NATURAL SCIENCES TRIPOS Part II

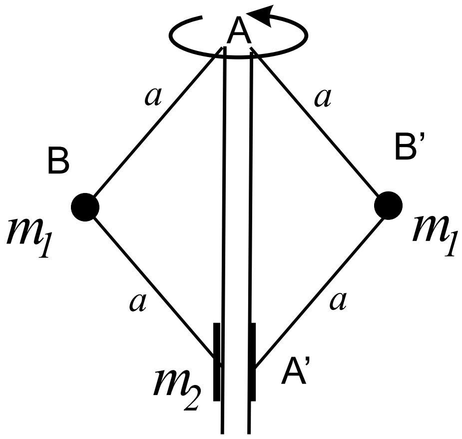
Wednesday 17 January 2001 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Describe briefly how Hamilton's principle of least action leads to Lagrange's equations of motion for a dynamical system having coordinates and velocities $(q_i, \dot{q}_i)$. [6]

A mechanical governor used to control the speed of a steam engine consists of the configuration shown in the figure:



(i) the vertical axis AA' rotates at a constant angular velocity Ω;
(ii) light rods $AB, AB', A'B, A'B'$ each of length a are freely pivoted at A, B, A', B';
(iii) the pivot at A is fixed, so that the pivot at A' moves as the angle θ changes;
(iv) masses m_1 are attached at B and B' and a mass m_2 is free to slide on the vertical axis at A'.

Show that the Lagrangian of the system is given by

$$L = m_1 a^2 (\Omega^2 \sin^2\theta + \dot{\theta}^2) + 2m_2 a^2 \dot{\theta}^2 \sin^2\theta + 2ag\cos\theta\,(m_1 + m_2) \ . \qquad [6]$$

Find the equation of motion of the system. [7]

Show that the system can rotate in equilibrium with $\theta = 0$ unless Ω exceeds a certain critical velocity. Determine the equilibrium angle θ_0 for the case when Ω is greater than this critical value. [8]

Show that the angular frequency of small oscillations about the equilibrium angle θ_0 is given by $\Omega\sin\theta_0/\sqrt{1 + 2(m_2/m_1)\sin^2\theta_0}$. [7]

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2 A dynamical system has Lagrangian $L(q_i, \dot{q}_i, t)$. Define the conjugate momenta p_i and the Hamiltonian $H(q_i, p_i, t)$. Write down Hamilton's equations of motion for the system. [6]

A particle of mass m moves in a spherically symmetric potential $V(r)$. Write down the Lagrangian using spherical polar coordinates (r, θ, ϕ) and find the conjugate momenta (p_r, p_θ, p_ϕ). Find the Hamiltonian H, expressing it in terms of the conjugate momenta and coordinates. [6]

Show that p_ϕ is a constant of the motion but that, in general, p_θ is not. [4]

Write $J^2 \equiv m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$ in terms of the canonical momenta and coordinates. Hence show that J^2 is another constant of the motion. [10]

Suppose that an additional dipole field is present, so that the potential then has the form

$$V(r, \theta) = V_0(r) + \frac{A\cos\theta}{r^2}$$

What can you say about the variation of p_ϕ and J^2? In particular:

(a) how does J^2 depend on θ?

(b) Can you find a new conserved quantity that reduces to J^2 for $A = 0$? [8]

3 Show that the Lagrangian

$$L = -\frac{m_0 c^2}{\gamma} - U(r)$$

gives the Euler-Lagrange equations for the motion of a relativistic particle of rest mass m_0 in a potential $U(\boldsymbol{r})$, where $\gamma \equiv (1 - |\dot{\boldsymbol{r}}|^2/c^2)^{-1/2}$. [6]

Write down L for planar orbits in a central potential $U(r)$ using plane polar coordinates (r, θ). Explain which features of L lead to the conservation laws

$$\begin{aligned} \gamma m_0 r^2 \dot{\theta} &= J = \text{ constant} \\ \gamma m_0 c^2 + U(r) &= E = \text{ constant} \end{aligned}$$ [8]

Using these conservation laws, show that the equation of the orbit is

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} = \frac{(E - U(r))^2 - m_0^2 c^4}{J^2 c^2} \, .$$ [12]

For the case $U(r) = -K/r$, where K is a positive constant, find the value of α such that the orbit has the form

$$l = r(1 + \epsilon\cos\alpha\theta) \, ,$$

where l and ϵ are further constants. [8]

[The equation $(\mathrm{d}u/\mathrm{d}\theta)^2 + \alpha^2(u - u_0)^2 = A^2\alpha^2$ has the solution $u = u_0 + A\cos(\alpha\theta)$.]

4 An electric charge density distribution $\rho(\boldsymbol{r})$ has the three-dimensional Fourier transform

$$\tilde{\rho}(\boldsymbol{k}) \equiv \int \mathrm{d}^3\boldsymbol{r}\ \rho(\boldsymbol{r}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \ .$$

Write down the formula for the inverse Fourier transform. [4]

The electrostatic potential $\varphi(\boldsymbol{r})$ is determined by the Poisson equation

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon_0} \ .$$

Determine the relationship between the Fourier transforms $\tilde{\rho}(\boldsymbol{k})$ and $\tilde{\varphi}(\boldsymbol{k})$. Explain how the potential can be found in terms of an integral over $\boldsymbol{k}$ if the charge density is known. [8]

A thin film of caesium metal, deposited on a substrate having dielectric constant equal to unity, occupies the region $-t \le z \le t$ and extends infinitely in x and y. A charge density wave $\rho(\boldsymbol{r}) = A\cos(Qx)$ is set up in the layer by perturbing the electron distribution. Calculate the Fourier transform $\tilde{\rho}(\boldsymbol{k})$, where $\boldsymbol{k} \equiv (k_x, k_y, k_z)$. [10]

Calculate the potential at the point $(x, 0, 0)$, expressing the answer in terms of $I(a)$, where

$$I(a) \equiv \int_{-\infty}^{\infty} \mathrm{d}k\ \frac{\sin k}{(a^2 + k^2)k} \ .$$ [5]

By using a contour integral, show that

$$I(a) = \frac{\pi}{a^2}\left(1 - \exp(-a)\right) \ .$$ [7]

5 Consider a one-dimensional quantum system described by a Hamiltonian $\widehat{\mathcal{H}}$. Describe how a propagator $G(x, x'; t)$ can be used to determine the wavefunction $\Psi(x, t)$ at time t from the initial wavefunction $\Psi(x, 0)$ at $t = 0$. [8]

By expanding the wave function $\Psi(x, t)$ in terms of a complete set of normalised eigenfunctions, $\widehat{\mathcal{H}}\,\phi_n = E_n\phi_n$, such that, $\Psi(x, t) = \sum_n c_n \phi_n e^{-\mathrm{i}E_n t/\hbar}$, verify that the propagator can be written as

$$G(x, x'; t) = \sum_n \phi_n(x)\phi_n^*(x') e^{-\mathrm{i}E_n t/\hbar} \quad \text{for } t > 0 \ .$$ [12]

Give an account of the path-integral representation of the quantum propagator $G(x, x'; t)$. Discuss also the behaviour of the propagator in the classical limit $\hbar \to 0$. [14]

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6 A system with one coordinate q is displaced at time $t = 0$ from its equilibrium position $q = 0$ to a position Q_0. Thereafter, the probability density $P(q,t)$ that it is to be found at position q at time t satisfies the evolution (Fokker-Planck) equation

$$\frac{\partial P}{\partial t} = D\left(\frac{\partial^2 P}{\partial q^2} + \frac{\partial}{\partial q}\left(P\frac{\partial U}{\partial q}\right)\right) ,$$

where U is a potential function and D a diffusion coefficient. Near equilibrium, the potential can be expressed as a quadratic form $U = \frac{1}{2}\alpha q^2$, where α is a constant.

By setting $\partial P/\partial t = 0$, verify that the equilibrium probability distribution is a Gaussian of mean $\langle q\rangle = 0$ and variance $\left\langle (q - \langle q\rangle^2\right\rangle = 1/\alpha$. [8]

Throughout the approach to equilibrium the probability distribution always has the Gaussian form

$$P(q,t) = \frac{1}{\sqrt{2\pi\Delta(t)}}\exp[-(q - Q(t))^2/2\Delta(t)] ,$$

where the only time-dependent quantities are the mean $Q(t)$ and the variance $\Delta(t)$.

Substitute this Gaussian form into the evolution equation and verify directly that the term on the LHS can be expressed as

$$\frac{\partial P}{\partial t} = \left[\left(-1 + \frac{(q-Q)^2}{\Delta}\right)\frac{1}{2\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{(q-Q)}{\Delta}\frac{\mathrm{d}Q}{\mathrm{d}t}\right] P .$$ [8]

By developing similar expressions for the terms on the RHS of the evolution equation and comparing powers of q, show that the mean $Q(t)$ and the variance $\Delta(t)$ evolve according to the ordinary differential equations:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} + D\alpha Q = 0$$

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} + 2D\alpha\Delta = 2D$$ [10]

Using the boundary conditions $Q(0) = Q_0, \Delta(0) = 0$, solve these equations for $Q(t)$ and $\Delta(t)$. [6]