1.3 Problem Set

1.3.1 Questions on Collective Modes and Field Theories

1. In obtaining the spectrum of collective phonon excitations for the lattice Lagrangian (1.1), a continuum approximation was employed. However, since the degrees of freedom are coupled linearly, the equations of motion can be solved explicitly, even for the discrete model. By constructing the equations of motion, identify the **normal modes** of the system and obtain the exact spectrum of excitations. Identify the limit in which the spectrum of the discrete lattice model coincides with that obtained for the continuum approximation of the model. In what limit does the continuum approximation fail and why?

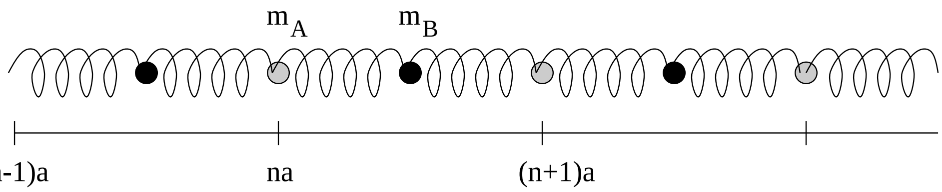

Figure 1.8: Lattice with two atoms of mass m_A and m_B per unit cell.

2. In lattices with two atoms (of different mass m_A and m_B) per unit cell (see fig. 1.8) the spectrum of elementary excitations splits into an **acoustic** and **optic** branch. For this model, show that the lattice Lagrangian can be written as

$$L = \sum_{n=1}^{N} \left[\frac{m_A}{2}(\dot{\phi}_n^{(A)})^2 + \frac{m_B}{2}(\dot{\phi}_n^{(B)})^2 - \frac{k_s}{2}\left(\phi_{n+1}^{(A)} - \phi_n^{(B)}\right)^2 - \frac{k_s}{2}\left(\phi_n^{(B)} - \phi_n^{(A)}\right)^2 \right].$$

Applying the Euler-Lagrange equation, obtain the equations of motion. Taking the boundary conditions to be periodic, and switching to Fourier representation, show that the exact (i.e. discrete) spectrum can be obtained from the 2×2 secular equation for each k value

$$\det \begin{vmatrix} m_A\omega_k^2 - 2k_s & k_s(1+e^{ika}) \\ k_s(1+e^{-ika}) & m_B\omega_k^2 - 2k_s \end{vmatrix} = 0.$$

By finding an expression for the spectrum, obtain the asymptotic dependence as $k \to 0$. In this limit, describe qualitatively the symmetry of the normal modes.

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3. Applying the Euler-Lagrange equation, obtain the equation of motion of the Lagrangian densities:

$$1. \qquad \mathcal{L}[\phi, \dot{\phi}, \partial_x\phi] = \frac{m\dot{\phi}^2}{2} - \frac{k_s a^2}{2}(\partial_x\phi)^2 - \frac{m}{2}\omega^2\phi^2$$

2. $$\mathcal{L}[\phi, \dot{\phi}, \partial_x^2\phi] = \frac{m\dot{\phi}^2}{2} - \frac{\kappa}{2}\left(\partial_x^2\phi\right)^2$$

3. $$\mathcal{L}[\phi, \dot{\phi}] = \frac{m\dot{\phi}^2}{2} - \frac{m}{2}\omega^2\phi^2 - \frac{\eta}{4}\phi^4$$

4. $$\mathcal{L}[\{\dot{\phi}_i\}, \{\partial_x\phi_i\}] = \sum_{i=1}^{n}\left[\frac{m}{2}\dot{\phi}_i^2 - \frac{1}{2}k_s a^2(\partial_x\phi_i)^2\right]$$

5. $$\mathcal{L}[\dot{\phi}, \partial_x\phi] = \frac{m}{2}|\dot{\phi}|^2 - \frac{1}{2}k_s a^2|\partial_x\phi|^2$$

[Note that in 5. the field ϕ is complex.] Suggest a physical significance of the last term in 1. What is the effect of this term on the excitation spectrum of the corresponding quantum Hamiltonian? Starting with the Lagrangian 2., obtain the Hamiltonian density.

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4. Following the discussion in the lectures, a periodic one-dimensional quantum elastic chain of length L is expressed by the Hamiltonian

$$\hat{H} = \int dx\left[\frac{1}{2m}\hat{\pi}^2 + \frac{k_s a^2}{2}(\partial_x\hat{\phi})^2\right]$$

where the field operators obey the canonical commutation relations

$$\left[\hat{\pi}(x), \hat{\phi}(x')\right] = -i\hbar\delta(x - x').$$

(a) Defining the Fourier representation,

$$\left\{\begin{array}{l}\hat{\phi}_k\\ \hat{\pi}_k\end{array}\right. \equiv \frac{1}{L^{1/2}}\int_0^L dx e^{\{\mp ikx}\left\{\begin{array}{l}\hat{\phi}(x)\\ \hat{\pi}(x)\end{array}\right., \qquad \left\{\begin{array}{l}\hat{\phi}(x)\\ \hat{\pi}(x)\end{array}\right. = \frac{1}{L^{1/2}}\sum_k e^{\{\pm ikx}\left\{\begin{array}{l}\hat{\phi}_k\\ \hat{\pi}_k\end{array}\right.,$$

where $\sum_k$ represents the sum over all quantised quasi-momenta $k = 2\pi m/L$, $m \in \mathcal{Z}$, show that the field operators obey the commutation relations $[\hat{\pi}_k, \hat{\phi}_{k'}] = -i\hbar\delta_{kk'}$.

(b) In the Fourier representation, show that the Hamiltonian takes the form

$$\hat{H} = \sum_k\left[\frac{1}{2m}\hat{\pi}_k\hat{\pi}_{-k} + \frac{k_s a^2}{2}k^2\hat{\phi}_k\hat{\phi}_{-k}\right].$$

(c) Defining

$$a_k \equiv \sqrt{\frac{m\omega_k}{2\hbar}}\left(\hat{\phi}_k + i\frac{1}{m\omega_k}\hat{\pi}_{-k}\right)$$

where $\omega_k = a(k_s/m)^{1/2}|k| = v|k|$ show that the field operators obey the canonical commutation relations $[a_k, a_{k'}^\dagger] = \delta_{kk'}$, and $[a_k, a_{k'}] = 0$.

(d) Finally, with this definition, show that the Hamiltonian can be expressed in the form

$$\hat{H} = \sum_k \hbar\omega_k\left(a_k^\dagger a_k + \frac{1}{2}\right).$$
