THEORETICAL PHYSICS I

Answer three questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains 3 sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

1. Two equal masses $m$ joined by a length $2a$ of light thin wire are in orbit about a planet of mass $M$. The wire is in the plane of the orbit at all times. Write down a Lagrangian for the system in terms of the polar coordinates $(r, \phi)$ of its midpoint and the angle $\theta$ between the wire and the line to the planet centre, assuming $a \ll$ distance from the planet and the gravitational attraction between the two little masses is negligible. 

![Diagram of orbit and wire]

Find the equation of motion for $\theta$.  
Determine the stability of equilibrium values of $\theta$.  
If the orbit is circular and the system is close to a stable value of $\theta$, how many oscillations does it perform in one orbital period?

2. Describe the terms in the Bernoulli equation in fluid dynamics, and the conditions when it applies.  
Consider a bath of cross-sectional area $0.6 \text{ m}^2$, filled to a depth of $20 \text{ cm}$. Estimate how long it takes to empty when the plug is removed, if the area of the plug hole is $10 \text{ cm}^2$.  
Two cylindrical jets of water, which have the same radius $a$ and velocity components $(0, 0, v)$ and $(0, 0, -v)$ respectively, meet head-on at the origin and spread to form a sheet in the $z = 0$ plane. Show the thickness of this sheet at distance $r$ from the origin is $a^2/r$.
3 The Lagrangian $L$ of a particle of mass $m$ and charge $e$ moving with velocity $v$ in an electrostatic potential $\phi$ is $\frac{1}{2}mv^2 - e\phi$. Using the requirement that $\int L\,dt$ should be Lorentz invariant, or otherwise, explain why the generalisation of the potential energy term in the Lagrangian $L = T - V$ to $V = e(\phi - v \cdot A)$ is required, where $(\phi, A)$ is the electromagnetic 4-potential.

Consider the non-relativistic motion of the charged particle in a constant magnetic field $B$ which is directed along $z$-axis. In cylindrical polar coordinates $(r, \theta, z)$:

(a) By using the Stokes theorem for $\int \mathbf{A} \cdot d\mathbf{l}$, or otherwise, show that

$\{A_\theta = \frac{1}{2}Br, A_r = 0, A_z = 0\}$ represents a magnetic field with a $z$-component $B$.

(b) Write down the Lagrangian and derive the equations of motion: in particular, show that

$$\dot{\theta} = -\frac{eB}{2m} - \frac{J}{r^2}$$

where $J$ is a constant.

(c) Show that the radius of the helical orbit must be proportional to $B^{-1/2}$ and the angular frequency of particle on this orbit is $|\dot{\theta}| = eB/m$.

(d) Further show that the helical pitch angle $\psi$ (the angle between $B$ and $v$) obeys $\tan \psi \propto B^{1/2}$.

4 Describe how contour integration methods can be used to evaluate definite integrals of functions with no poles. Illustrate your answer by showing that:

$$\int_0^\infty dx \cos(x^2) = \int_0^\infty dx \sin(x^2) = \frac{\sqrt{2\pi}}{4}$$

(Hint: use a wedge-shaped contour with angle $\pi/4$, not forgetting that $e^{i\pi/4} = (1 + i)/\sqrt{2}$.)

How are poles on the real axis treated in contour integration? Illustrate your answer by showing that:

$$\int_{-\infty}^{\infty} dx \frac{\sin(x)}{x} = \pi$$

(Hint: $\sin z = (e^{iz} - e^{-iz})/2i$)

How are branch cuts dealt with in contour integration? Illustrate your answer by showing that:

$$\int_0^{\infty} dx \frac{x^\alpha}{1 + \sqrt{2}x + x^2} = \sqrt{2\pi} \frac{\sin(\alpha\pi/4)}{\sin(\alpha\pi)} .$$

for $-1 < \alpha < 1$.

5 Derive the Kramers-Kronig relations between the real and imaginary parts of the generalized susceptibility for a perturbation potential of the form $V = -x.f$ where $x(t)$ is a position coordinate and $f(t)$ a force.
The equation of motion for a damped harmonic oscillator has the form:

\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = f(t) \]

Derive the relationship between the Green’s function for this differential equation \( G(t - t') \), the position coordinate \( x \) and the force \( f \).

Derive an expression for the Fourier Transform of the Green’s function \( G(\omega) \) and write down the Kramers-Kronig relations for its real and imaginary parts.

Convert the principal-value integrals to contour integrals and find their poles.

By evaluating one of these integrals show that the corresponding Kramers-Kronig relation is obeyed by this Green’s function. (Hint: \( \text{P}\int_{-\infty}^{\infty} f(x)dx = \lim_{\epsilon \to 0}[\int_{-\epsilon}^{-\infty} + \int_{x_0}^{\infty}] f(x)dx \) for a pole at \( x = 0 \).)

Thermal (Johnson) noise in a resistor is modelled as a white-noise voltage source, with a mean-square amplitude \( \langle V^2 \rangle = 2Rk_B T \), connected in series with an ideal resistor \( R \). Consider a circuit consisting of such a resistor connected across a capacitor \( C \). Show that the equation of motion for the charge on the capacitor obeys a Langevin equation.

Describe the standard form for the Langevin (stochastic) equation,

\[ \dot{q}_\alpha = F_\alpha(q) + G^i_\alpha(q)A_k(t) , \]

and its relation to classical Brownian motion.

Outline principles leading to the Fokker-Planck (kinetic) equation for the probability density \( f(q,t) \):

\[ \frac{\partial f(q,t)}{\partial t} = \left\{ -\frac{\partial K_\alpha(q)}{\partial q_\alpha} + \frac{1}{2} \frac{\partial^2}{\partial q_\alpha \partial q_\beta} Q_{\alpha\beta}(q) \right\} f(q,t) , \]

with \( K_\alpha = F_\alpha + \frac{1}{2} \frac{\partial G^i_\alpha}{\partial q_\beta} G^k_\beta \delta_{ik} \); \( Q_{\alpha\beta} = G^i_\alpha G^k_\beta \delta_{ik} \).

Obtain the Fokker-Planck equation for the probability density \( f(q,t) \) of the charge on the capacitor.

Show that the equilibrium probability density \( f(q) \) of the charge on the capacitor is proportional to \( \exp[-q^2/2Ck_BT] \) and comment on its form.