THEORETICAL PHYSICS I

Answer three questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains 4 sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

1 A dynamical system has position co-ordinates \( q_i \) and canonical momenta \( p_i \). Write the Hamiltonian \( H(q_i, p_i, t) \) in terms of the system Lagrangian \( L(q_i, \dot{q}_i, t) \) and these quantities. A non-relativistic particle of mass \( m \) and charge \( q \) moves in an electromagnetic field produced by an electrostatic potential \( \phi \) and magnetic vector potential \( A \). Show that the Hamiltonian is

\[
H = \frac{|p - qA|^2}{2m} + q\phi.
\]

In Cartesian coordinates \((x, y, z)\) the electric field is \( E = (0, -m\omega_0^2 y, 0) \) and the magnetic field is \( B = (0, 0, B) \). Show that \( \phi = m\omega_0^2 y^2/2, A = (-By, 0, 0) \) are suitable choices for the potentials.

For a particle moving in this field, show that the momenta \( p_x, p_z \) and the Hamiltonian \( H \) are constants of the motion.

Find Hamilton’s equations of motion for the variables \( p_y, x, y \) and \( z \) and show that

\[
\ddot{y} + (\omega^2 + \omega_0^2)y = \frac{p_y}{m},
\]

where \( \omega \equiv qB/m \).

Hence find the general solutions for \( x(t), y(t) \) and describe the motion in the \( x, y \) plane of a particle initially moving with velocity \( v = (v_x, 0, 0) \).

2 Describe briefly how the principle of least action leads to Lagrange’s equations of motion for a dynamical system having coordinates and velocities \((q_i, \dot{q}_i)\).

A mechanical governor used to control the speed of a steam engine consists of the configuration shown in the figure:
(i) the vertical axis $AA'$ rotates at a constant angular velocity $\Omega$;
(ii) light rods $AB, AB', A'B, A'B'$ each of length $a$ are freely pivoted at $A, B, A', B'$;
(iii) the pivot at $A$ is fixed, so that the pivot at $A'$ moves as the angle $\theta$ changes;
(iv) masses $m_1$ are attached at $B$ and $B'$ and a mass $m_2$ is free to slide on the vertical axis at $A'$.

Show that the Lagrangian of the system is given by

$$L = m_1 a^2 (\Omega^2 \sin^2 \theta + \dot{\theta}^2) + 2m_2 a^2 \dot{\theta}^2 \sin^2 \theta + 2ag \cos \theta (m_1 + m_2).$$

Find the equation of motion of the system.

Show that the system can rotate in equilibrium with $\theta = 0$ unless $\Omega$ exceeds a certain critical velocity. Determine the equilibrium angle $\theta_0$ for the case when $\Omega$ is greater than this critical value.

Show that the angular frequency of small oscillations about the equilibrium angle $\theta_0$ is given by $\Omega \sin \theta_0 / \sqrt{1 + 2(m_2/m_1) \sin^2 \theta_0}$.

3 Consider the following generalisation of the Lagrangian for a simple relativistic particle:

$$L = -m_0 \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}},$$

where $(dx^0, dx^1, dx^2, dx^3) = (c dt, dx, dy, dz)$, and $g_{\mu\nu}$ is a symmetric tensor which varies with position and time, and $m_0$ is a constant.

For the simplifying case of only time plus one dimension of space, with $g_{00} = -g_{11} = g(x)$, independent of time, and also $g_{10} = g_{01} = 0$, show that the Euler-Lagrangian equations reduce to the form

$$\frac{d}{dt}(\Gamma m_0 v) = -\frac{m_0}{\Gamma} \frac{\partial \phi}{\partial x},$$

and give an explicit expression for the function $\Gamma$ and the potential $\phi$ in terms of $g(x)$ and $v = dx/dt$.

For the general case show that the equations of motion are given by

$$\frac{d}{dt} \left( \gamma g_{k\nu} \frac{dx^\nu}{dt} \right) = \frac{1}{2} \gamma \frac{g_{\mu\nu}}{dt} \frac{dx^\mu}{dt} \frac{\partial g_{\mu\nu}}{\partial x^k},$$

giving the explicit expression for $\gamma$ and indicating carefully what values are taken by the indices $k, \mu, \nu$. 

The three-dimensional Fourier transform of an electric charge density distribution $\rho(r)$ can be written as

$$\tilde{\rho}(k) \equiv \int d^3r \, \rho(r) \exp(ik \cdot r).$$

Write down the formula for the inverse Fourier transform.

If it is placed within a dielectric medium with dielectric constant $\epsilon_0$ the associated electrostatic potential $\varphi(r)$ is determined by the Poisson equation

$$\nabla^2 \varphi = \frac{-\rho}{\epsilon_0}.$$

Find the relationship between the Fourier transforms $\tilde{\rho}(k)$ and $\tilde{\varphi}(k)$. Explain how the potential can be found in terms of an integral over $k$ if the charge density is known.

A uniform metallic layer occupying the region $-t \leq z \leq t$ and extending infinitely in the $x, y$ plane is embedded in a dielectric medium with $\epsilon_0 = 1$. A charge density wave $\rho(r) = A \cos(Qx)$ is set up in the layer by perturbing the electron distribution. Calculate the Fourier transform $\tilde{\rho}(k)$, where $k = (k_x, k_y, k_z)$.

Calculate the potential at the point $(x, 0, 0)$, expressing the answer in terms of $I(a)$, where

$$I(a) \equiv \int_{-\infty}^{\infty} dk \, \frac{\sin k}{(a^2 + k^2)k}.$$

By using a contour integral, show that

$$I(a) = \frac{\pi}{a^2} (1 - \exp(-a)).$$

Describe how the Cauchy integral theorem can be used to evaluate contour integrals in the complex plane. Illustrate your answer by showing that

$$\int_{-1}^{1} \sqrt{1 - x^4} \, dx = \frac{\pi}{2}.$$

(Hint: you may find it useful to note that the integrand has a pole at $x \to \infty$ and a branch cut between $x = -1..1$)

The Cauchy integral theorem can be used to evaluate infinite series. By considering the identity

$$\oint_C \frac{\pi \cotan \pi z}{(a + z)^2} \, dz = 0,$$

where the contour $C$ is a circle of infinite radius in the complex plane centered about $z = 0$, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a + n)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$
Discuss the concept of discrete transition probability $w(k, k')$ for a discrete one-dimensional random walk.

Consider an ensemble of small identical spherical Brownian particles, of radius $a$ and density $\rho$, suspended in a container filled with water (density $\rho_0$). Derive the modified diffusion equation for the probability $P(z, t)$ of finding a particle at a height $z$ taking into account only first-order corrections in powers of $\frac{\tilde{m} ga}{k_B T}$ (assumed small):

$$\frac{\partial P}{\partial t} = \frac{1}{2} D \left( \frac{\partial^2 P}{\partial z^2} + \frac{\tilde{m} g}{k_B T} \frac{\partial P}{\partial z} \right)$$

where $\tilde{m} = 4\pi (\rho - \rho_0) a^3 / 3$.

Derive the equilibrium Boltzmann distribution of these particles along the vertical $z$-axis.

For $\rho = 1.1 \times 10^3$ kg m$^{-3}$ and $\rho_0 = 1 \times 10^3$ kg m$^{-3}$ estimate the order of magnitude of the radius $a$ of a particle for which the effect of Brownian diffusion is relevant, such that the trajectory of moving particle deviates significantly from a straight line.

[At room temperature $k_B T \sim 4 \times 10^{-21}$ J.]