5.5.2 Answers

1. (a) Substituting the change of field variables, and rearranging some terms by integrating by parts, one obtains

$$\mathcal{Z} = \int D(p_k, q_k) \exp\left[- \int_0^\beta d\tau \sum_k \left(\frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega^2 q_k q_{-k} - \frac{i}{\hbar} p_{-k} \partial_\tau q_k \right) \right].$$

With the change of variables $\tau = it/\hbar$, one obtains the Hamiltonian formulation of the path integral.

(b) Performing the Gaussian integral over the fields p_k, one obtains

$$\mathcal{Z} = \int Dq_k \exp\left[- \int_0^\beta d\tau \sum_k \left(\frac{1}{2\hbar^2} m|\dot{q}_k|^2 + \frac{1}{2} m\omega_k^2 |q_k|^2 \right) \right].$$

With $\omega_k^2 = k^2 k_s a^2/m$, turning to the real space representation, one obtains the required result.

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2. Starting with the expression for the action given in the question, differentiating S with respect to Δ one obtains

$$\frac{1}{\beta L^3} \frac{dS}{d\Delta} = -\nu(\mu) \int d\xi \int \frac{d\omega}{2\pi} \frac{\Delta^*}{\omega^2 + |\Delta|^2 + \xi^2} + \frac{\Delta^*}{g}.$$

Although the integral is formally divergent, recalling that the microscopic mechanism derives from the phonon-mediated interaction, we introduce a high energy cut-off at the Debye frequency, ω_D. As a result one obtains

$$\begin{aligned} \frac{1}{\beta L^3} \frac{dS}{d\Delta} &= -\frac{\nu(\mu)}{2} \int_{-\omega_D}^{\omega_D} \frac{\Delta^* d\xi}{(\xi^2 + |\Delta|^2)^{1/2}} + \frac{\Delta^*}{g} = -\nu(\mu)\Delta^* \sinh^{-1}\left(\frac{\omega_D}{|\Delta|} \right) + \frac{\Delta^*}{g} \\ &\simeq -\Delta^* \left[\nu(\mu) \ln\left(\frac{\omega_D}{|\Delta|} \right) - \frac{1}{g} \right]. \end{aligned}$$

Setting $dS/d\Delta = 0$, we obtain the zero temperature gap equation.

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3. (a) Defining the unitary transformation,

$$\begin{pmatrix} \alpha^\dagger_{\mathbf{k}\uparrow} \\ \alpha_{-\mathbf{k}\uparrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow} \end{pmatrix},$$

the quadratic Hamiltonian is diagonalised when $\tan(2\theta_{\mathbf{k}}) = -\Delta/\xi_{\mathbf{k}}$

(b) Taking the definition from the question

$$\begin{aligned}\langle \text{g.s.}|\text{g.s.}\rangle &= \langle\Omega|\prod_{\mathbf{k}}(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow})(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow})|\Omega\rangle \\ &= \prod_{\mathbf{k}}\left(\cos^2\theta_{\mathbf{k}} + \sin^2\theta_{\mathbf{k}}\right)\langle\Omega|\Omega\rangle.\end{aligned}$$

The corresponding ground state energy is given by

$$\langle \text{g.s.}|\hat{H}|\text{g.s.}\rangle = \sum_{\mathbf{k}}(\xi_{\mathbf{k}} - \lambda_{\mathbf{k}}) + \frac{\Delta^2}{g}.$$

(c) Making use of the relation between the bare electron operators and the quasi-particle operators, the self-consistency equation yields

$$\begin{aligned}\Delta &= \left\langle\Omega\left|\left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\right) c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}\right)\right|\Omega\right\rangle \\ &= -g\sum_{\mathbf{k}}\sin\theta_{\mathbf{k}}\cos\theta_{\mathbf{k}} = \frac{g}{2}\sum_{\mathbf{k}}\frac{\Delta}{(\Delta^2+\xi^2_{\mathbf{k}})^{1/2}} \\ &\simeq \frac{g\Delta}{2}\nu(\mu)\int_{-\omega_D}^{\omega_D}\frac{d\xi}{(\Delta^2+\xi^2)^{1/2}}.\end{aligned}$$

From this result we obtain

$$g\nu(\mu)\ln\left(\frac{2\omega_D}{\Delta}\right) = 1, \qquad \Delta = 2\omega_D\exp\left[-\frac{1}{g\nu(\mu)}\right].$$