Exact Description of Spectral Correlators by a Quantum One-Dimensional Model with Inverse-Square Interaction

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Abstract

The universal correlations which describe the response of energy levels of a disordered metallic grain to an arbitrary perturbation are shown to be equivalent to time-dependent correlations of a one-dimensional quantum Hamiltonian with inverse-square interaction. These results establish a direct connection between a strongly interacting quantum Hamiltonian, the non-linear σ-model of disordered electronic systems, and quantum chaotic spectra. As a consequence we have an expression for the correlation function of the quantum Hamiltonian which we believe to be exact for all space and time.

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Spectral correlations of complex quantum systems ranging from those with strong correlations and many degrees of freedom [1] to the motion of particles in irregular [2] or disordered potentials [3] frequently display the same universal behavior. The universality, which appears to follow from non-integrability has become the signature of quantum chaos. In such cases correlators of level spacings obey Wigner-Dyson statistics [4], characterized by the mean-level spacing, Δ. Recently two of us [5] have found that the response of the energy levels, $E_i(U)$ to an arbitrary perturbation, U such as a potential or magnetic field are also described by universal functions, and are characterized by a well defined "conductance." The universality is defined by a rescaling of the eigenvalues, and perturbation parameter, U [6,5],

$$\epsilon_i(u) = E_i(U)/\Delta, \qquad u^2 = C(0)U^2. \tag{1}$$

where $C(0) = (1/\Delta^2)\langle(\partial E_i(U)/\partial U)^2\rangle$, and without loss of generality we have assumed $\langle \partial E_i(U)/\partial U\rangle = 0$. The statistical average, denoted by $\langle\cdots\rangle$ can be performed over U, and/or over a range of levels (ergodicity). After rescaling the statistical properties of the random functions $\epsilon_i(u)$ become independent of the detailed properties of the system. A physical interpretation of $C(0)$ can be found through a universal fluctuation-dissipation theorem [6] which shows it to behave as a generalized conductance.

It has been known for some time [7] that the Wigner-Dyson distribution of eigenvalues, $P_{N\beta}[\{\epsilon_i\}]$ coincides with the probability distribution of the N-particle coordinates, $P_{N\beta}[\{r_i\}]$ of the quantum ground state of a one-dimensional Hamiltonian interacting via an inverse-square pairwise interaction. By confining the particles with a harmonic potential the ground states correspond to the Gaussian random matrix ensembles [8] and display a semi-circular distribution of density. More conveniently, the particles can be confined by constraining their motion to a ring when the ground states becomes equivalent to the equilibrium distribution of the circular ensemble [9]. Since we will be interested in the thermodynamic limit ($N \to \infty$) in which the statistical properties of both ensembles become equivalent, we will concentrate on the periodic Sutherland Hamiltonian given by,

$$H_S = -\sum_i \frac{\partial^2}{\partial r_i^2} + \beta(\beta/2-1)\Big(\frac{\pi}{N}\Big)^2 \sum_{i>j} \frac{1}{\sin^2[\pi(r_i - r_j)/N)]}, \tag{2}$$

where the spatial coordinates, defined in units of the mean interparticle distance correspond directly to the eigenvalues of the circular ensemble, $\epsilon_i \equiv r_i$. The pairwise interaction scales as the inverse-square of the chord length between the particles on the ring. The coupling constant of the interaction depends on the Dyson index, which can take the values $\beta = 1, 2$, or 4 corresponding to orthogonal, unitary, or symplectic symmetry respectively.

The above correspondence relies only of the fact that the ground state of Eq. (2) is of the Jastrow form,

$$|0\rangle = \Big(\frac{[\Gamma(\beta/2+1)]^N}{N^N\Gamma(N\beta/2+1)}\Big)^{1/2} \prod_{i>j} |e^{2\pi i r_i/N} - e^{2\pi i r_j/N}|^{\beta/2}. \tag{3}$$

Our main point is that the connection between the Hamiltonian and the spectra is much deeper. We are motivated by the following picture. The dispersion of the energy levels, ϵ_i as a function of u resembles the world lines of N particles at positions r_i plotted as a function of imaginary time $\tau = -it$. At any given τ, the distribution of $\{r_i\}$ is given by the ground state wavefunction of Eq. (2). It is then natural to question whether the time evolution of $\{r_i\}$ also corresponds to the dispersion of $\{\epsilon_i\}$ as a function of u. In this paper we present analytical results which provide strong evidence that the correspondence is in fact exact if we associate separations in time τ and position r in the Sutherland model with differences in the perturbation u and energy ω in the spectral problem through the relation,

$$u^2 = 2\tau, \qquad \omega = r. \tag{4}$$

To illustrate this connection we will examine the autocorrelator of the density of states (DOS) fluctuations. For brevity we will focus only on the unitary and orthogonal ensembles. The former differs from the latter in that the coupling constant vanishes for $\beta = 2$ and the Sutherland model becomes equivalent to a system of non-interacting fermions.

The autocorrelator in both energy and perturbation of the DOS fluctuations is defined by the statistical average,

$$k(\omega, u) = \Big\langle \sum_{ij} \delta(\epsilon - \epsilon_i(\bar{u}))\ \delta(\epsilon - \omega - \epsilon_j(\bar{u} + u))\Big\rangle - 1. \tag{5}$$

By examining the spectra of disordered metallic grains subject to an external perturbation the universal correlator for unitary ensembles was found to be [5],

$$k^u(\omega, u) = \frac{1}{2}\mathrm{Re}\int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_1 \exp[-\pi^2 u^2(\lambda_1^2 - \lambda^2)/2 + i\pi\omega(\lambda_1 - \lambda + i\delta)], \tag{6}$$

while for orthogonal ensembles,

$$\begin{aligned} k^o(\omega, u) = \mathrm{Re}\int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_1 \int_{1}^{\infty} d\lambda_2 \frac{(1-\lambda^2)(\lambda - \lambda_1\lambda_2)^2}{(2\lambda\lambda_1\lambda_2 - \lambda^2 - \lambda_1^2 - \lambda_2^2 + 1)^2} \\ \times \exp[-\pi^2 u^2(2\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1)/4 + i\pi\omega(\lambda - \lambda_1\lambda_2 + i\delta)]. \end{aligned} \tag{7}$$

Using Eq. (4), $k(r, \tau)$ becomes equivalent to the autocorrelator in position and imaginary time of particle density fluctuations from the ground state. Since $P_{N\beta}[\{\epsilon_i\}]$ and $P_{N\beta}[\{r_i\}]$ are equivalent the correspondence at $\tau = 0$ is clear. We will show that at non-vanishing τ, the correlation function $k(r, \tau)$, which depends not just on the ground state but also on the excited states, reproduces Eqs. (6-7).

We will begin by establishing the correspondence of $k(\omega, u)$ to the Sutherland model. Later, we will examine the non-linear σ-model of weakly disordered metallic systems from which the universal correlation functions (6-7) were derived. We first discuss the unitary case where H_S describes the free fermion Hamiltonian. The wavefunctions can be written as the Slater determinant of a set of non-interacting eigenstates,

$$|\mathbf{p}\rangle = \frac{1}{N!}\mathrm{det}|\exp[ip_i r_j]|, \tag{8}$$

defined by the integral quantum numbers $\{n_i\}$ through the relation $p_i = 2\pi n_i/N$, and having energy equal to $E(\mathbf{p}) = \sum_i p_i^2$. The ground state wavefunction, $|\mathbf{p}_0\rangle$ defined by the set of quantum numbers which occupy the lowest N states has the form of a Vandermonde determinant allowing it to be written as a Jastrow function (3). Inserting a complete set of states into Eq. (5) we obtain,

$$k(r,\tau) = \frac{1}{N^2} \sum_{\mathbf{p}\neq\mathbf{p}_0} e^{-\tau(E(\mathbf{p})-E(\mathbf{p}_0))} \sum_{ij} \langle 0|\delta(r-r_i)|\mathbf{p}\rangle \, \langle \mathbf{p}|\delta(r_j)|0\rangle. \tag{9}$$

Since the action of the density operator is to change the momentum of one of the particles it is clear that only states with a single particle excited above the Fermi surface contribute to the sum. We therefore obtain the expression,

$$k(r,\tau) = \frac{2}{N^2} \sum_{|p|\leq p_f} \sum_{|p_1|>p_f} \exp[-\tau(p_1^2 - p^2) + i(p_1 - p)r], \tag{10}$$

where the Fermi momentum $p_f = \pi$. Taking the thermodynamic limit, $N \to \infty$ and making the change of variables $\lambda = p/p_f$, and $\lambda_1 = p_1/p_f$ we obtain the universal expression given by Eq. (6).

The orthogonal case corresponds to a non-vanishing value of the coupling constant. The absence of a closed expression for the excited state wavefunctions makes a proof analogous to the unitary case less accessible and we confirm the correspondence by deriving expressions for $k(r,\tau)$ at leading order in τ and $1/\tau$, with the latter corresponding to the hydrodynamic limit.

Sutherland has obtained the complete spectrum of H_S together with an expression for the ground state wavefunction (3), and a method of constructing excited states at arbitrary values of β [7]. The wavefunctions of H_S were shown to be factorizable into the ground state and a superposition of free boson basis states making the particles distinguishable and allowing their statistics to be chosen arbitrarily by means of a Jordan-Wigner transformation. By choosing a boson representation the leading order term in τ can be determined. T-invariance of the Sutherland model causes the coefficient of this term to vanish at all values of $r \neq 0$. At this order, $k(r,\tau)$ is independent of β ("f-sum rule" [10]) and equal to,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqr} \Big[k(r,\tau) - k(r,0)\Big] dr = -\tau q^2 \theta(q)/\pi + \mathcal{O}(\tau^2), \tag{11}$$

which up to the change of variables (4) agrees with the estimate from $k(\omega,u)$. Performing the inverse transform we obtain,

$$k(\omega,u) = k(\omega,0) + \frac{u^2}{2} \frac{\partial^2}{\partial\omega^2} \delta(\omega) + \mathcal{O}(u^4), \tag{12}$$

where the expression for $k(\omega,0)$ can be found in Ref. [4]. Explicit calculation at still higher orders becomes non-trivial due to the complexity of the integral expression for $k(\omega,u)$, which is manifested in the Sutherland model as a rapidly increasing number of contributions.

For the fermionic model, the form of the leading order term has a natural interpretation in terms of T-invariance. The disappearance of this term for all but $\omega = 0$ in the eigenvalue problem is less trivial. The density correlator is comprised of two sorts of contribution; the first from the same level, and the second from neighboring levels. The leading order term singles out the non-analytic contribution of the former at $\omega = 0$.

Since the Sutherland model displays a gapless linear low energy excitation spectrum it belongs to the universality class of Luttinger liquids [11]. The $1/\tau$ asymptotic can therefore be determined by the method of bosonization [12] when we obtain,

$$k(r,\tau) = -\frac{1}{2\pi^2\beta}\sum_{\sigma=\pm 1}\frac{1}{(r+i\sigma v_s\tau)^2}, \tag{13}$$

where the sum is over right and left movers, and the sound-wave velocity of the density fluctuations is given by $v_s = \pi\beta$ [7]. Comparing this to the large ω asymptotics of Eqs. (6-7) given by,

$$k(\omega,u) = \frac{1}{\beta\pi^2}\mathrm{Re}\frac{1}{(i\omega+\pi\beta u^2/2)^2}, \tag{14}$$

and using Eq. (4) the correspondence is verified.

We believe that the proof of the equivalence for the unitary ensemble, together with the results at $\tau = 0$ and both the large and small τ asymptotics for the orthogonal ensemble provides compelling evidence for the correspondence of the Sutherland model to the eigenvalue problem. If we accept this correspondence, Eq. (7) then provides an expression for the density-density correlator of the Sutherland model with an attractive coupling constant $\beta(\beta/2-1) = -1/2$ which is exact for all t and r. In principle an expression for $k(\omega,u)$ can be calculated for the symplectic ensemble when the appropriate coupling constant is repulsive and equal to 4.

To understand the connection of the Sutherland model to the non-linear σ-model of weakly disordered metallic systems it is instructive to briefly review the existing derivation of Eqs. (6-7). Although the Wigner-Dyson distribution is thought to apply to the spectra of all non-integrable systems, formally it can be derived only in a limited number of cases. Typically, this is where statistical averaging can be performed by first averaging over an ensemble of systems whose statistical properties are equivalent, such as random matrices [4] or disordered metallic systems [13]. In the same way, we believe that the autocorrelation function, $k(\omega,u)$ applies to all non-integrable systems but are required to examine specific examples to perform its derivation. Instead of using random matrix theory, where the Wigner-Dyson distribution was first derived [4], $k(\omega,u)$ is determined by investigating disordered metallic grains subject to a variety of perturbations [5,14]. For this purpose we make use of the supersymmetry approach [13].

The Hamiltonian that we investigate describes the motion of a spinless particle confined to a cylinder of circumference L with a background disorder potential, $W(\mathbf{r})$ and subject to one of two types of perturbation, U. The first describes an Aharonov-Bohm flux, ϕ through the cylinder while the second describes an additional background potential $Vs(r)$, where the fixed spatial dependence $s(r)$ can be chosen arbitrarily. The Hamiltonian is given by,

$$H = \frac{1}{2m}\Big(\mathbf{p} - \frac{e}{c}\mathbf{A}(\phi)\Big)^2 + W(\mathbf{r}) + Vs(\mathbf{r}), \tag{15}$$

where only the azimuthal component of the vector potential is non-vanishing and equal to $A = \phi/L$. If we assume that the particle scatters elastically from the disorder potential its motion is diffusive with a classical diffusion constant $D = v^2\tau_s/d$. The mean time between collisions, denoted by τ_s, depends on the nature of the disorder. For a $\delta-$correlated white-noise impurity potential, τ_s is determined through the ensemble average, $\langle W(\mathbf{r})W(\mathbf{r}')\rangle = L^d\Delta\delta(\mathbf{r}-\mathbf{r}')/2\pi\tau_s$, where L^d denotes the volume of the system. To consider only diffusive motion it is necessary to assume that $2/mv^2 \ll \tau_s \ll L/v$ where v denotes the velocity of the particle on the torus.

By representing $k(\omega, u)$ in terms of a supersymmetric Lagrangian containing equal numbers of commuting and anticommuting fields, the interaction generated by the ensemble averaging is decoupled by a Hubbard-Stratonovitch transformation with the introduction of 8×8 supermatrix fields, $Q(\mathbf{r})$ [13]. To order $2/mv^2\tau_s$ small fluctuations about the saddle-point are controlled by a functional non-linear $\sigma-$model [5,14],

$$k(\Omega,\phi,V) = -\frac{1}{2}\Big[\int Q_{11}^{11}(\mathbf{r})\ Q_{11}^{22}(\mathbf{r}')\exp(-F[Q])\ \mathcal{D}Q\ L^{-2d}\ d\mathbf{r}\ d\mathbf{r}' + 1\Big], \tag{16}$$

where $F[Q]$ denotes the free energy,

$$F[Q] = \frac{\pi}{8L^d\Delta}\int \mathrm{STr}\Big[D(\nabla Q - \frac{ie}{c}[Q,\mathbf{A}\tau_3])^2 + 2i(\Omega + Vs(\mathbf{r}))\Lambda Q\Big]d\mathbf{r}, \tag{17}$$

with the constraint that $Q^2 = 1$, and where we have assumed that $(\Delta\tau_s)^{-1} \ll 1$. $\mathbf{A}^{\zeta\zeta'}$ represents a diagonal matrix which depends only on the upper indices with signature $(\bar\phi, \bar\phi + \phi)/L$. For brevity we refer to Ref. [13] for the symmetry properties of Q together with the definition of τ_3, Λ and the supertrace, denoted Str. The excitations of Q correspond to the diffusion modes of the particles [15].

We will examine each type of perturbation separately. In both cases the Goldstone modes of the transverse fluctuations in Q are destroyed at non-vanishing values of the perturbation. It is therefore necessary to retain only the symmetries of the supermatrix which contribute most significantly to the average. The fluctuation of the supermatrix can be classified according to whether or not the elements commute with the matrix τ_3 which appears when time-reversal symmetry is violated. Those which do correspond to the diffuson degrees of freedom while those which do not describe the Cooperons [15]. Firstly, for the potential perturbation, $U\equiv VL^2/D$ both degrees of freedom contribute and the fluctuations of Q display the full orthogonal symmetry. For the flux perturbation, $U\equiv e\phi/hc$ the Cooperon modes are frozen out for non-vanishing $\bar\phi$, even when $\phi = 0$. In this case, the symmetry of the remaining diffuson modes belong to the unitary ensemble.

Within the diffusive regime, it is permissible to neglect all but the lowest spatial harmonic at the saddle-point in $\mathbf{r}$. For $U\equiv e\phi/hc$ this implies a spatially independent Q [16], while for $U\equiv VL^2/D$ it is necessary to calculate the spatial dependence at the saddle-point explicitly. In each case we obtain a definite integral expression for $k(\Omega, U)$ which has the functional form of Eqs. (6-7) appropriate to the symmetry. From the expression for $k(\Omega, U)$ it is possible to determine the conductances [5,14]

$$C(0) = \begin{cases} 4\pi g & U\equiv e\phi/hc, \\ g/3\pi & U\equiv VL^2/D, \end{cases} \tag{18}$$

where $g = D/\Delta L^2$ denotes the dimensionless conductance. Applying the rescaling (1) to U and Ω we obtain the universal expressions (6-7). We note that when $u > \sqrt{g}$, or $\omega > g$ corrections from higher modes become important and the universality breaks down.

The expansion at large u (or ω) shown in Eq. (14) is equivalent to the lowest order of diagrammatic perturbation theory [17], and is reproduced by the hydrodynamic limit of the Sutherland model. The propagators from the right and left moving particles in the Luttinger model therefore correspond to the exchange of two diffusons or Cooperons lines between two closed loops.

Why the Sutherland and σ-model should display equivalent correlation functions remains mysterious. That the potential should be inverse-square is natural since it allows the coupling to be dimensionless. Any other power would demand a rescaling with the particle density. Whether the Sutherland model possesses a hidden symmetry [18] at the special values of the coupling constant corresponding to the three Dyson ensembles remains an open question. A naive continuation of Eq. (14) to arbitrary values of β coincides with the hydrodynamic limit of the Sutherland model (13), which is valid for all values of β. However, a physical interpretation of arbitrary β for the eigenvalue problem is unclear. One possibility might be to generate other values from fractional spin in analogy with spin-orbit interactions where spin degeneracy leads to $\beta = 4$.

In conclusion, we have demonstrated that the time-dependent correlation functions of a many-particle interacting one-dimensional model have an exact correspondence with the universal spectral correlations of disordered metallic grains and other quantum chaotic systems. A deeper understanding of the connection of the one-dimensional model, random matrix theory, and the supersymmetry method used to derive the universal functions remains to be found.

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