NATURAL SCIENCES TRIPOS Part II

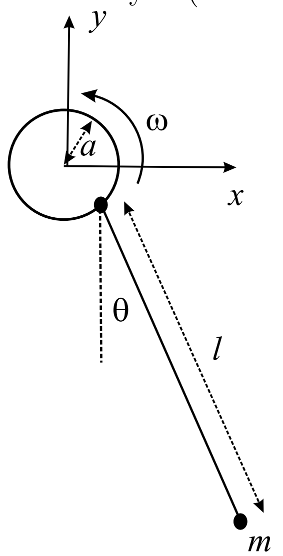
Wednesday 15 January 2003 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Describe briefly how Hamilton's principle of least action leads to Lagrange's equations of motion for a dynamical system having generalised coordinates q_i and velocities $\dot{q}_i$. [4]

A pendulum consists of a mass m at the end of a rigid massless rod of length l, and moves in the $x-y$ plane, making an angle θ with the vertical. This pendulum is attached by means of a free hinge to a ring of radius a, which rotates with a constant angular velocity ω (see sketch).



Find the Lagrangian for this dynamical system. [10]

Derive the Euler-Lagrange equation of motion for θ. [10]

Examine the form of the equation of motion in the limit of small oscillations, that is, when both conditions $\theta \ll 1$ and $a^2\omega^2 \ll gl$ are satisfied. [6]

Find the general solution of this simplified equation and show in particular that there is a resonance at $\omega^2 = g/l$. [4]

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2 A dynamical system has Lagrangian $L(q_i, \dot{q}_i, t)$. Define the canonical momenta p_i and the Hamiltonian $H(q_i, p_i, t)$. A non-relativistic particle of mass m and charge q moves in an electromagnetic field produced by an electrostatic potential ϕ and magnetic vector potential $\boldsymbol{A}$. Show that the Hamiltonian is

$$H = \frac{|\boldsymbol{p} - q\boldsymbol{A}|^2}{2m} + q\phi \; . \qquad [8]$$

In Cartesian coordinates (x, y, z) the electric field is $(\boldsymbol{E} = E, 0, 0)$ and the magnetic field is $\boldsymbol{B} = (0, 0, B)$. Show that $\phi = -Ex$, $\boldsymbol{A} = (0, Bx, 0)$ are suitable choices for the potentials. [4]

For a particle moving in this field, show that the momenta p_y, p_z and the Hamiltonian H are constants of the motion. [4]

Find Hamilton's equations of motion for the variables p_x, x, y and z and show that

$$\ddot{x} + \omega_0^2 x = \frac{qBp_y}{m^2} + \frac{qE}{m} \; ,$$

where $\omega_0 \equiv qB/m$. [10]

Hence find the general solutions for $x(t), y(t)$ and demonstrate that the particle has mean velocity $-E/B$ in the y direction. [8]

3 A system comprises N particles. The ith particle has rest mass m_i, is at position $\boldsymbol{x}_i$ and is moving with relativistic velocity $\boldsymbol{v}_i \equiv \dot{\boldsymbol{x}}_i$. Show that the action $S = \int \mathrm{d}t \; L$ is Lorentz-invariant, where

$$L = -\sum_i m_i c^2/\gamma_i$$

and $\gamma_i \equiv (1 - |\boldsymbol{v}_i|^2/c^2)^{-1/2}$. Find the canonical momenta $\boldsymbol{p}_i$ and the Hamiltonian H. [12]

A ring of rest mass m_0 has zero net momentum in frame F, but rotates relativistically with angular velocity ω. Determine the Lagrangian L, the angular momentum J and the Hamiltonian H. [10]

The system is now viewed in frame F', which moves with velocity $\boldsymbol{v}$ with respect to frame F. Determine how the transformed quantities S', L', ω', H' and J' are related to their values in frame F. [12]

4 Describe how the Cauchy integral theorem can be used to evaluate contour integrals in the complex plane. Illustrate your answer by showing that

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^2+1} = \pi \ . \qquad [10]$$

Calculate the integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^4+1} \qquad [12]$$

and

$$\int_{-\infty}^{\infty} \mathrm{dx} \, \frac{\cos\, ax}{x^2+b^2} \ . \qquad [12]$$

5 Define the Fourier transform $\tilde{f}(\omega)$ of a function $f(t)$ and write down the expression for the inverse transform. Describe how the Fourier transform can be used to solve linear dynamical equations. [8]

Find the causal Green function $G(t;t')$ for a damped harmonic oscillator described by the equation

$$\ddot{y} + \gamma\dot{y} + \omega_{\rm o}^2 y = \delta(t-t')$$

with $\gamma < 2\omega_{\rm o}$. [10]

Show how the Green function can be used to determine the response of the system to an external force which is constant $f(t') = f_0$ during the period $0 < t' < \tau$, and zero otherwise. For the case $t < \tau$ show that the response is

$$y(t) = \frac{1}{2\omega_0^2}\left(2\Omega - e^{-\frac{1}{2}\gamma t}\left(2\Omega\cos\Omega t + \gamma\sin\Omega t\right)\right) \qquad [10]$$

where $\Omega = \sqrt{\omega_{\rm o}^2 - \gamma^2/4}$.

Find also the response for the case $t > \tau$. It is sufficient to express the answer as a definite integral. [6]

[You may quote the following indefinite integral:

$$\int \mathrm{d}x \; e^{-\frac{1}{2}\gamma x} \sin\Omega x = -\frac{1}{2\omega_{\rm o}^2} e^{-\frac{1}{2}\gamma x}\left(2\Omega\cos\Omega x + \gamma\sin\Omega x\right)$$

where $\Omega = \sqrt{\omega_{\rm o}^2 - \gamma^2/4}$.]

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6 Discuss the concept of discrete transition probability $w(k, k')$ for a discrete one-dimensional random walk. [6]

Consider an ensemble of small identical spherical Brownian particles, of radius a and density ρ, suspended in a container filled with water (density ρ_0). Derive the modified diffusion equation for the probability $P(z, t)$ of finding a particle at a height z taking into account only first-order corrections in powers of $\tilde{m}ga/k_{\mathrm{B}}T$ (assumed small): [12]

$$\frac{\partial P}{\partial t} = \frac{1}{2}D\left(\frac{\partial^2 P}{\partial z^2} + \frac{\tilde{m}g}{k_{\mathrm{B}}T}\frac{\partial P}{\partial z}\right)$$

where $\tilde{m} = 4\pi(\rho - \rho_0)a^3/3$.

Derive the equilibrium Boltzmann distribution of these particles along the vertical z-axis. [6]

For $\rho = 1.1 \times 10^3$ kg m^{-3} and $\rho_0 = 1 \times 10^3$ kg m^{-3} estimate the order of magnitude of the radius a of a particle for which the effect of Brownian diffusion is relevant, such that the trajectory of moving particle deviates significantly from a straight line. [10]

[At room temperature $k_{\mathrm{B}}T \sim 4 \times 10^{-21}$ J.]