \end{aligned}$$

1

1

Where $\nu = p_x/m$ and $\beta = v_x - \nu\,\omega_0/\Omega$.

These solutions have a number of special cases. For $\beta = 0$ the trajectory in the x, y plane is a straight line with $y = -\nu\,\omega/\Omega^2$. For small β the trajectory is a sinusoidal oscillation around this value of y. Larger values of β produce a helical trajectory. For large B the kinetic energy in the x direction is quenched and the trajectory tends towards a closed circle.

(4)

Q2. Hamilton's principle states that $\delta \int dt\, L(q, \dot{q}, t) = 0$ and leads to (via calculus of variations)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \qquad (4)$$

i.e. a collection of N 2nd order equations for the coordinates q_i.

The kinetic energy of the masses at B and B' is $2 \times \frac{1}{2} m_1 a^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta)$. The mass at A' has the velocity $2a\dot{\theta}\sin\theta$ and so contributes the kinetic energy $2m_2 a^2 \sin^2\theta\,\dot{\theta}^2$.The potential energy is made of two contributions from the m_1 masses and one from m_2, giving $V = -ga\cos\theta(2m_1 + 2m_2)$. This gives the Lagrangian in question, $L = T - V$, with the only variable $\theta(t)$.

The corresponding canonical momentum $p = \partial L/\partial\dot{\theta} = a^2\dot{\theta}(2m_1 + 4m_2\sin^2\theta)$. The equation of motion is (note the partial cancellation of the $\dot{\theta}$ term)

$$a^2(2m_1 + 4m_2\sin^2\theta)\ddot{\theta} + 4a^2 m_2 \sin\theta\cos\theta\,\dot{\theta} = 2a\sin\theta(m_1 a\Omega^2\cos\theta - g[m_1 + m_2]) \qquad (5)$$

In equilibrium the l.h.s. is zero and so

$$\cos\theta = \frac{g(m_1 + m_2)}{m_1 a \Omega^2} \qquad (6)$$

The stable position has to be at $\theta = 0$ unless $\cos\theta_0 \le 1$, which gives the critical spinning velocity

$$\Omega_c^2 = \frac{g(m_1 + m_2)}{m_1 a}$$

For small oscillations about $\theta = \theta_0$ we ignore the $\dot{\theta}^2$ term and expand the r.h.s. of the dynamic equation, obtaining

$$a(m_1 + 2m_2 \sin^2\theta_0)\ddot{\theta} \approx \delta\theta(m_1 a\Omega^2[\cos^2\theta_0 - \sin^2\theta_0] - g[m_1 + m_2]\cos\theta_0) \quad (7)$$

where $\delta\theta = \theta - \theta_0$ is the small deviation from equilibrium. Substituting $g(m_1 + m_2)$ from the expression above leads to cancellation of $\cos^2$ terms and the final equation

$$a(m_1 + 2m_2 \sin^2\theta_0)\delta\ddot{\theta} + m_1 a\Omega^2 \sin^2\theta_0\, \delta\theta \quad (8)$$

The frequency of the resulting small oscillations is, therefore, as given in the question.

Q3. First of all, let's write down the Lagrangian in the simplifying case. Now $(dx^0, dx^1) = (cdt, dx)$ and

$$g_{\mu\nu} = \begin{pmatrix} g(x) & 0 \\ 0 & -g(x) \end{pmatrix}$$

which gives, after multiplication under the root, [4]

$$L = -m_0\sqrt{c^2 g(x) - \dot{x}^2 g(x)} = -m_0 c\sqrt{g}\sqrt{1 - v^2/c^2}$$

The l.h.s. of the Euler-Lagrange equation will then take the form [4]

$$\frac{d}{dt}\left(m_0\sqrt{g}\frac{\dot{x}}{\sqrt{c^2 - \dot{x}^2}}\right) = \frac{d}{dt}\left(m_0 v\frac{\sqrt{g}}{\sqrt{c^2 - v^2}}\right)$$

(the factor following the $m_0 v$ is therefore denoted as Γ in the question. The r.h.s. is

$$\frac{\partial L}{\partial x} = -m_0 c\sqrt{1 - v^2/c^2}\left(\frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x}\right) = -\frac{m_0}{\Gamma}\frac{\partial}{\partial x}[\frac{1}{2}g(x)]$$

where ϕ is the expression in square brackets. [8]

For the general case of $L = -m_0\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ we just need to be careful with components and indices. For the three spatial components of the 4-vector variable, we'll have in the l.h.s. of the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt}\left(-m_0\frac{2g_{i\mu}\dot{x}^\mu}{2\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right) \equiv \frac{d}{dt}\left(\gamma g_{i\mu}\dot{x}^\mu\right)$$

Here $i = (1, 2, 3)$ and $\mu, \nu = (0, 1, 2, 3)$. Now evaluating the derivatives in the r.h.s. we should group terms together into $\gamma = -m_0/\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ (or, equivalently, without m_0 as this cancels on both sides of the linear equation): [10]

$$\frac{\partial L}{\partial x_i} = -m_0 \frac{(\partial g_{\mu\nu}/\partial x_i)\dot{x}^\mu\dot{x}^\nu}{2\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \equiv \frac{1}{2}\gamma\left(\frac{\partial g_{\mu\nu}}{\partial x_i}\right)\dot{x}^\mu\dot{x}^\nu.$$

[8]

Q4. The inverse transform is

$$\rho(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \mathrm{d}^3\boldsymbol{k}\ \tilde{\rho}(\boldsymbol{k})\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})$$ [4]

The relation between the Fourier transforms is

$$|\boldsymbol{k}|^2\tilde{\varphi} = \frac{\tilde{\rho}}{\epsilon_0}$$ [4]

so we can (in the absence of noise) find the potential via the relation

$$\varphi(\boldsymbol{r}) = \frac{1}{(2\pi)^3\epsilon_0}\int_{-\infty}^{\infty} \mathrm{d}^3\boldsymbol{k}\ \frac{\tilde{\rho}(\boldsymbol{k})}{|\boldsymbol{k}|^2}\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})$$ [4]

For the case $\rho(\boldsymbol{r}) = A\cos(Qx)$ for the layer $-t \le z \le t$, we have the Fourier transform

$$\tilde{\rho}(x, y, z) = \int_{-t}^{t}\mathrm{d}z\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}x\ A\cos(Qx)\exp(i(k_x x + k_y y + k_z z))$$

Writing $\cos(Qx) = \frac{1}{2}(\exp(iQx) + \exp(-iQx))$, using $\int_{-\infty}^{\infty}\mathrm{d}x\ \exp(-ikx) = 2\pi\delta(k)$ for the x and y integrals and doing the z integral explicitly, we find

$$\tilde{\rho}(\boldsymbol{k}) = (2\pi)^2 A\delta(k_y)\left(\delta(k_x - Q) + \delta(k_x + Q)\right)\frac{\sin(k_z t)}{k_z}\ .$$ [10]

The back-transform is only required for $y = z = 0$ (the potential is independent of y anyway, but the variation in z is quite interesting...), so, after the trivial k_y integral, we have

$$\varphi(x, 0, 0) = \frac{A}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\mathrm{d}k_z\int_{-\infty}^{\infty}\mathrm{d}k_x\left(\delta(k_x - Q) + \delta(k_x + Q)\right)\frac{\sin(k_z t)}{k_z}\frac{\exp(-i(k_x x))}{k_x^2 + k_z^2}\ . \tag{14}$$

Doing the k_x integral leaves

$$\varphi(x, 0, 0) = \frac{A\cos(Qx)}{\pi\epsilon_0}\int_{\infty}^{\infty}\mathrm{d}k_z\ \frac{\sin(k_z t)}{k_z(k_z^2 + Q^2)} = \frac{A\cos(Qx)}{\pi\epsilon_0}t^2 I(Qt)\ ,$$ [5]

$\phi(x,0,0) = \dfrac{A\cos(QX)}{\pi\varepsilon_0}t^2 I(Qt)$

$\phi = \dfrac{A\cos QX\,(1 - \exp(-Qt))}{Q}$ $\quad I(Qt) = \dfrac{\pi(1 - \exp(-(Qt)))}{t^2}$

using the definition of $I(a)$ given.

To do the integral, you can either write $\sin k = (\exp(ik) - \exp(-ik))/2i$ and close over the top for the first term and underneath for the second one, or express it as $\Im(\exp(ik))$ and just use the pole at $k = ia$, which has residue $\exp(-a)/2a^2$. There is an slight subtlety with the pole at the origin, which has residue $1/a^2$, but only contributes $\pi i \times$ residue because it is exactly on the path of integration.

The final answer is $\epsilon_0\varphi(x,0,0) = A\cos(Qx)\left(1 - \exp(-Qt)\right)/Q^2$.

Q5. We wish to evaluate

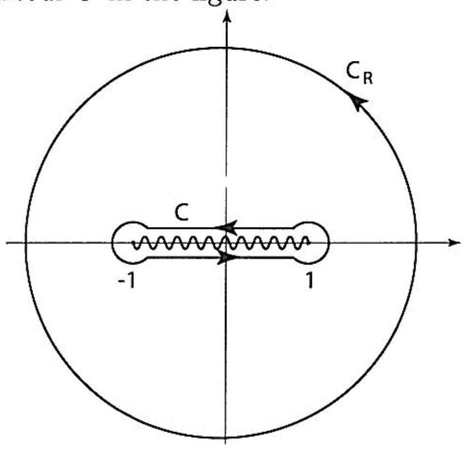
$$I = \int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

using contour integration.

Consider $(z^2-1)^{1/2}$ with a branch cut from -1 to 1. For $z = x$ on the real axis, just above the cut we have

$$(z^2-1)^{1/2} = i\sqrt{1-x^2}$$

Consider the contour C in the figure:



$$\begin{aligned}\oint (z^2-1)^{1/2}\mathrm{d}z &= \int_{-1}^{1} i\sqrt{1-x^2}\,\mathrm{d}x \\ &= -2iI\end{aligned}$$

We can deform the contour C to the contour C_R, the circle of radius R, as there are no singularities betwen C and C_R. Hence

$$\begin{aligned}I = \frac{1}{2}i\oint_{C_R}(z^2-1)^{1/2}\mathrm{d}z &= \int_{-1}^{1} i\sqrt{1-x^2}\,\mathrm{d}x \\ &= -2iI\end{aligned}$$

Let $\zeta = 1/z$, and let $C_{1/R}$ be the circle of radius $1/R$ traversed *clockwise*, so that $C_{1/R}$ is the image of C_R under the transformation $z \to \zeta$. Then $z = -\zeta^{-2}\mathrm{d}\zeta$, s making the substitution

$$I = \frac{1}{2} i \oint_{C_{1/R}} (\zeta^{-2} - 1)^{1/2}(-\zeta^{-2})\mathrm{d}\zeta$$

Now,

$$(\zeta^{-2} - 1)^{1/2}(-\zeta^{-2}) = -\zeta^{-3}(1 - \frac{1}{2}\zeta^2 + \ldots)$$

so the integrand has a singularity at $\zeta = 0$ with residue $1/2$. Therefore (introducing a minus sign because $C_{1/R}$ is traversed in the negative sense),

$$I = -\frac{1}{2} i \times 2\pi i \times \frac{1}{2} \quad = \pi/2$$ [17]

We wish to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

using the identity

$$\oint_C \frac{\pi \operatorname{cotan} \pi z}{(a+z)^2} \mathrm{d}z = 0,$$

The integrand of the identity has (i) simple poles at $z = n$ where n is any integer and (ii) a double pole at $z = -a$.

To find the residue of $\cot \pi z$, put $z = n + \xi$ for small ξ:

$$\cot \pi z = \frac{\cos(n\pi + \xi\pi)}{\sin(n\pi + \xi\pi)} \approx \frac{\cos n\pi}{(cos n\pi)\xi\pi} = \frac{1}{\xi\pi}$$

The residue of the integrand a $z = n$ is thus $\pi(a+n)^{-2}\pi^{-1}$.

Putting $z = -1 + \xi$ for small ξ and determining the coefficient of ξ^{-1}

$$\begin{aligned}
\frac{\pi \cot \pi z}{(a+z)^2} &= \frac{\pi}{\xi^2} \cot(-a\pi + \xi\pi) \\
&= \frac{\pi}{\xi^2} \left\{ \cot(-a\pi) + \xi \left[\frac{\mathrm{d}}{\mathrm{d}z} \cot \pi z \right]\Bigg|_{z=-a} + \ldots \right\}
\end{aligned}$$

so that the residue at the double pole $z = -a$ is

$$\pi \left[-\pi \mathrm{cosec}^2 \pi z \right]_{z=-a} = -\pi^2 \mathrm{cosec}^2 \pi a.$$

Collecting together these terms and using the residue theorem gives

$$I = \oint_C \frac{\pi \cot \pi z}{(a+z)^2} \mathrm{d}z = 2\pi i \left[\sum_{-N}^{N} \frac{1}{(a+n)^2} - \pi^2 \mathrm{cosec}^2 \pi a \right]$$

where N equals the integer part of R. But as the radius R of C tends to ∞, $\cot \pi z \to \pm i$ (depending on whether $\Im(z)$is greater or less than zero respectively. Hence,

$$I < \int k \frac{\mathrm{d}}{(a+z)^2}$$

which tends to zero as $R \to \infty$. Thus $I \to 0$ as R (and hence $N \to \infty$. We therefore have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

[17]

Q6. Need to describe, for a discrete one-dimensional process with length scale a and timescale τ, the idea that the transitions rates into $P_{N+1}(m)$ are given by $w(m, m')P_N(m')$.

Principle of detailed balance is then $w(m, m')P(m') = w(m', m)P(m)$ for each pair m, m'

The idea of the derivation presented in the notes was to consider the case when transitions are made only from m to $m \pm 1$, so that

$$\begin{aligned} P_{N+1}(m) &= w(m, m+1)P_N(m+1) - w(m+1, m)P_N(m) \\ &\quad + w(m, m-1)P_N(m-1) - w(m-1, m)P_N(m) \end{aligned} \tag{18}$$

If the diffusion is symmetric $w = 1/2$, and we get the diffusion equation with coefficient $D = a^2/\tau$

If there is a vertical asymmetry due to gravity, then transitions to $k-1$ are preferred over those to $k+1$, giving the first-derivative term in

$$\frac{\partial P}{\partial t} = \frac{1}{2} D \left(\frac{\partial^2 P}{\partial z^2} + \frac{\tilde{m}g}{k_\mathrm{B}T} \frac{\partial P}{\partial z} \right) \tag{19}$$

The argument leading to the coefficient on this term will probably be circular (appeal to Boltzmann factors...), but never mind. An alternative is to use the formal Fokker-Planck equation derivation for a constant force $F = mg$.

The steady-state solution of this equation is

$$P(z) \propto \exp(-\tilde{m}gz/kT) \tag{20}$$

The critical size of particle is that for which $\tilde{m}ga/kT \sim 1$. Evaluating this for the given parameters we find $a \sim 10^{-6}$ m.