Matrix Models, One-Dimensional Fermions, and Quantum Chaos

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Abstract

Recent studies of universal parametric correlations in quantum chaotic spectra have revealed an astonishing connection to an integrable one-dimensional quantum system. We introduce a continuous matrix model which establishes a direct connection between the quantum Hamiltonian and the exact field theoretic description of spectral correlations. This reveals a common mathematical structure which underlies quantum chaos, matrix models, and a quantum Hamiltonian.

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Studies of correlations in the spectra of quantum chaotic systems revealed a striking connection [1] to an *integrable* one-dimensional fermionic model [2]. However, this connection was made as a conjecture which was supported by comparing the asymptotics of two-point functions. In this paper we prove the conjecture by showing that properties of the integrable quantum Hamiltonian and the spectra of non-integrable systems can be obtained from a continuous matrix model. This provides a new way of determining universal correlations in chaotic spectra which is extendible to higher order correlation functions, and completes an investigation which reveals a common mathematical structure underlying branches of theoretical physics ranging from one-dimensional interacting fermionic systems, matrix models (of random surfaces), to the universal spectral correlations of quantum chaotic systems. To assist our discussion Fig. 1 shows a schematic diagram summarizing the web of connections of different models. Different links in the diagram are referenced (I-X) throughout the text.

We begin with a brief review of universality in quantum chaos. Suppose a non-integrable Hamiltonian, H is acted upon by an external perturbation controlled by a parameter, U. For example we might examine,

$$H(U) = \Phi + U\Phi_c, \tag{1}$$

where Φ belongs to one of the Dyson ensembles [3], and Φ_c is some fixed hermitian traceless matrix from the same ensemble. Non-integrability of H causes the energy levels, $E_i(U)$ to disperse as a function of U into a manifold of separated levels. According to Wigner-Dyson statistics [3] rescaling the energy levels by the local mean level spacing, $\epsilon_i = E_i/\Delta$ eliminates the dependence of equal U correlations on detailed properties of the system. By further rescaling the perturbation by the mean-square gradient of the levels [4],

$$C(0) = \Big\langle \Big(\frac{\partial \epsilon_i(U)}{\partial U}\Big)^2 \Big\rangle, \qquad u^2 = C(0)U^2, \tag{2}$$

the statistical properties of the entire random functions $\epsilon_i(u)$ become universal and dependent only on the symmetry of the Dyson ensemble. The statistical averaging denoted by $\langle \cdots \rangle$ can be performed over a typical range of levels and/or U [5]. The universality has been verified numerically (I) for several chaotic systems [6] including quantum billiards [4,7], hydrogen in a magnetic field [8], and strongly correlated many-particle systems [9].

However, if averaging can be performed over a statistical ensemble, the universality can be confirmed analytically (II). After rescaling both random matrix ensembles [3] and weakly disordered metals [10,4] display the same universal two-point density of states autocorrelator,

$$k(\omega, u) = \sum_{ij} \langle \delta(\epsilon - \omega - \epsilon_i(\bar{u} - u)) \ \delta(\epsilon - \epsilon_j(\bar{u})) \rangle - 1. \tag{3}$$

In fact, $k(\omega, u)$ and its higher order generalizations can be expressed through a non-linear supermatrix σ-model (III) which shows universality to extend to correlations of more than two densities of states.

Remarkably, the expression for $k(\omega, u)$ describes exactly the two-point *particle* density correlator of an integrable one-dimensional system of fermions interacting through an inverse-square pairwise potential [1] when the separation in position, λ and imaginary time, τ have the association (IV),

$$\omega \leftrightarrow \lambda, \qquad u^2 \leftrightarrow 2\tau. \tag{4}$$

The correspondence of the two-point function was checked only in the limits of large and small τ. Its origin, and with it the equivalence of higher-point functions remained an open question.

Our main result will be to show that both the fermionic Hamiltonian and the σ-model can be derived from a continuous matrix model. Furthermore the correspondence extends to all higher order correlation functions, thus proving that the connection of the fermionic model to the problem of level statistics in quantum chaotic systems is *exact*.

The partition function $\mathcal{Z} = \int \mathcal{D}\Phi(\tau) \exp\{-S[\Phi(\tau)]\}$ with the classical Euclidean action

$$S[\Phi(\tau)] = \int_{-\infty}^{\infty} d\tau \ \mathrm{Tr} \ \Big[\frac{1}{4}\Big(\frac{\partial \Phi(\tau)}{\partial \tau}\Big)^2 + V(\Phi)\Big], \tag{5}$$

describes the free propagation of $N \times N$ random matrices in a potential $V(\Phi)$. The potential together with the symmetry of Φ, denoted by the Dyson index β, is left unspecified. In the limit of large N only planar diagrams contribute [11], and for certain choices of $V(\Phi)$ the logarithm of $\mathcal{Z}$ generates the sum of connected surfaces [12].

If we interpret the matrix model as a path integral it describes the quantum mechanics of an $N \times N$ matrix Φ, governed by the Hamiltonian,

$$H = \mathrm{Tr} \ \Big[-\Big(\frac{\delta}{\delta \Phi}\Big)^2 + V(\Phi)\Big]. \tag{6}$$

According to the symmetry of Φ, the Hamiltonian is invariant under global $SO(N)$, $SU(N)$, or $SU(N) \times SU(N)$ rotations for orthogonal ($\beta = 1$), unitary ($\beta = 2$), or symplectic ($\beta = 4$) ensembles respectively.

In the limit $N \to \infty$ the angular degrees of freedom of the matrices which diagonalize Φ give an exponentially small contribution and attention can be restricted to the "singlet" component [13] (for a review see Ref. [14]). The Hamiltonian then reduces to a one-dimensional system of interacting particles describing the N eigenvalues, λ_i of Φ. Up to a constant Eq. (6) can be rewritten,

$$\begin{aligned} H &= \prod_{i<j} |\lambda_i - \lambda_j|^{-\beta/2} \ H_C \ \prod_{i<j} |\lambda_i - \lambda_j|^{\beta/2}, \\ H_C &= -\sum_i \frac{\partial^2}{\partial \lambda_i^2} + \beta\Big(\frac{\beta}{2} - 1\Big) \sum_{i<j} \frac{1}{(\lambda_i - \lambda_j)^2} + \sum_i V(\lambda_i). \end{aligned} \tag{7}$$

H_C describes the motion of particles with an inverse square pairwise interaction in the presence of a background potential $V(\lambda_i)$. A Jordan-Wigner transformation can be used to assign arbitrary statistics to the *hard-core* particles. When the potential is Gaussian, $V(\lambda_i) = \omega_0^2 \ \lambda_i^2$ Calogero [15] has shown that the model is separable, and the spectrum integrable.

The ground state wavefunction of H_C [15],

$$|0\rangle \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta/2} \exp\Big[-\frac{\omega_0}{2} \sum_i \lambda_i^2 \Big], \tag{8}$$

generates a probability distribution which coincides with that of Gaussian random matrix ensembles. The mean particle density corresponds to Wigner's semi-circular density of states distribution [3], and the mean interparticle spacing at the origin, Δ is determined by,

$$\Delta^2 = \frac{\pi^2}{4N}\frac{\beta}{\omega_0}. \tag{9}$$

Although the Hamiltonian depends explicitly on the form of $V(\lambda_i)$, in the thermodynamic limit ($N \to \infty$) correlation functions become universal on a scale comparable to the mean particle spacing. The role of ω_0 is merely to fix Δ. Equivalently, to order $1/N$ the form of the correlation functions can be preserved by removing the confining potential and constraining the particles to move on a ring. The resulting distribution corresponds to Dyson's circular ensemble [3]. More generally the potential smoothly regulates the average density with particles filling the potential energy surface up to some Fermi energy. To order $1/N$ the local value of Δ is fixed and correlation functions again coincide with those of the circular ensemble. Formally the universality of the large N limit can be seen from renormalization group arguments as a flow to an attractive Gaussian fixed point of the potential [16].

As $N \to \infty$ we thus expect the correlation functions of H_C to coincide with the Sutherland Hamiltonian [2],

$$H_S = -\sum_i \frac{\partial^2}{\partial\lambda_i^2} + \beta\Big(\frac{\beta}{2} - 1\Big)\sum_{i<j}\frac{(\pi/N)^2}{\sin^2[\pi(\lambda_i - \lambda_j)/N]}, \tag{10}$$

where henceforth we set the mean spacing $\Delta = 1$. This achieves our first goal of establishing a precise connection between the continuous matrix model, $S[\Phi(\tau)]$ and the one-dimensional integrable quantum Hamiltonian H_S (V). Time-dependent correlation functions of the Sutherland Hamiltonian can therefore be represented by the path integral,

$$\langle 0|\cdots|0\rangle \equiv \mathcal{Z}^{-1}\int \mathcal{D}\Phi(\tau)\cdots e^{-S[\Phi(\tau)]}. \tag{11}$$

In the second part of the paper we will describe how a supersymmetrical field theory can be used to determine the m-point correlator of Green functions,

$$\langle 0|f_m|0\rangle = \Big\langle 0\Big|\prod_{\alpha=1}^m \mathrm{Tr}\,\Big(\frac{1}{\lambda_\alpha + i\sigma_\alpha 0^+ - \Phi(\tau_\alpha)}\Big)\Big|0\Big\rangle, \tag{12}$$

where $\{\sigma_\alpha = \pm 1\}$ defines whether the resolvents in the product are retarded or advanced, and we have used Greek subscripts to distinguish the parameters of the m-point functions from the eigenvalues. f_m can be expressed through a generating function,

$$f_m = \prod_{\alpha=1}^m \sigma_\alpha \frac{\partial}{\partial J_\alpha}\int d[\psi]\exp\Big[i\sum_{\alpha=1}^m \psi_\alpha^\dagger\sigma_\alpha\Big(\lambda_\alpha + i\sigma_\alpha 0^+ - \Phi(\tau_\alpha) + \frac{J_\alpha I}{2}\Big)\psi_\alpha\Big]\Big|_{J_\alpha=0}, \tag{13}$$

where $\psi_{\alpha\mu} = (\, S_{\alpha\mu} \quad \chi_{\alpha\mu}\,)$ is a $m \times 2 \times N$ component supervector with χ/S denoting the anti/commuting components, α and μ indexing the m Green functions and the N components of Φ respectively, and the matrix $I = \mathrm{diag}\,(\,-1 \quad 1\,)$ in the graded subspace. For simplicity we allow the symmetry of Φ to belong to either the orthogonal or unitary ensemble. The analysis can be easily generalized to the symplectic ensemble.

If we assume that,

$$\lambda_\alpha \sim O(N^0), \qquad |\tau_\alpha - \tau_\beta| \sim O(N^0), \tag{14}$$

we can obtain an expression for $\langle 0|f_m|0\rangle$ which is exact to order $1/N$. The method requires a generalization of the supersymmetry approach [10] in which the problem of evaluating averages reduces to the study of a non-linear supermatrix σ-model (for a pedagogical description of this approach as applied to the problem of random matrix ensembles we refer to Ref. [17]). The details of the calculation will be presented elsewhere.

Performing the path integral over Φ generates an interaction of the fields ψ. The interaction can be decoupled by the introduction of $4m \times 4m$ Hubbard-Stratonovitch fields Q. By making use of Eq. (14) we obtain an expression for $\langle 0|f_m|0\rangle$ in the form of a supermatrix σ-model,

$$\langle 0|f_m|0\rangle = \prod_\alpha^m \sigma_\alpha \frac{\partial}{\partial J_\alpha} \int d[Q] e^{-F[Q]}\Big|_{J_\alpha=0},$$
$$F[Q] = i\frac{\pi}{2}\sum_\alpha^m \mathrm{STr}\Big[\Big(\lambda_\alpha + i\sigma_\alpha 0^+ + \frac{J_\alpha I}{4}\Big)Q_{\alpha\alpha}\Big] + \frac{\pi^2\beta}{8}\sum_{\alpha,\beta}^m |\tau_\alpha - \tau_\beta| \mathrm{STr}\ [Q_{\alpha\beta}Q_{\beta\alpha}]. \tag{15}$$

The integration is restricted to $Q^2 = 1$, and for unitary ensembles, ($\beta = 2$) we require the additional constraint that $[Q, \tau_3] = 0$. The constant matrix τ_3 breaks time-reversal symmetry. The definition of τ_3, together with the supertrace, denoted STr can be found in Ref. [17].

The method relies on a saddle-point approximation which gives rise to the constraint on Q. Eq. (15) describes the leading order expansion about the saddle-point and the higher order corrections are smaller by a factor of $1/N$. Since we are focusing on the thermodynamic limit when the quantum Hamiltonian converges to H_S, Eq. (15) may be regarded as the *exact* time-dependent m-point correlator of the Sutherland model (VI).

An analogous approach can be used to determine the universal m-point correlator for a non-integrable Hamiltonian such as Eq. (1),

$$\langle f_m\rangle = \Big\langle \prod_{\alpha=1}^m \mathrm{Tr}\ \Big(\frac{1}{\omega_\alpha + i\sigma_\alpha 0^+ - H(u_\alpha)}\Big)\Big\rangle, \tag{16}$$

where $\langle\cdots\rangle$ denotes spectral averaging. (In this case $C(0) = 2\ \mathrm{Tr}[\Phi_c^2]/\beta N^2$ and we have set the mean level spacing $\Delta = 1$.) With the correspondence,

$$\omega_\alpha \leftrightarrow \lambda_\alpha, \qquad (u_\alpha - u_\beta)^2 \leftrightarrow 2|\tau_\alpha - \tau_\beta|, \tag{17}$$

the universal expression for $\langle f_m\rangle$ (valid when ω_α and $(u_\alpha - u_\beta)^2 \sim O(N^0)$) becomes equivalent to Eq. (15).

More generally the Hamiltonian can depend parametrically on m matrices with $H(\{U_\alpha\}) = \Phi + \sum_\alpha^m U_\alpha \Phi_c^{(\alpha)}$ when the correlator $\langle f_m\rangle$ depends on $m(m-1)/2$ parameters $p_{\alpha\beta} = 2\ \mathrm{Tr}[U_\alpha\Phi_c^{(\alpha)} - U_\beta\Phi_c^{(\beta)}]^2/\beta N^2$ rather than the $m-1$ independent elements $(u_\alpha - u_\beta)^2$.

The correlators of the Sutherland Hamiltonian and the quantum chaotic spectra of Eq. (1) can be regarded as special cases for which there are $(m-1)(m-2)/2$ constraints on $p_{\alpha\beta}$. However, for each case the set of constraints are different. While the solution to the general problem uniquely determines the correlators of H_S and Eq. (1) we note that the reverse statement does not apply.

This achieves our main objective of showing that the problem of universal level statistics in quantum chaotic systems and the ground state correlations of H_S are related through the σ-model. We emphasize that the continuous matrix model is qualitatively different from the random matrix model of Eq. (1). The non-linear relationship between u_α and τ_α is a manifestation of this difference. For the two-point function, $k(\omega, u)$ a parameterization of Q can be found in which the definite integration can be performed analytically. The results of such a calculation can be found in Ref. [4].

To complete our discussion of the connections in Fig. 1 we focus on alternative approaches to level dynamics. Pechukas has shown [18] that the dispersion of the energy levels of a non-integrable Hamiltonian in response to an external perturbation can be expressed as a set of first order differential equations (VII), which were later shown to be integrable [19]. The equilibrium statistical mechanics of the "Pechukas gas" becomes equivalent to random matrix theory (VIII) if we accept a statistical hypothesis [20]. Recent studies by Narayan and Shastry [21] have shown that the eigenvalues of a *perturbed* random matrix obey the same Fokker-Planck equation which governs the Brownian dynamics of the levels (IX) (Dyson's "Coulomb gas" model [22]). Since correlation functions of the Fokker-Planck equation can be expressed through the Sutherland Hamiltonian [2] the final connection (X) in Fig. 1 is established.

From a pedagogical point of view it is instructive to reinterpret Fig. 1 in terms of the appropriate perturbation theory at large $(\omega, u) \leftrightarrow (\lambda, \tau)$. For the problem of random matrices and weakly disordered metallic systems the Cooperon and diffusion modes of diagrammatic perturbation theory provide the leading order contributions. The behavior in this regime is reproduced by the hydrodynamic limit of the Brownian motion model [23], and the Sutherland (or Luttinger Liquid) Hamiltonian [24,1]. The leading order of perturbation theory we view as the inclusion of all periodic trajectories in phase space, but with the exclusion of tunneling events between trajectories. This same semi-classical interpretation we believe provides the basis of the Gutzwiller trace formula [25].

In conclusion we have shown that a continuous random matrix model provides a precise correspondence between the universal non-linear $\sigma-$model description of spectral correlations in non-integrable systems and an integrable one-dimensional quantum Hamiltonian. These results establish a precise connection between interacting quantum systems, matrix models and quantum chaos.

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FIGURES

A schematic diagram showing how the matrix model introduced here draws together different branches of physics and relates them through the non-linear supermatrix $\sigma-$model. Connections V and VI are established in this paper. Details for the other connections can be found for (I) in Ref. [6], (II) in Refs. [3,10], (III) in Refs. [10,4], (IV) in Ref. [1], (VII) in Ref. [18], (VIII) in Ref. [20], (IX) in Refs. [22,21], and (X) in Ref. [2]. The dotted line (IV) is established only in limiting cases, and the dashed lines (I,II,VIII) require a statistical hypothesis which has been checked numerically.