Theoretical Physics 1
Answers to Examination 2004

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Q1. The Lagrangian, depending on positions and velocities of all particles is

$$L = \frac{M}{2}\dot{\boldsymbol{R}}^2 + \frac{m}{2}\sum_{\alpha=1}^{n}\dot{\boldsymbol{R}}_\alpha^2 - U \tag{1}$$

A brief discussion of $L = T - U$, depending on $\boldsymbol{q}, \dot{\boldsymbol{q}}$ should be here.The (holonomic) constraint of fixed centre of mass reads: [6]

$$M\boldsymbol{R} + m\sum_\alpha \boldsymbol{R}_\alpha = 0. \tag{2}$$

In suggested relative coordinates, $\boldsymbol{r}_\alpha = \boldsymbol{R}_\alpha - \boldsymbol{R}$, one can directly express

$$(M+mn)\boldsymbol{R} + m\sum_\alpha \boldsymbol{r}_\alpha = 0, \quad \text{or} \quad \boldsymbol{R} = -\frac{m}{M+mn}\sum_\alpha \boldsymbol{r}_\alpha. \tag{3}$$

Substituting this into the Lagrangian and expanding the square under the sum, after two lines of algebra we can obtain [10]

$$L = \frac{m}{2}\sum_\alpha \boldsymbol{v}_\alpha^2 - \frac{1}{2}\frac{m^2}{M+mn}\left(\sum_\alpha \boldsymbol{v}_\alpha\right)^2 - U, \tag{4}$$

which only has n independent variables $\boldsymbol{r}_\alpha$. The canonical momenta are obtained directly: [6]

$$\boldsymbol{p}_\alpha = \frac{\partial L}{\partial \boldsymbol{v}_\alpha} = m\,\boldsymbol{v}_\alpha - \frac{m^2}{M+mn}\left(\sum_\beta \boldsymbol{v}_\beta\right). \tag{5}$$

The Hamiltonian is, by definition, $H = \sum_\alpha \boldsymbol{p}_\alpha \dot{\boldsymbol{r}}_\alpha - L$, but in order to complete the change of variables to $(\boldsymbol{p}_\alpha, \boldsymbol{r}_\alpha)$ we need to express $\boldsymbol{v}_\alpha = \dot{\boldsymbol{r}}_\alpha$ from eq.(5). This may be done in many ways, one is to sum the eq.(5) over α to express $\sum_\alpha \boldsymbol{p}_\alpha = \frac{mM}{M+mn}\sum_\alpha \boldsymbol{v}_\alpha$. After this, one easily obtains

$$\dot{\boldsymbol{r}}_\alpha = \frac{1}{m}\boldsymbol{p}_\alpha + \frac{1}{M}\left(\sum_\beta \boldsymbol{p}_\beta\right) \tag{6}$$

and, after substitution into the definition of Hamiltonian and another line of algebra, the final result: [12]

$$H = \frac{1}{2m}\sum_\alpha \boldsymbol{p}_\alpha^2 + \frac{1}{2M}\left(\sum_\alpha \boldsymbol{p}_\alpha\right)^2 + U. \tag{7}$$

(There are simpler ways of obtaining this expression directly.)

Q2. You may or may not remember that the relevant angular velocity in this case is equal to $(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$. The hint is designed to help those who don't: the full kinetic energy is $\frac{1}{2}I_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + \frac{1}{2}I_3\Omega_3^2$, in principal axes. With $I_1 = I_2 = I_\perp$ and $I_3 = 0$ the Lagrangian reads:

$$L = \frac{1}{2}I_\perp(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) - \frac{1}{2}\kappa(\ell\sin\theta/2)^2 \tag{8}$$

[since the potential energy $U = \frac{1}{2}\kappa(\Delta x)^2$]. [10]

Canonical momenta:

$$\begin{aligned} p_\theta &= I_\perp\dot\theta \\ p_\phi &= I_\perp\dot\phi\sin^2\theta, \quad \text{so } \dot\phi = \frac{p_\phi}{I_\perp\sin^2\theta} \end{aligned} \tag{9}$$

The Hamiltonian: [4]

$$H = \frac{p_\theta^2}{2I_\perp} + \frac{p_\phi^2}{2I_\perp\sin^2\theta} + \frac{\kappa}{2}\ell^2\sin^2\theta/2 \tag{10}$$

The Hamilton equations ($\dot p = -\partial H/\partial q,\ \dot q = \partial H/\partial p$ take the form:

$$\begin{aligned} \dot p_\theta &= \frac{p_\phi^2\cos\theta}{I_\perp\sin^3\theta} - \frac{\kappa\ell^2\sin\theta}{4} \\ \dot p_\phi &= 0 \end{aligned} \tag{11}$$

(the second equation suggests the conservation of z-angular momentum, but it is not equivalent to saying $p\dot hi$ =const). [4]

Substituting the eq.(9) into this, we can obtain the dynamic equation

$$I_\perp\ddot\theta = \sin\theta\left[I_\perp\dot\phi^2\cos\theta - \frac{\kappa\ell^2}{4}\right] \tag{12}$$

The steady state is possible when the bracket in the r.h.s. is held at zero. [10]
For a constant $\dot\phi = \Omega$ this is achieved when

$$\cos\theta_0 = \frac{\kappa\ell^2}{4I_\perp\Omega^2} \equiv \frac{3\kappa}{m\Omega^2} \le 1 \tag{13}$$

(in this stable equilibrium state $\dot\theta$ =const=0).

To find the small oscillations about this equilibrium, expand the r.h.s. in powers of small deviation: $\theta = \theta_0 + \Delta(t)$. It is easier than it may look, because only the leading, linear term is required. The result is [6]

$$\ddot\Delta = -\Delta\Omega^2\sin^2\theta_0 = -\Delta\Omega^2\left(1 - \frac{9\kappa^2}{m^2\Omega^4}\right). \tag{14}$$

Q3. For the constant force F, the potential energy is $U = -Fq$ (giving the coordinate the name q). The relativistic Lagrangian function is [8]

$$L = -\frac{m_0c^2}{\gamma} - U = -m_0c^2\sqrt{1-\dot{q}^2/c^2} + Fq \tag{15}$$

(writing the kinetic energy from memory would be sufficient, but you can derive it, if you've forgotten its form).

Straight from the lecture notes and exercises, the canonical momentum is

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{m_0 v}{\sqrt{1-v^2/c^2}}, \quad \text{so } v^2 = \frac{c^2p^2}{m_0^2c^2+p^2}$$

Substituting this into the Hamiltonian, $H = p\dot{q} - L$, you will easily obtain [8]

$$H = c\sqrt{p^2+m_0^2c^2} - Fq, \quad \text{so } \mathcal{E}_0 = m_0c^2 \tag{16}$$

To prove the energy conservation (which you expect, since no explicit time dependence is present), you must write the full derivative

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}.$$

This is zero when the Hamilton equations hold: [4]

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = F \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{cp}{\sqrt{p^2+m_0^2c^2}} \end{aligned} \tag{17}$$

The first equation integrates directly, to give $p = Ft$ (with the given initial condition). Substituting this $p = p(t)$ into the second equation, we obtain

$$q = \int \frac{cFt\ dt}{\sqrt{F^2t^2+m_0^2c^2}}.$$

The integration is very easy; taking care of the initial condition $q(0) = 0$ gives the answer [8]

$$q = \frac{m_0c^2}{F}\left[-1+\sqrt{1+\frac{F^2t^2}{m_0^2c^2}}\right] \tag{18}$$

The time derivative of this looks a bit messy, but in the limits of short and long time it takes the expected forms: [6]

$$v \approx (F/m_0)t \ \ (t \ll \frac{m_0c}{F}) \qquad v \approx c - \frac{m_0^2c^3}{2F^2t^2} \ \ (t \to \infty) \tag{19}$$

(just declaring that $v \approx c$ would do as well).

Q4. The first step is to Fourier-transform the force in the r.h.s. Please don't be confused by the (much more complicated) FT of the step-function that was discussed in the lectures. The problem to overcome there, and the $1/\omega$ singularity, is due to the infinite limit of integration of oscillating function – but here we have a completely regular expression:

$$f_\omega = \int_0^a f_0 e^{i\omega t} dt = -\frac{i f_0}{\omega}(e^{i\omega a} - 1).$$

Accordingly, the required expression for $x_\omega = G_\omega f_\omega$ is [8]

$$x_\omega = \frac{1}{\omega^2 + i\omega\gamma - \Omega^2} \frac{i f_0}{\omega}(e^{i\omega a} - 1) \tag{20}$$

The discussion of contour integration and causality must include the arguments about closing the contour in the integral $x(t) = \int_{-\infty}^{\infty} x_\omega e^{-i\omega t} d\omega/2\pi$ in the top- or bottom-half plane and how the result is related to the position of singularities on the complex plane. [8]

In this problem, we have:

$$x(t) = -i f_0 \int_{-\infty}^{\infty} \frac{(1 - e^{i\omega a})e^{-i\omega t}}{\omega(\omega^2 + i\omega\gamma - \Omega^2)} \frac{d\omega}{2\pi} \tag{21}$$

It may look like there is a pole at $\omega = 0$, but in fact the force f_ω is completely regular at this point. Only the two simple poles of the Green function matter in the bottom half-plane, at $\omega_{1,2} = -\frac{1}{2} i\gamma \pm \sqrt{\Omega^2 - \frac{1}{4}\gamma^2}$. However, the closing of the contour with $\omega = -Re^{i\phi}$ is only clear-cut when $t - a > 0$. At shorter times (while the force $f(t)$ is still present), the two exponentials in the numerator have to be treated separately: one requires the closure in the bottom-, the other in the top-half plane. Once they are separated (the bracket $(1 - e^{i\omega a})$ expanded), the point $\omega = 0$ becomes an issue – it will require a careful treatment since the contour passes through this singularity. Yo do not need to do this, just outlining the points above is all that's required. [8]

When $t \gg a$ the closing of integration contour in the bottom half-plane is unambiguous (note the contour direction is clockwise) and the result is

$$x(t) = i f_0 (2\pi i) \left(\frac{(1 - e^{i\omega_1 a})e^{-i\omega_1 t}}{\omega_1(\omega_1 - \omega_2)} + \frac{(1 - e^{i\omega_2 a})e^{-i\omega_2 t}}{\omega_2(\omega_2 - \omega_1)} \right) \frac{1}{2\pi}$$

After a little bit of algebra (pulling out the common factors and uniting trigonometric functions), the full result is

$$\begin{aligned} x \;=\; & -\frac{f_0\gamma}{2\Omega^2\sqrt{\Omega^2 - \frac{1}{4}\gamma}} \left(e^{-\frac{1}{2}\gamma t} \sin\sqrt{\Omega^2 - \tfrac{1}{4}\gamma} t - e^{-\frac{1}{2}\gamma(t-a)} \sin\sqrt{\Omega^2 - \tfrac{1}{4}\gamma}(t-a) \right) \\ & -\frac{f_0\gamma}{\Omega^2} \left(e^{-\frac{1}{2}\gamma t} \cos\sqrt{\Omega^2 - \tfrac{1}{4}\gamma} t - e^{-\frac{1}{2}\gamma(t-a)} \cos\sqrt{\Omega^2 - \tfrac{1}{4}\gamma}(t-a) \right). \end{aligned} \tag{22}$$

The limit $t \gg a$ is all that's required. It can be implemented (the expansion, retaining only the leading term – the result is zero at $a \to 0$) at any stage, giving the final approximate result [10]

$$x \approx \frac{a\, f_0}{\sqrt{\Omega^2 - \frac{1}{4}\gamma}} e^{-\frac{1}{2}\gamma t} \sin \sqrt{\Omega^2 - \frac{1}{4}\gamma}t.$$

Q5. The first integration is very easy, but you need to draw the complex plane and the contours on it. We need to evaluate

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\epsilon f(x)}{(x-y)^2 + \epsilon^2} \frac{dx}{\pi}$$

The denominator has two roots at $x_{1,2} = y \pm i\epsilon$, above and below the real axis. [4]

You can close the contour with a semi-circle at $R \to \infty$ in either of the half-planes, taking care of the direction of the contour and the resulting sign. In both cases only one pole would be encircled. [4]

The upper half-plane contour gives [6]

$$\lim_{\epsilon \to 0} 2i \frac{\epsilon f(x_1)}{x_1 - x_2} = \lim_{\epsilon \to 0} 2i \frac{\epsilon f(y + i\epsilon)}{2i\epsilon} = f(y)$$

as required.

The second integration is not trivial at all, but the two hints should guide you. Write the product of two gamma functions as a double integral over $dt\, ds$:

$$\Gamma(x)\Gamma(1-x) = \int_0^{\infty} t^{x-1} e^{-t} dt \int_0^{\infty} s^{[1-x]-1} e^{-s} ds\ .$$

The recommended substitution does wonders

$$\int \int_0^{\infty} u^{x-1} e^{-us} s^{x-1} s du s^{-x} e^{-s} ds = \int_0^{\infty} u^{x-1} e^{-s(u+1)} ds du \qquad (23)$$

The first step is achieved by integrating over s. [6]

It is necessary to design a contour such as shown in the question because we need to evaluate the integral between 0 and ∞. (You may equivalently choose a contour with the cut along the positive axis and the original integral with the pole at $u = -1$, but the one suggested gives the easier value of residue.) The whole close-contour integral

$$\oint \frac{z^{x-1}}{1-z} dz = -2\pi i.$$

It consists of two integrals over the big and the small circles, both tending to zero for $0 < x < 1$, and two integrals along the cut (with $z = u\, e^{\pm i\pi}$):

$$\int_{-\pi}^{\pi} \frac{R^x e^{i\phi x}}{1 - Re^{i\phi}} d\phi + \int_R^{\epsilon} \frac{u^{x-1} e^{i\pi x} du}{1+u} + \int_{\pi}^{-\pi} \frac{\epsilon^x e^{i\phi x}}{1 - \epsilon e^{i\phi}} d\phi + \int_{\epsilon}^{R} \frac{u^{x-1} e^{-i\pi x} du}{1+u}$$
$$= \left(e^{-i\pi x} - e^{i\pi x}\right) \int_0^{\infty} \frac{u^{x-1} du}{1+u} = -2\pi i \quad (24)$$

Identifying the $\sin \pi x$ and dividing through, the required result is obtained. [14]

Q6. The first two parts are straight from the lecture notes: The description of terms should include the mention of dynamic and stochastic forces and the statistical properties of white noise $A(t)$, its second moment is either Γ or defined as 1, with the prefactor $G_{\alpha}^{k} = \sqrt{\Gamma}$. For the free Brownian particle: $\dot{\boldsymbol{v}} = -\gamma \boldsymbol{v} + \boldsymbol{A}(t)$. Strictly, there are two Langevin equations (the second is $\dot{\boldsymbol{x}} = \boldsymbol{v}$) but with no potential forces, the first is sufficient. [6]

To get full marks here you need to mention the steps of derivation: continuity equation for f_A, substitution of $\dot{\boldsymbol{v}}$, Taylor expansion of the exponential containing $A(t)$, averaging over the stochastic force, Wick's theorem, etc. [6]

If you identified all the terms correctly, then the F-P equation is written for you (it is also (8.27) in the course handout booklet): [8]

$$\frac{\partial f(\boldsymbol{v}, t)}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}(\gamma \boldsymbol{v} f) + \frac{\Gamma}{2} \frac{\partial^2 f}{\partial \boldsymbol{v}^2}. \quad (25)$$

To obtain the classical diffusion equation you do need the coordinate dependence (see above). Either from the full $(\boldsymbol{x}, \boldsymbol{v})$-description, substituting the Maxwell $f(v)$, or separately starting from describing the overdamped motion and a "new" Langevin eqn, $\gamma \dot{\boldsymbol{x}} = \boldsymbol{A}(t)$, you should be able to write down [6]

$$\frac{\partial f(\boldsymbol{x}, t)}{\partial t} = \frac{\Gamma}{2\gamma^2} \frac{\partial^2 f}{\partial \boldsymbol{x}^2}, \quad \text{so } D = \Gamma/\gamma^2. \quad (26)$$

Returning back to the eq.(25) and setting its l.h.s. to zero you can easily integrate to obtain the equilibrium $f(\boldsymbol{v})$ [with no net velocity, which would arise from an integration constant]:

$$\frac{df}{f} = -\frac{2\gamma}{\Gamma} \boldsymbol{v} d\boldsymbol{v}, \quad f \propto \exp\left[-\frac{\gamma}{\Gamma} \boldsymbol{v}^2\right].$$

Identifying the exponent with $-\frac{1}{2} m \boldsymbol{v}^2 / kT$, you obtain [8]

$$\Gamma = \frac{2\gamma kT}{m} \quad \text{and} \quad D = \frac{2kT}{\gamma m}.$$