THEORETICAL PHYSICS I

Answer three questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains 4 sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

1  Describe briefly how Hamilton’s principle of least action leads to Lagrange’s equations of motion for a dynamical system having coordinates and velocities \((q_i, \dot{q}_i)\).  
A mechanical governor used to control the speed of a steam engine consists of the configuration shown in the figure:

(i) the vertical axis \(AA'\) rotates at a constant angular velocity \(\Omega\);
(ii) light rods \(AB, AB', A'B, A'B'\) each of length \(a\) are freely pivoted at \(A, B, A', B'\);
(iii) the pivot at \(A\) is fixed, so that the pivot at \(A'\) moves as the angle \(\theta\) changes;
(iv) masses \(m_1\) are attached at \(B\) and \(B'\) and a mass \(m_2\) is free to slide on the vertical axis at \(A'\).

Show that the Lagrangian of the system is given by

\[
L = m_1 a^2 (\Omega^2 \sin^2 \theta + \dot{\theta}^2) + 2m_2 a^2 \dot{\theta}^2 \sin^2 \theta + 2ag \cos \theta (m_1 + m_2) .
\]

Find the equation of motion of the system.

Show that the system can rotate in equilibrium with \(\theta = 0\) unless \(\Omega\) exceeds a certain critical velocity. Determine the equilibrium angle \(\theta_0\) for the case when \(\Omega\) is greater than this critical value.

Show that the angular frequency of small oscillations about the equilibrium angle \(\theta_0\) is given by \(\Omega \sin \theta_0/\sqrt{1 + 2(m_2/m_1) \sin^2 \theta_0}\).

(TURN OVER)
2  A dynamical system has Lagrangian $L(q_i, \dot{q}_i, t)$. Define the conjugate 
momenta $p_i$ and the Hamiltonian $H(q_i, p_i, t)$. Write down Hamilton’s equations of 
motion for the system.

A particle of mass $m$ moves in a spherically symmetric potential $V(r)$. Write 
down the Lagrangian using spherical polar coordinates $(r, \theta, \phi)$ and find the 
conjugate momenta $(p_r, p_\theta, p_\phi)$. Find the Hamiltonian $H$, expressing it in terms of 
the conjugate momenta and coordinates.

Show that $p_\phi$ is a constant of the motion but that, in general, $p_\theta$ is not. 
Write $J^2 \equiv m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)$ in terms of the canonical momenta and 
coordinates. Hence show that $J^2$ is another constant of the motion.

Suppose that an additional dipole field is present, so that the potential then 
has the form

$$V(r, \theta) = V_0(r) + \frac{A \cos \theta}{r^2}$$

What can you say about the variation of $p_\phi$ and $J^2$? In particular:

(a) how does $J^2$ depend on $\theta$?

(b) Can you find a new conserved quantity that reduces to $J^2$ for $A = 0$?

3  Show that the Lagrangian

$$L = -\frac{m_0 c^2}{\gamma} - U(r)$$

gives the Euler-Lagrange equations for the motion of a relativistic particle of rest 
mass $m_0$ in a potential $U(r)$, where $\gamma \equiv (1 - |\mathbf{v}/c|^2)^{-1/2}$.

Write down $L$ for planar orbits in a central potential $U(r)$ using plane polar 
coordinates $(r, \theta)$. Explain which features of $L$ lead to the conservation laws

$$\gamma m_0 r^2 \dot{\theta} = J = \text{constant}$$

$$\gamma m_0 c^2 + U(r) = E = \text{constant}$$

Using these conservation laws, show that the equation of the orbit is

$$\left( \frac{d}{d\theta} \left( \frac{1}{r} \right) \right)^2 + \frac{1}{r^2} = \frac{(E - U(r))^2 - m_0^2 c^4}{J^2 c^2}.$$ 

For the case $U(r) = -K/r$, where $K$ is a positive constant, find the value of 
$\alpha$ such that the orbit has the form

$$l = r(1 + \epsilon \cos \alpha \theta),$$

where $l$ and $\epsilon$ are further constants.

[The equation $(du/d\theta)^2 + \alpha^2 (u - u_0)^2 = A^2 \alpha^2$ has the solution $u = u_0 + A \cos(\alpha \theta)$.]
4 An electric charge density distribution \( \rho(\mathbf{r}) \) has the three-dimensional Fourier transform

\[
\tilde{\rho}(\mathbf{k}) \equiv \int d^3 \mathbf{r} \, \rho(\mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r}) .
\]

Write down the formula for the inverse Fourier transform.

The electrostatic potential \( \varphi(\mathbf{r}) \) is determined by the Poisson equation

\[
\nabla^2 \varphi = -\frac{\rho}{\varepsilon_0} .
\]

Determine the relationship between the Fourier transforms \( \tilde{\rho}(\mathbf{k}) \) and \( \tilde{\varphi}(\mathbf{k}) \). Explain how the potential can be found in terms of an integral over \( \mathbf{k} \) if the charge density is known.

A thin film of caesium metal, deposited on a substrate having dielectric constant equal to unity, occupies the region \(-t \leq z \leq t\) and extends infinitely in \(x\) and \(y\). A charge density wave \( \rho(\mathbf{r}) = A \cos(Qx) \) is set up in the layer by perturbing the electron distribution. Calculate the Fourier transform \( \tilde{\rho}(\mathbf{k}) \), where \( \mathbf{k} \equiv (k_x, k_y, k_z) \).

Calculate the potential at the point \((x, 0, 0)\), expressing the answer in terms of \( I(a) \), where

\[
I(a) \equiv \int_{-\infty}^{\infty} dk \frac{\sin k}{(a^2 + k^2)^{1/2}} .
\]

By using a contour integral, show that

\[
I(a) = \frac{\pi}{a^2} (1 - \exp(-a)) .
\]

5 Consider a one-dimensional quantum system described by a Hamiltonian \( \hat{\mathcal{H}} \). Describe how a propagator \( G(x, x'; t) \) can be used to determine the wavefunction \( \Psi(x, t) \) at time \( t \) from the initial wavefunction \( \Psi(x, 0) \) at \( t = 0 \).

By expanding the wave function \( \Psi(x, t) \) in terms of a complete set of normalised eigenfunctions, \( \hat{\mathcal{H}} \phi_n = E_n \phi_n \), such that, \( \Psi(x, t) = \sum_n c_n \phi_n e^{-iE_n t/\hbar} \), verify that the propagator can be written as

\[
G(x, x'; t) = \sum_n \phi_n(x) \phi_n^*(x') e^{-iE_n t / \hbar} \text{ for } t > 0 .
\]

Give an account of the path-integral representation of the quantum propagator \( G(x, x'; t) \). Discuss also the behaviour of the propagator in the classical limit \( \hbar \to 0 \).

(TURN OVER)
A system with one coordinate $q$ is displaced at time $t = 0$ from its equilibrium position $q = 0$ to a position $Q_0$. Thereafter, the probability density $P(q, t)$ that it is to be found at position $q$ at time $t$ satisfies the evolution (Fokker-Planck) equation

$$\frac{\partial P}{\partial t} = D \left( \frac{\partial^2 P}{\partial q^2} + \frac{\partial}{\partial q} \left( P \frac{\partial U}{\partial q} \right) \right),$$

where $U$ is a potential function and $D$ a diffusion coefficient. Near equilibrium, the potential can be expressed as a quadratic form $U = \frac{1}{2} \alpha q^2$, where $\alpha$ is a constant.

By setting $\partial P/\partial t = 0$, verify that the equilibrium probability distribution is a Gaussian of mean $\langle q \rangle = 0$ and variance $\langle (q - \langle q \rangle)^2 \rangle = 1/\alpha$. Throughout the approach to equilibrium the probability distribution always has the Gaussian form

$$P(q, t) = \frac{1}{\sqrt{2\pi \Delta(t)}} \exp\left[-(q - Q(t))^2 / 2\Delta(t)\right],$$

where the only time-dependent quantities are the mean $Q(t)$ and the variance $\Delta(t)$.

Substitute this Gaussian form into the evolution equation and verify directly that the term on the LHS can be expressed as

$$\frac{\partial P}{\partial t} = \left[ \left(-1 + \frac{(q - Q)^2}{\Delta} \right) \frac{1}{2\Delta} \frac{d\Delta}{dt} + \frac{(q - Q)}{\Delta} \frac{dQ}{dt} \right] P.$$

By developing similar expressions for the terms on the RHS of the evolution equation and comparing powers of $q$, show that the mean $Q(t)$ and the variance $\Delta(t)$ evolve according to the ordinary differential equations:

$$\frac{dQ}{dt} + D\alpha Q = 0$$

$$\frac{d\Delta}{dt} + 2D\alpha \Delta = 2D$$

Using the boundary conditions $Q(0) = Q_0, \Delta(0) = 0$, solve these equations for $Q(t)$ and $\Delta(t)$.