PHYSICS TRIPOS Part III

16 April 2007

PHASE TRANSITIONS AND COLLECTIVE PHENOMENA

2004 Minor Options Examination Paper

1 Close to the critical point of a classical Ferromagnet, it is proposed that the singular part of the free energy density assumes the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^{\mathit{\Delta}}}\right)$$

where $t = (T - T_c)/T_c$ represents the reduced temperature, and h denotes the dimensionless magnetic field.

(a) Starting with the expression for the free energy density, show that the magnetisation assumes a homogeneous form. From this result, determine the relation between the scaling exponents of the magnetisation density $m(t, h = 0) \sim t^{\beta}$ and $m(t = 0, h) \sim h^{1/\delta}$ and the critical exponents α and $\mathit{\Delta}$. [5]

(b) Using the expression for the magnetisation density, obtain the relation between the scaling exponent γ of the susceptibility $\chi(t) \sim t^{-\gamma}$ and the exponents α and $\mathit{\Delta}$. [5]

(c) According to the hyperscaling hypothesis, close to the critical point, it is proposed that the correlation length assumes the homogeneous form

$$\xi(t,h) = t^{-\nu} g_{\xi}\left(\frac{h}{t^{\mathit{\Delta}}}\right).$$

Explain why this result is compatible with the hyperscaling identity $d\nu = 2 - \alpha$. [5]

(d) Finally, according to the scaling hypothesis, the correlation function takes the form

$$\langle m(\boldsymbol{x})m(0)\rangle = \frac{1}{|\boldsymbol{x}|^{d-2+\eta}} g\left(\frac{|\boldsymbol{x}|}{\xi(t,h)}\right).$$

From this result, obtain the susceptibility and prove the exponent identity $\gamma = (2 - \eta)\nu$. [5]

2 Write detailed notes on **one** of the following topics: [?]

(a) Mermin-Wagner Theorem and the Lower-Critical Dimension;

(b) Toplogical defects and the Kosterlitz-Thouless transition;

(c) Feynman path integral and its connection with statistical mechanics *illustrating your discussion with the example of the one-dimensional Ising model.*

3 Outline concisely the conceptual basis of the *Renormalisation Group* (RG) method. [5]

In the Gaussian approximation, the Ginzburg-Landau Hamiltonian for the disordered phase of a 'smectic liquid crystal' takes the form

$$\beta H[m(\boldsymbol{x})] = \int \mathrm{d}x_{\parallel} \int \mathrm{d}^{d-1}\boldsymbol{x}_{\perp} \left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla_{\parallel} m)^2 + \frac{L}{2}(\nabla_{\perp}^2 m)^2 - hm\right]$$

where $m(\boldsymbol{x})$ represents a one-component field depending on a d-dimensional set of coordinates $\boldsymbol{x} = (x_{\parallel}, \boldsymbol{x}_{\perp})$, and the coefficients K, L, and t assume positive values.

(a) Transforming to the Fourier basis, reexpress the Hamiltonian $\beta H[m]$ in terms of the fields $m(q_{\parallel}, \boldsymbol{q}_{\perp})$. [3]

(b) Construct a Renormalisation Group transformation for the Hamiltonian $\beta H[m]$ by (i) applying an anisotropic rescaling of the coordinates such that $q'_{\parallel} = b\, q_{\parallel}$ and $\boldsymbol{q}'_{\perp} = c\, \boldsymbol{q}_{\perp}$, and (ii) applying the field renormalisation $m' = m/z$. How do the parameters t, K, L, and h scale under the RG transformation? [10]

(c) For what values of c and z (as a function of b) do the parameters K and L remain fixed? For the remaining coefficients t and h, show that the corresponding Gaussian fixed point is associated with the exponents $y_t = 2$ and $y_h = (d+5)/4$ respectively. [3]

(d) By establishing the relationship between the free energies $f(t, h)$ and $f(t', h')$ of the original and rescaled Hamiltonians, show that the free energy assumes the homogeneous form

$$f(t, h) = t^{2-\alpha} g_f(h/t^{\Delta}).$$

Identify the exponents α and Δ. [4]

2004 Minor Options Examination Answers

1 **Scaling Theory:**

(a) From the free energy density, one may obtain the magnetisation density as

$$m(t,h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^\Delta),$$

where $g_m(x)$ represents a homogeneous function of its argument. In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t,h=0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2-\alpha-\Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t=0,h) \sim t^{2-\alpha-\Delta}(h/t^\Delta)^p$. Since this limit is independent of t, we must have $p\Delta = 2-\alpha-\Delta$. Hence $m(t=0,h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2-\alpha-\Delta) = \Delta/\beta$). [8]

(b) From the magnetisation, one may obtain the susceptibility as

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^\Delta) \Rightarrow \chi(t,h=0) \sim t^{2-\alpha-2\Delta},$$

from which it follows that $\gamma = 2\Delta - 2 + \alpha$. [4]

(c) Close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t,h)$ is dimensionless and extensive (i.e. proportional to the volume, L^d), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t,h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^\Delta)$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity [7]

$$2 - \alpha = d\nu$$

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(d) Using the correlation function, one obtains the susceptibility

$$\begin{aligned}\chi(t,h) &\sim \int d\boldsymbol{x}\ \langle m(\boldsymbol{x})m(0)\rangle \sim \int_0^\xi dx \frac{x^{d-1}}{x^{d-2+\eta}} \sim \xi^{2-\eta} \\ &\sim t^{-(2-\eta)\nu} g_\xi\left(\frac{h}{t^\Delta}\right)\end{aligned}$$

We thus obtain the exponent identity $\gamma = (2-\eta)\nu$. [6]

2 **Essay Question:**

(a) *Mermin-Wagner Theorem and the Lower-Critical Dimension:* In moving through a second order phase transition, the order parameter for the transition grows continuously from zero. The development of order below the transition is accompanied by a spontaneous symmetry breaking — the symmetry of the low temperature ordered phase is lower than the symmetry of the high temperature disordered phase. An example is provided by the classical ferromagnet where the appearance of net magnetisation breaks the symmetry $m \mapsto -m$. If the number of components is larger than unity, the symmetry is continuous and its breaking is accompanied by the appearance of low-energy (i.e. 'massless') Goldstone mode excitations. In the magnet, these excitations are known as spin waves.

To assess the importance of these fluctuations on the validity of the mean-field, let us consider a two-component order parameter, $\boldsymbol{m}(\boldsymbol{x}) = m(\boldsymbol{x})(\cos\theta(\boldsymbol{x}), \sin\theta(\boldsymbol{x}))$. Focussing on the magnetic system, massless transverse fluctuations of the order parameter are described by the Ginzburg-Landau Hamiltonian

$$\mathcal{Z}_{\text{fluct.}} = \int D\theta \exp\left[-\frac{1}{2}\int \mathrm{d}^d\boldsymbol{x} K\bar{m}^2(\partial\theta)^2\right].$$

Although the phase θ is multivalued, at low temperatures ($K \gg 1$), one may neglect the constraint and apply the rules of Gaussian functional integration. In doing so, one finds that $\langle\theta(\boldsymbol{x})\rangle = 0$, and the correlation function assumes the form

$$G(\boldsymbol{x},\boldsymbol{x}') \equiv \langle\theta(\boldsymbol{x})\theta(0)\rangle = -\frac{C_d(\boldsymbol{x})}{\bar{K}}, \qquad \nabla^2 C_d(\boldsymbol{x}) = \delta^d(\boldsymbol{x})$$

where C_d denotes the Coulomb potential for a δ-function charge distribution. Exploiting the symmetry of the field, and employing Gauss', $\int d\boldsymbol{x}\ \nabla^2 C_d(\boldsymbol{x}) = \oint dS \cdot \nabla C_d$, one finds that C_d depends only on the radial coordinate x, and

$$\frac{dC_d}{dx} = \frac{1}{x^{d-1}S_d}, \qquad C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + \text{const.},$$

where $S_d = 2\pi^{d/2}/(d/2-1)!$ denotes the total d-dimensional solid angle.

Using this result, one finds that

$$\left\langle[\theta(\boldsymbol{x})-\theta(0)]^2\right\rangle = 2\left[\langle\theta(0)^2\rangle - \langle\theta(\boldsymbol{x})\theta(0)\rangle\right] \stackrel{|\boldsymbol{x}|\geq a}{=} \frac{2(|\boldsymbol{x}|^{2-d}-a^{2-d})}{\bar{K}(2-d)S_d},$$

where the cut-off, a is of the order of the lattice spacing. (Note that the case where $d=2$, the combination $|\boldsymbol{x}|^{2-d}/(2-d)$ must be interpreted as $\ln|\boldsymbol{x}|$.

This result shows that the long distance behaviour changes dramatically at $d=2$. For $d>2$, the phase fluctuations approach some finite constant as $|\boldsymbol{x}|\to\infty$, while they become asymptotically large for $d\leq 2$. Since the phase is bounded by 2π, it implies that long-range order (predicted by the mean-field theory) is destroyed.

Applied to the two-point correlation function of $\boldsymbol{m}$, an application of the rules of Gaussian functional integral, obtains

$$\langle\boldsymbol{m}(\boldsymbol{x})\cdot\boldsymbol{m}(0)\rangle = \bar{m}^2\mathrm{Re}\,\left\langle e^{i[\theta(\boldsymbol{x})-\theta(0)]}\right\rangle = \bar{m}^2\exp\left[-\frac{1}{2}\langle[\theta(\boldsymbol{x})-\theta(0)]^2\rangle\right].$$

We thus obtain

$$\langle\boldsymbol{m}(\boldsymbol{x})\cdot\boldsymbol{m}(0)\rangle = \bar{m}^2\exp\left[-\frac{(|\boldsymbol{x}|^{2-d}-a^{2-d})}{\bar{K}(2-d)S_d}\right],$$

implying a power-law decay of correlations in $d=2$, and an exponential decay in $d<2$. From this result we find

$$\lim_{|\boldsymbol{x}|\to\infty}\langle\boldsymbol{m}(\boldsymbol{x})\cdot\boldsymbol{m}(0)\rangle = \begin{cases} m_0^2 & d>2,\\ 0 & d\leq 2.\end{cases}$$

This result encapsulates the Mermin-Wagner Theorem: In dimensions $d\leq d_c^l$, where d_c^l denotes the lower critical dimension, the proliferation of Goldstone mode fluctuations destroys long-range order at any non-zero temperature.

(b) *Toplogical defects and the Kosterlitz-Thouless transition:* According to the Mermin-Wagner theorem, spontaneous symmetry breaking of a continuous symmetry leads to the appearance of Goldstone modes which destroy long-range order in dimensions $d\leq 2$. However, in two-dimensions, there exists a low temperature phase of quasi long-range order in which the correlations decay algebraically at long-distances. This leaves open the room for a phase transition at some intermediate temperature in which the correlation function crosses over to exponential decay.

To understand the nature of the transition, it is necessary to take into account the existence of topological defects, vortex configurations of the fields. The elementary defect which has a unit charge involves a 2π twist of θ as one encircles the defect. More formally,

$$\oint\nabla\theta\cdot d\ell = 2\pi n \quad\Longrightarrow\quad \nabla\theta = \frac{n}{r}\hat{\boldsymbol{e}}_r\times\hat{\boldsymbol{e}}_z,$$

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where $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_z$ are unit vectors respectively in the plane and perpendicular to it. This (continuum) approximation fails close to the centre (core) of the vortex, where the lattice structure is important.

The energy cost from a single vortex of charge n has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius a to distinguish the two, i.e.

$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d^2\boldsymbol{x}(\nabla\theta)^2 = \beta E_n^0(a) + \pi K n^2 \ln\left(\frac{L}{a}\right).$$

The dominant part of the energy comes from the region outside the core and diverges logarithmically with the system size L. The large energy cost associated with the defects prevents their spontaneous formation close to zero temperature. The partition function for a configuration with a single vortex of charge n is

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right],$$

where the factor of $(L/a)^2$ results from the configurational entropy of possible vortex locations in an area of size L^2. The entropy and energy of a vortex both grow as $\ln L$, and the free energy is dominated by one or the other. At low temperatures, large K, energy dominates and $\mathcal{Z}_1$, a measure of the weight of configurations with a single vortex, vanishes. At high enough temperatures, $K < K_n = 2/(\pi n^2)$, the entropy contribution is large enough to favour spontaneous formation of vortices. On increasing temperature, the first vortices that appear correspond to $n = \pm 1$ at $K_c = 2/\pi$. Beyond this point many vortices appear and the equation above is no longer applicable.

In fact this estimate of K_c represents only a lower bound for the stability of the system towards the condensation of topological defects. This is because pairs (dipoles) of defects may appear at larger couplings. Consider a pair of charges ± 1 separated by a distance d. Distortions far from the core $|\boldsymbol{r}| \gg d$ can be obtained by superposing those of the individual vortices

$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \approx 2\boldsymbol{d}\cdot\nabla\left(\frac{\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_z}{|\boldsymbol{r}|}\right),$$

which decays as $d/|\boldsymbol{r}|^2$. Integrating this distortion leads to a finite energy, and hence dipoles appear with the appropriate Boltzmann weight at any temperature. The low temperature phase should therefore be visualised as a gas of tightly bound dipoles, their density and size increasing with temperature. The high temperature phase constitutes a plasma of unbound vortices.

3 The divergence of the correlation length at a second order phase transition suggests that, in the vicinity of the transition, the microscopic length-scales are irrelevant. The critical behaviour is dominated by fluctuations that are statistically self-similar up to the length scale ξ. Self-similarity allows the gradual elimination of the correlated degrees of freedom at length scales $|\boldsymbol{x}| \ll \xi$, until one is left with the relatively simple uncorrelated degrees of freedom at the scale of the correlation length ξ. [5]

(a) In the Fourier representation the Hamiltonian takes the diagonal form

$$\beta H = \frac{1}{2}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d} G^{-1}(\boldsymbol{q})|m(\boldsymbol{q})|^2 - hm(\boldsymbol{q}=0),$$

where the anisotropic propagator is given by [3]

$$G^{-1}(\boldsymbol{q}) = t + Kq_{\parallel}^2 + L\boldsymbol{q}_{\perp}^4.$$

(b) To implement the RG procedure, the first step is to apply a course-graining by integrating over the fast field fluctuations. Setting

$$m(\boldsymbol{q}) = \begin{cases} m_<(\boldsymbol{q}) & 0 < |q_{\parallel}| < \Lambda/b \text{ and } 0 < |\boldsymbol{q}|_{\perp} < \Lambda/c, \\ m_>(\boldsymbol{q}) & \Lambda/b < |q_{\parallel}| < \Lambda \text{ or } \Lambda/c < |\boldsymbol{q}|_{\perp} < \Lambda, \end{cases}$$

the fast fluctuations separate from the slow identically for the Gaussian Hamiltonian. As such, an integration over the fast fluctuations obtains

$$\mathcal{Z} = \mathcal{Z}_> \int Dm_< \exp\left[-\frac{1}{2}\int_0^{\Lambda/b}(\mathrm{d}q_{\parallel})\int_0^{\Lambda/c}(\mathrm{d}^{d-1}\boldsymbol{q}_{\perp})G^{-1}(\boldsymbol{q})|m_<(\boldsymbol{q})|^2 + hm_<(0)\right],$$

where the constant $\mathcal{Z}_>$ is obtained from performing the functional integral over $m_>$. Applying the rescaling $q_{\parallel}' = bq_{\parallel}$ and $\boldsymbol{q}_{\perp}' = c\boldsymbol{q}_{\perp}$, the cut-off in the domain of momentum integration is restored. Finally, applying the renormalisation $m'(\boldsymbol{q}) = m_<(\boldsymbol{q})/z$ to the Fourier field amplitudes, one obtains

$$\mathcal{Z} = \mathcal{Z}_> \int Dm'(\boldsymbol{q}')e^{-(\beta H)'[m'(\boldsymbol{q}')]},$$

where the renormalised Hamiltonian takes the form

$$(\beta H)' = \frac{1}{2}\int(\mathrm{d}^d\boldsymbol{q})b^{-1}c^{-(d-1)}z^2\left(t + Kb^{-2}{q_{\parallel}'}^2 + Lc^{-4}{\boldsymbol{q}_{\perp}'}^4\right)|m'(\boldsymbol{q}')|^2 - zhm'(0).$$

From the result, we obtain the renormalisation of the coefficients [10]

$$\begin{cases} t' = tb^{-1}c^{-(d-1)}z^2, \\ K' = Kb^{-3}c^{-(d-1)}z^2, \\ L' = Lb^{-1}c^{-(d+3)}z^2, \\ h' = hz. \end{cases}$$

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(c) Choosing parameters $c = b^{1/2}$ and $z = b^{(d+5)/4}$ ensures that $K' = K$ and $L' = L$ and implies the scaling exponents $y_t = 2$, $y_h = (d+5)/4$. [3]

(d) From this result we obtain the renormalisation of the free energy density

$$f(t, h) = b^{-(d+1)/2} f(b^2 t, b^{(d+5)/4} h).$$

Setting $b^2 t = 1$, we can identify the exponents $2 - \alpha = (d+1)/4$ and $\Delta = y_h/y_t = (d+5)/8$. [4]

2005 Minor Options Examination Paper

1 In the leading approximation, the influence of lattice compressibility on the ferromagnetic transition can be explored within the framework of the Ginzburg-Landau Hamiltonian

$$\beta H[m,\phi] = \int d^3\boldsymbol{x}\left[\frac{t}{2}m^2 + um^4 + vm^6 + \frac{K}{2}(\nabla m)^2 - hm + g\phi\, m^2 + \frac{c}{2}\phi^2\right],$$

where $\phi(\boldsymbol{x})$ denotes the (scalar) strain field, and the parameters u and v are both assumed positive.

(a) Integrating out strain field fluctuations $\phi(\boldsymbol{x})$, show that the partition function for the magnetisation field is controlled by the effective Hamiltonian [4]

$$\beta H_{\text{eff}}[m] = \int d^3\boldsymbol{x}\left[\frac{t}{2}m^2 + \left(u - \frac{g^2}{2c}\right)m^4 + vm^6 + \frac{K}{2}(\nabla m)^2 - hm\right].$$

(b) Working in the Landau theory approximation, by sketching the m dependence of the Landau Hamiltonian for different values of the parameters, describe *qualitatively* the magnetic phase diagram for $h = 0$. In particular, discuss what happens when $\frac{g^2}{2c} > u$. [4]

(c) When $\frac{g^2}{2c} = u$, obtain the average magnetisation $\bar{m}(t, h=0)$, $\bar{m}(t=0, h)$, and susceptibility $\chi(t, h=0) = \frac{\partial \bar{m}}{\partial h}\Big|_{h=0}$ within the Landau theory approximation. [6]

2 Write a detailed essay on **one** of the following topics:

(a) the scaling theory of critical phenomena; [20]

(b) the Kosterlitz-Thouless phase transition; [20]

(c) the connection between statistical and quantum field theory illustrating your discussion with specific examples. [20]

3 If we define a Hamiltonian $\beta H[\phi] = \beta H_0[\phi] + U[\phi]$ as the sum of a free theory $\beta H_0[\phi]$ and a perturbation $U[\phi]$, show that the renormalisation group (RG) transformation resulting from field integration over fast field fluctuations $\phi_>$ results in the following renormalised Hamiltonian for the slow field fluctuations $\phi_<$,

$$\beta H'[\phi_<] = -\mathcal{Z}^0_> + \beta H_0[\phi_<] - \ln\langle e^{-U[\phi_<,\phi_>]}\rangle_>$$

where $\mathcal{Z}^0_> = \int D\phi_> e^{-\beta H_0[\phi_>]}$ and $\langle \cdots \rangle_> = \frac{1}{\mathcal{Z}_>}\int D\phi_> \cdots e^{-\beta H_0[\phi_>]}$. [5]

The two-dimensional sine-Gordon theory describes a free scalar field $\phi(\boldsymbol{x})$ perturbed by a periodic potential,

$$\beta H[\phi] = \int d^2\boldsymbol{x}\left[\frac{K}{2}(\nabla\phi)^2 + g\cos(\lambda\phi)\right],$$

where $K > 0$.

(a) Treating the periodic potential as a perturbation of the free Gaussian theory, and applying the perturbative momentum shell RG, show that the renormalised Hamiltonian takes the form [2]

$$\beta H'[\phi_<] = -\mathcal{Z}^0_> + \beta H_0[\phi_<] + \int d^2\boldsymbol{x}\, g\langle\cos[\lambda(\phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x}))]\rangle_> + \mathcal{O}(g^2)$$

(b) Working to first order in g, show that, under the RG transformation, the parameters obey the scaling relations [7]

$$\begin{cases} K(b) = Kz^2b^4 \\ g(b) = gb^2\exp\left[-\frac{\lambda^2}{4\pi K}(1-b^{-1})\right] \\ \lambda(b) = \zeta\lambda \end{cases}$$

[*Your discussion should indicate the significance of the parameters z, b and ζ in the RG. For a free Gaussian theory, you may assume the identity $\langle e^{i\lambda\phi(\boldsymbol{x})}\rangle = e^{-\lambda^2\langle\phi^2(\boldsymbol{x})\rangle/2}$.*]

(c) Focusing on the fixed Hamiltonian $K(b) = K$ (i.e. $z = b^{-2}$), it may be confirmed that $\lambda(b) = \lambda$. In this case, setting $b = e^\ell \simeq 1 + \ell + \cdots$, show that the differential recursion relations translate to the form

$$\frac{dg}{d\ell} = g\left(2 - \frac{\lambda^2}{4\pi K}\right).$$

Identify the fixed point and sketch the renormalisation group flow. Comment briefly on the physical implications of the result. [6]

2005 Minor Options Examination Answers

1 The Hamiltonian given in the question represents the canonical form of the Ginzburg-Landau Hamiltonian for a second order phase transition. In the Landau theory, the functional integral for the classical partition function is approximated by its value at the Hamiltonian minimum, viz. [2]

$$\mathcal{Z} \equiv e^{-\beta F} = \int Dm(\boldsymbol{x}) e^{-\beta H[m(\boldsymbol{x})]} \simeq \exp\left[-\min_{m(\boldsymbol{x})} \beta H[m(\boldsymbol{x})]\right].$$

For $K > 0$, the minimal Hamiltonian is given by $m(\boldsymbol{x}) = \bar{m}$, constant. In this approximation, the free energy density is given by $f = \frac{\beta F}{V} = \beta H[\bar{m}]$, where $\bar{m} + 4u\bar{m}^3 - h = 0$. In particular, for $h = 0$, the magnetisation acquires a non-zero expectation value when $t < 0$ with $\bar{m} = \sqrt{-t/4u}$. Similarly, for $t = 0$, the magnetisation varies as $m = (h/3u)^{1/3}$. From this result, one can infer a phase diagram in which a line of first order transitions along $h = 0$ terminates at the critical point $t = 0$. Finally, differentiating the condition on $\bar{m}$ with respect to $\bar{m}$, one obtains the susceptibility [4]

$$\chi(t, h = 0) = \frac{\partial m}{\partial h}\Big|_{h=0} = \begin{cases} 1/t & t > 0 \\ -1/2t & t < 0. \end{cases}$$

[*Full credit will be given even if the specific heat is not derived.*]

(a) In the presence of the strain field, the partition function is given by

$$\mathcal{Z} = \int Dm\, D\phi\, e^{-\beta H[m,\phi]}.$$

Being Gaussian in ϕ, the integral may be performed exactly and obtains

$$\int D\phi e^{-\int d^3\boldsymbol{x}\left[\frac{c}{2}\phi^2 + g\phi m^2\right]} = \int D\phi e^{-\int d^3\boldsymbol{x}\left[\frac{c}{2}(\phi - \frac{gm^2}{c})^2 - \frac{g^2}{2c}m^4\right]} = \text{const.} \times e^{\int d^3\boldsymbol{x}\left[\frac{g^2}{2c}m^4\right]},$$

leading to the suggested reduction in the quartic coefficient. [4]

(b) While the quartic coefficient $u' = u - g^2/2c$ remains positive, the Landau theory continues to predict a second order transition at $h = t = 0$. However, when the sign is reversed, the Landau Hamiltonian ($h = 0$)

$$\psi(m) = \frac{t}{2}m^2 + u'm^4 + vm^6$$

develops additional minima. By sketching $\psi(m)$ for different parameter values, one may see that, for $u' < 0$ and $t = 0$ the (degenerate) global minimum lies at some non-zero value of $\bar{m}$ while, for t large, the global minimum lies a $\bar{m} = 0$. In between, there exists a line of first order transitions which merges with the line of second order critical points at the tricritical point $t = u' = 0$. [*By careful calculation, one may show that the first order boundary follows the line $t = u'^2/2v$.*] [4]

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(c) Near the tricritical point ($u' = 0$ and $h = 0$), one obtains

$$\frac{\partial \psi}{\partial m} = m(t + 6vm^4) = 0, \qquad \bar{m}(t, h = 0) = \begin{cases} 0 & t > 0, \\ (-t/6v)^{1/4} & t < 0, \end{cases}$$

implying an exponent $\beta = 1/4$. Similarly, for $t = 0$, one obtains

$$h = 6v\bar{m}^5, \qquad \bar{m}(t = 0, h) = (h/6v)^{1/5}$$

i.e. $\delta = 5$. Finally, for finite h and t, differentiating the defining equation for $\bar{m}$, one obtains the susceptibility [4]

$$\chi(t, h = 0) = \frac{\partial \bar{m}}{\partial h}\Big|_{h=0} = (t + 30v\bar{m}^4)^{-1},$$

implying that $\chi \sim 1/|t|$ for $t < 0$ and $t > 0$. Thus we find the exponent $\gamma = 1$. [2]

2 Essay question

3 Separating the field fluctuations into fast and slow degrees of freedom, $\phi(\boldsymbol{x}) = \phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x})$,

$$\begin{aligned} \mathcal{Z} &= \int D\phi_< e^{-\beta H_0[\phi_<]} D\phi_> e^{-\beta H[\phi_>] - U[\phi_<, \phi_>]} \\ &= \mathcal{Z}^0_> \int D\phi_< e^{-\beta H_0[\phi_<]} \langle e^{-U[\phi_<, \phi_>]} \rangle_> \\ &= \mathcal{Z}^0_> \int D\phi_< e^{-\beta H_0[\phi_<]} + \ln \langle e^{-U[\phi_<, \phi_>]} \rangle_>. \end{aligned}$$

From this result, one obtains the required renormalised Hamiltonian. [5]

(a) Applying the perturbative expansion,

$$-\ln \langle e^{-U[\phi_<, \phi_>]} \rangle_> \simeq \langle U[\phi_<, \phi_>] \rangle_> + O(U^2).$$

to the sine-Gordon theory, one obtains the required expression for the Hamiltonian. [2]

(b) Integrating over the fast field fluctuations, [2]

$$\begin{aligned} \langle g \cos[\lambda(\phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x}))] \rangle_> &= g\mathrm{Re}\left[e^{i\lambda\phi_<(\boldsymbol{x})} \langle e^{i\lambda\phi_>(\boldsymbol{x})} \rangle_> \right] \\ &= g e^{-\lambda^2 \langle \phi_>^2(\boldsymbol{x}) \rangle / 2} \cos(\lambda \phi_<(\boldsymbol{x})) \end{aligned}$$

Then, making use of the identity, [2]

$$\langle \phi_>^2(\boldsymbol{x}) \rangle = \int_> \frac{d^2\boldsymbol{q}}{(2\pi)^d} \frac{1}{Kq^2} = \frac{1}{2\pi K}(1 - b^{-1})$$

and applying the rescalings, [2]

$$\boldsymbol{q}' = \boldsymbol{q}/b, \qquad \phi'(\boldsymbol{q}') = \phi_<(\boldsymbol{q})/z, \qquad \phi'(\boldsymbol{x}') = \phi_<(\boldsymbol{x})/\zeta,$$

one obtains the renormalised Hamiltonian [1]

$$\beta H'[\phi'] = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2}\frac{K(b)}{2}\boldsymbol{q}'^2|\phi(\boldsymbol{q})|^2 + \int d^2\boldsymbol{x}'\, g(b)\cos\left[\lambda(b)\phi'(\boldsymbol{x}')\right]$$

where the coefficients are as stated.

(c) Using the expansion, [3]

$$\begin{aligned} g(\ell) &= g(0)e^{2\ell}\exp\left[-\frac{\lambda^2}{4\pi K}(1-e^{-\ell})\right] \\ g(0) + \ell\frac{dg}{d\ell} + \cdots &= g(0)\left[1 + 2\ell - \frac{\lambda^2}{4\pi K}\ell + \cdots\right] \end{aligned}$$

one recovers the required differential recursion relation. For $\lambda^2 > 8\pi K$, $g(\ell)$ dimishes under the RG and the system flows towards a free massless theory. Conversely, for $\lambda^2 < 8\pi K$, $g(\ell)$ grows under RG leading to a confined or massive theory. When $\lambda_*^2 = 8\pi K$, the Hamiltonian is fixed and the theory critical. [3]

2006 Minor Options Examination Paper

1 In the restricted solid on solid model, the Hamiltonian of a rough surface is specified by

$$H = K \sum_{\langle \boldsymbol{l}\boldsymbol{m}\rangle} |h_{\boldsymbol{l}} - h_{\boldsymbol{m}}|^{\infty} ,$$

where the discrete coordinates $\boldsymbol{l}$ and $\boldsymbol{m}$ index the sites of a two-dimensional square lattice, and the height variable $h_{\boldsymbol{l}}$ can take positive and negative integer values. Here we have used the notation $\langle \boldsymbol{l}\boldsymbol{m}\rangle$ to indicate that the sum involves only neighbouring sites of the lattice.

(a) Considering βH, show that the height difference between neighbouring sites can only assume values of ± 1 or zero. [3]

(b) As a consequence, show that the $N \times N$ site Hamiltonian may be recast in terms of the $2 \times N \times N$ variables $n_{\boldsymbol{l}\boldsymbol{m}} = h_{\boldsymbol{l}} - h_{\boldsymbol{m}}$ indexing the bonds between neighbouring sites. Explain why the sum of $n_{\boldsymbol{l}\boldsymbol{m}}$ around each square plaquette of the lattice is constrained to be zero, i.e. defining $\hat{\boldsymbol{e}}_x = (1,0)$ and $\hat{\boldsymbol{e}}_y = (0,1)$, [4]

$$n_{\boldsymbol{l},\boldsymbol{l}+\hat{\boldsymbol{e}}_x} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_x,\boldsymbol{l}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y,\boldsymbol{l}+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_y,\boldsymbol{l}} = 0 .$$

(c) Imposing these constraints using the identity $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{\pm in\theta} = \delta_{n,0}$ for integer n, show that the partition function can be written as [6]

$$\mathcal{Z} = \left(\prod_{\boldsymbol{l}} \int_0^{2\pi} \frac{\mathrm{d}\theta_{\boldsymbol{l}}}{2\pi} \right) \exp\left\{ \sum_{\langle \boldsymbol{l}\boldsymbol{m}\rangle} \ln\left[1 + 2\mathrm{e}^{-\beta K} \cos(\theta_{\boldsymbol{l}} - \theta_{\boldsymbol{m}})\right] \right\} .$$

(d) At low temperatures (i.e. $\beta K \gg 1$), show that the system becomes isomorphic to that of the classical two-dimensional XY spin model. Without resorting to detailed calculation, discuss the significance of this correspondence for the phase behaviour of the restricted solid on solid model? [7]

2 Write a detailed essay on **one** of the following topics:

(a) the scaling theory of critical phenomena; [20]

(b) Goldstone modes and the lower critical dimension; [20]

(c) the upper critical dimension and the Ginzburg Criterion. [20]

3 Starting with the Ginzburg-Landau Hamiltonian

$$\beta H = \int \mathrm{d}^d x \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 - hm + \frac{L}{2} (\nabla^2 \phi)^2 + v \nabla m \cdot \nabla \phi \right] ,$$

involving the two one-component fields $m(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$:

(a) recast βH in terms of the Fourier elements $m(\boldsymbol{q}) = \int \mathrm{d}^d x \, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} m(\boldsymbol{x})$ and $\phi(\boldsymbol{q}) = \int \mathrm{d}^d x \, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} \phi(\boldsymbol{x})$. [3]

(b) Construct a renormalisation group transformation by rescaling distances such that $\boldsymbol{q}' = b\boldsymbol{q}$, and the fields such that $m'(\boldsymbol{q}') = m(\boldsymbol{q})/z$ and $\phi'(\boldsymbol{q}') = \phi(\boldsymbol{q})/y$. [8]

(c) At the fixed point $K' = K$ and $L' = L$, obtain the exponents y_t, y_h and y_v. [3]

(d) The singular part of the free energy has a scaling form $f(t, h, v) = t^{2-\alpha} g(h/t^{\Delta}, v/t^{w})$ for t, h and v close to zero. Find α, Δ and w. [3]

(e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point and how do they scale? [3]

2006 Minor Options Examination Answers

1 (a) For $h_i = h_j$ the site energy of a link $\beta H_{ij} = 0$; for $h_i = h_j \pm 1$ $\beta H_{ij} = K$; and $\beta H_{ij} \to \infty$ otherwise. Therefore, the former $n_{ij} = h_i - h_j = 0, \pm 1$ are the only allowed field configurations. [3]

(b) Taking into account the constraint $n_{ij} = 0, \pm 1$, one may note that the sum of n_{ij} around a plaquette $\sum_{\mathrm{ij}\in\mathrm{plaquette}} n_{ij} = 0$. Such a constraint ensures that the sum of n_{ij} around any closed loop must vanish since any loop can be decomposed into a set of elementary plaquettes. [4]

(c) Then, making use of the identity given in the question to impose the constraint, the partition function may be written as [6]

$$\mathcal{Z} = \sum_{n_{ij}=0,\pm 1} e^{-K|n_{ij}|} \left(\prod_{\boldsymbol{i}} \int_0^{2\pi} d\theta_{\boldsymbol{i}} e^{i(n_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{e}}_x} + n_{\boldsymbol{i}+\hat{\boldsymbol{e}}_x,\boldsymbol{i}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{i}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y,\boldsymbol{i}+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{i}+\hat{\boldsymbol{e}}_y,\boldsymbol{i}})\theta_{\boldsymbol{i}}} \right),$$

where the product runs over all lattice sites $\boldsymbol{i}$. Noting that each site $\boldsymbol{i}$ is associated with two bonds along direction $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_y$, the partition function may be rearranged as

$$\begin{aligned}\mathcal{Z} &= \left(\prod_{\boldsymbol{i}} \int_0^{2\pi} d\theta_{\boldsymbol{i}} \right) \left[\sum_{n=0,\pm 1} e^{-K|n|} e^{i(\theta_{\boldsymbol{i}} + \theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_y})n} \right] \left[\sum_{n=0,\pm 1} e^{-K|n|} e^{i(\theta_{\boldsymbol{i}} + \theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_x})n} \right] \\ &= \left(\prod_{\boldsymbol{i}} \int_0^{2\pi} d\theta_{\boldsymbol{i}} \right) \exp\left[\ln(1 + 2e^{-K}\cos(\theta_{\boldsymbol{i}} + \theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_y})) + \ln(1 + 2e^{-K}\cos(\theta_{\boldsymbol{i}} + \theta_{\boldsymbol{i}-\hat{\boldsymbol{e}}_x})) \right].\end{aligned}$$

Finally, setting $\theta_{\boldsymbol{i}} \mapsto -\theta_{\boldsymbol{i}}$ on alternate lattice sites, one obtains

$$\mathcal{Z} = \left(\prod_{\boldsymbol{i}} \int_0^{2\pi} d\theta_{\boldsymbol{i}} \right) \exp\left[\sum_{\langle \boldsymbol{ij} \rangle} \ln(1 + 2e^{-K}\cos(\theta_{\boldsymbol{i}} - \theta_{\boldsymbol{j}})) \right].$$

(d) At low temperatures ($K \gg 1$), the logarithm may be expanded as

$$\mathcal{Z} = \left(\prod_{\boldsymbol{i}} \int_0^{2\pi} d\theta_{\boldsymbol{i}} \right) \exp\left[2e^{-K} \sum_{\langle \boldsymbol{ij} \rangle} \cos(\theta_{\boldsymbol{i}} - \theta_{\boldsymbol{j}}) \right].$$

The latter can be identified as the partition function of a two-dimensional XY model with exchange constant $J = 2e^{-K}$. This correspondence allows us to infer that the proliferation of massless fluctuations of the fields $\theta_{\boldsymbol{i}}$ leads to a disordering of the system for any non-zero temperature, i.e. spatial correlations of the height degrees of freedom allow for divergent fluctuations. However, since the present system lies at the lower critical dimension, one can infer that the restricted solid on solid model exhibits a topological Kosterlitz-Thouless phase transition from a phase with power-law correlations of the order parameter to exponential correlations. [7]

2 Write a detailed essay on **one** of the following topics:

(a) **the scaling theory of critical phenomena**; [20]

In the Landau mean-field theory of critical phenomena, the free energy of a system close to a critical point can be shown to take a homogeneous form. According to the scaling hypothesis, when one goes beyond the mean-field approximation, homogeneity of the singular form of the free energy (and of any other thermodynamic quantity) retains the homogeneous form [5]

$$f_{\text{sing.}}(t,h) = t^{2-\alpha} g_f(h/t^{\Delta}),$$

where the actual exponents α and Δ depend on the critical point being considered. With this Ansatz, one obtains a number of exponent identities:

(i) From the free energy, one obtains the magnetisation as [4]

$$m(t,h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^{\Delta}).$$

In the limit $x \to 0$, $g_m(x)$ is a constant, and $m(t, h=0) \sim t^{2-\alpha-\Delta}$ (i.e. $\beta = 2-\alpha-\Delta$). On the other hand, if $x \to \infty$, $g_m(x) \sim x^p$, and $m(t=0,h) \sim t^{2-\alpha-\Delta}(h/t^{\Delta})^p$. Since this limit is independent of t, we must have $p\Delta = 2-\alpha-\Delta$. Hence $m(t=0,h) \sim h^{(2-\alpha-\Delta)/\Delta}$ (i.e. $\delta = \Delta/(2-\alpha-\Delta) = \Delta/\beta$).

(ii) From the magnetisation, one obtains the susceptibility [3]

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^{\Delta}) \Rightarrow \chi(t,h=0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha\,.$$

(iii) Close to criticality, the correlation length ξ is solely responsible for singular contributions to thermodynamic quantities. Since $\ln \mathcal{Z}(t,h)$ is [5] dimensionless and extensive (i.e. $\propto L^d$), it must take the form

$$\ln \mathcal{Z} = \left(\frac{L}{\xi}\right)^d \times g_s + \left(\frac{L}{a}\right)^d \times g_a\,.$$

where g_s and g_a are non-singular functions of dimensionless parameters (a is an appropriate microscopic length). (A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length. Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as $(L/\xi)^d$. The singular part of the free energy comes from the first term and behaves as

$$f_{\text{sing.}}(t,h) \sim \frac{\ln \mathcal{Z}}{L^d} \sim \xi^{-d} \sim t^{d\nu} g_f(t/h^{\Delta})\,.$$

As a consequence, comparing with the homogeneous expression for the free energy, one obtains the Josephson identity

$$2 - \alpha = d\nu\,.$$

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(iv) Finally, using the correlation function one obtains the susceptibility [3]

$$\begin{aligned}\chi(t,h) &\sim \int d\boldsymbol{x}\ \langle m(\boldsymbol{x})m(0)\rangle \\ &\sim \int_0^\xi dx \frac{x^{d-1}}{x^{d-2+\eta}} \sim \xi^{2-\eta} \\ &\sim t^{-(2-\eta)\nu} g_\xi \left(\frac{h}{t^\Delta}\right).\end{aligned}$$

We thus obtain the exponent identity $\gamma = (2-\eta)\nu$.

(b) **Goldstone modes and the lower critical dimension**; [20]

According to the Ginzburg-Landau theory of phase transitions, the d-dimensional Hamiltonian describing an n-component order parameter takes the form

$$\beta H[\boldsymbol{m}] = \int d^d\boldsymbol{x} \left[\frac{t}{2}\boldsymbol{m}^2 + u\boldsymbol{m}^4 + \frac{K}{2}(\nabla\boldsymbol{m})^2\right].$$

Here, to be concrete, one might think of a ferromagnetic phase involving the magnetisation field $\boldsymbol{m}(\boldsymbol{x})$. [4]

Treated within the Landau mean-field approximation, a saddle-point analysis of the partition function $\mathcal{Z} = \int D\boldsymbol{m} e^{-\beta H}$ predicts a transition to an ordered phase when $t = 0$, i.e. for $n > 1$ and $t < 0$, there is a spontaneous breaking of the rotational symmetry in spin space and the system condenses into a phase with, e.g., $\boldsymbol{m} = (0, \cdots, 1)\bar{m}$. The breaking of this continuous symmetry leads to low-energy (massless) excitations corresponding to a slowly varying rotations of the field in the spin space — Goldstone modes. In the magnetic system the Goldstone modes are known as spin-waves. [4]

Focusing for simplicity on the two-component system, the parameterisation $\boldsymbol{m} = \bar{m}(\cos\theta, \sin\theta)$ leads to a Ginzburg-Landau Hamiltonian of XY spin model type,

$$\beta H[\theta(\boldsymbol{x})] = \beta H_0 + \frac{\bar{K}}{2}\int d^d\boldsymbol{x}\ (\nabla\theta)^2$$

where $\bar{K} = K\bar{m}^2/2$. Superficially quadratic, the multi-valued nature of the transverse field $\theta(\boldsymbol{x})$ makes the evaluation of the partition function problematic. However, at low temperatures, taking the fluctuations of the fields to be small $\theta(\boldsymbol{x}) \ll 2\pi$, the functional integral can be taken as Gaussian. [4]

Applying the rules of Gaussian integration, one obtains $\langle\theta(\boldsymbol{x})\rangle = 0$, and the correlation function takes the form

$$\langle\theta(\boldsymbol{x})\theta(\boldsymbol{x}')\rangle = -\frac{C_d(\boldsymbol{x}-\boldsymbol{x}')}{\bar{K}}, \qquad \nabla^2 C_d(\boldsymbol{x}) = \delta^d(\boldsymbol{x}),$$

where

$$C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + \text{const.},$$

denotes the Coulomb potential for a δ-function charge distribution, and $S_d = 2\pi^{d/2}/(d/2-1)!$ denotes the total d-dimensional solid angle.

Applied to the two-point correlation function,

$$\langle \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(0) \rangle = \bar{m}^2 \text{Re} \left\langle e^{i[\theta(\boldsymbol{x})-\theta(0)]} \right\rangle .$$

one obtains

$$\langle \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(0) \rangle = \bar{m}^2 \exp\left[-\frac{1}{2}\langle[\theta(\boldsymbol{x})-\theta(0)]^2\rangle\right] = \bar{m}^2 \exp\left[-\frac{(x^{2-d}-a^{2-d})}{\bar{K}(2-d)S_d}\right],$$

implying a power-law decay of correlations in $d=2$, and an exponential decay in $d<2$, [4]

$$\lim_{|\boldsymbol{x}|\to\infty} \langle \boldsymbol{m}(\boldsymbol{x}) \cdot \boldsymbol{m}(0) \rangle = \begin{array}{ll} \bar{m}'^2 & d>2\,, \\ 0 & d \le 2\,. \end{array}$$

The saddle-point approximation to the order parameter, $\bar{m}$ was obtained by neglecting fluctuations. The result above demonstrates that the inclusion of phase fluctuations leads to a reduction in the degree of order in $d=2$, and to its complete destruction in $d<2$. This result typifies a more general result known as the Mermin-Wagner Theorem. The theorem states that there is no spontaneous breaking of a continuous symmetry in systems with short-range interactions in dimensions $d \le 2$. The borderline dimensionality of two is known as the lower critical dimension. [4]

(c) **the upper critical dimension and the Ginzburg Criterion**. [20]

According to the Ginzburg-Landau theory of phase transitions, the d-dimensional Hamiltonian describing an one-component order parameter takes the form

$$\beta H[\boldsymbol{m}] = \int d^d\boldsymbol{x} \left[\frac{t}{2}m^2 + um^4 + \frac{K}{2}(\nabla m)^2\right] .$$

Here, to be concrete, one might think of a ferromagnetic phase involving the magnetisation field $m(\boldsymbol{x})$. [4]

According to the Landau mean-field theory, in the saddle-point approximation, the magnetisation field takes the constant value $\bar{m}$ where

$$\bar{m} = \begin{array}{ll} 0 & t>0\,, \\ \sqrt{\frac{-t}{4u}} & t<0\,. \end{array}$$

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The corresponding free energy density takes the form

$$f = -\frac{\ln \mathcal{Z}}{V} = \frac{t}{2}\bar{m}^2 + u\bar{m}^4\,,$$

while the heat capacity is given by [4]

$$C \propto -\frac{\partial^2 f}{\partial t^2} = \begin{matrix} 0 & t > 0\,, \\ \frac{1}{8u} & t < 0\,. \end{matrix}$$

Taking into account fluctuations around the saddle-point solution, viz. $m = \bar{m} + \phi$, an expansion to quadratic order obtains the Hamiltonian

$$\beta H = \beta H[\bar{m}] + \frac{K}{2}\int d^d\boldsymbol{q}(\xi^{-2} + \boldsymbol{q}^2)|\phi(\boldsymbol{q}|^2$$

where $\xi^{-2} \propto |t|/K$ specifies the correlation length. Integrating over fluctuations, one obtains the revised free energy density

$$f = -\frac{\ln \mathcal{Z}}{V} = \frac{t}{2}\bar{m}^2 + u\bar{m}^4 + \frac{1}{2}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d}\;\ln[K(\boldsymbol{q}^2 + \xi^{-2})].$$

Inserting the dependence of the correlation lengths on reduced temperature, the heat capacity on either side of the critical point becomes renormalised by a factor

$$\delta C \propto \int \frac{d^2\boldsymbol{q}}{(2\pi)^d}\;\frac{1}{(Kq^2 + c|t|)^2}$$

for some constant c. [4]

The behaviour of the integral correction changes dramatically at $d = 4$. For $d > 4$ the integral diverges at large $\boldsymbol{q}$ and is dominated by the upper cut-off $\Lambda \approx 1/a$, while for $d < 4$, the integral is convergent in both limits. It can be made dimensionless by rescaling $\boldsymbol{q}$ by ξ^{-1}, and is hence proportional to ξ^{4-d}. Therefore

$$\delta C \simeq \frac{1}{K^2}\begin{matrix} a^{4-d} & d > 4\,, \\ \xi^{4-d} & d < 4\,. \end{matrix}$$

In dimensions $d > 4$ fluctuation corrections to the heat capacity add a constant term to the background on each side of the transition. However, the primary form of the discontinuity in C is unchanged. For $d < 4$, the divergence of $\xi \propto t^{-1/2}$ at the transition leads to a correction term that dominates the original discontinuity. Indeed, the correction term suggests an exponent $\alpha = (4 - d)/2$. But even this is not reliable — a treatment of higher order corrections will lead to yet more severe divergences. In fact the divergence of δC implies that the conclusions drawn from the saddle-point approximation are simply no longer reliable in dimensions $d < 4$. One says that Ginzburg-Landau models which belong to this universality class exhibit an upper critical dimension $d_u = 4$. [4]

The analysis above suggests that fluctuations become important when the correlation length begins to diverge. Within the saddle-point approximation, the correlation length diverges as $\xi \simeq \xi_0 |t|^{-1/2}$, where $\xi_0 \approx \sqrt{K}$ represents the microscopic length scale. The importance of fluctuations can be assessed by comparing the saddle-point discontinuity $\Delta C_{\rm sp} \propto 1/u$, and the correction, δC. Since $K \propto \xi_0^2$, and $\delta C \propto \xi_0^d t^{(4-d)/2}$, fluctuations become important when

$$\left(\frac{\xi_0}{a}\right)^{-d} t^{(d-4)/2} \gg \left(\frac{\Delta C_{\rm sp}}{k_B}\right) \Longrightarrow |t| \ll t_G \approx \frac{1}{[(\xi_0/a)^d(\Delta C_{\rm sp}/k_B)]^{2/(4-d)}}$$

This inequality is known as the Ginzburg Criterion. Naturally, in $d < 4$ it is always satisfied sufficiently close to the critical point. However, the resolution of the experiment may not be good enough to get closer than the Ginzburg reduced temperature t_G. If so, the apparent singularities at reduced temperatures $t > t_G$ may show saddle-point behaviour. [4]

3 (a) Switching to the Fourier basis, the Hamiltonian takes the form [3]

$$\beta H = \int \frac{d^d q}{(2\pi)^d}\left[\frac{1}{2}(t+q^2)|m(\boldsymbol{q}^2)|^2 + \frac{L}{2}q^4|\phi(\boldsymbol{q})|^2 + vq^2 m(\boldsymbol{q})\phi(-\boldsymbol{q})\right] - hm(\boldsymbol{q}=0).$$

(b) Under the renormalisation group, the first step of the procedure is to coarse-grain the Hamiltonian by separating fluctuations into fast and slow degrees of freedom setting [2]

$$m(\boldsymbol{q}) = \begin{cases} m_<(\boldsymbol{q}) & 0 < |\boldsymbol{q}| < \Lambda/b \\ m_>(\boldsymbol{q}) & \Lambda/b < |\boldsymbol{q}| < \Lambda \end{cases}, \qquad \phi(\boldsymbol{q}) = \begin{cases} \phi_<(\boldsymbol{q}) & 0 < |\boldsymbol{q}| < \Lambda/b \\ \phi_>(\boldsymbol{q}) & \Lambda/b < |\boldsymbol{q}| < \Lambda \end{cases}.$$

Quadratic in the fields, the Hamiltonian separates into fast and slow degrees of freedom as $\beta H = \beta H[m_<, \phi_<] + \beta H[m_>, \phi_>]$. Integrating over the fast degrees of freedom, one obtains [2]

$$\mathcal{Z} = \int Dm_< D\phi_< e^{-\beta H[m_<,\phi_<] + \ln \mathcal{Z}_>}$$

where $\mathcal{Z}_> = \int Dm_> D\phi_> e^{-\beta H[m_>,\phi_>]} = \text{const}$. Step 2 of the RG involves the rescaling and step 3 involves the renormalisation as proposed in the question. In this case, one obtains [2]

$$\beta H' = \int \frac{d^d q'}{(2\pi)^d} b^{-d}\left[\frac{1}{2}(t + Kb^{-2}q'^2)z^2|m'(\boldsymbol{q}'^2)|^2 + \frac{L}{2}b^{-4}y^4 q'^4 |\phi'(\boldsymbol{q}')|^2 \right.$$
$$\left. + vb^{-2}zyq^2 m(\boldsymbol{q})\phi(-\boldsymbol{q})\right] - hzm(\boldsymbol{q}=0).$$

Finally, we obtain the RG equations [2]

$$\begin{cases} t' = b^{-d} z^2 t \\ K' = b^{-d-2} z^2 K \\ L' = b^{-d-4} y^2 L \\ v' = b^{-d-2} zyv \\ h' = zh \end{cases}$$

(c) For the fixed point $K' = K$ and $L' = L$, one obtains $z = b^{1+d/2}$ and $y = b^{2+d/2}$. In this case [3]

$$\begin{cases} t' = b^2 t \\ v' = bv \\ h' = b^{1+d/2} h \end{cases}$$

i.e. $y_t = 2$, $y_v = 1$, and $y_h = 1 + d/2$.

(d) Applied to the dimensionless free energy density, [3]

$$f(t, v, h) = \frac{\beta F}{V} = -\frac{\ln \mathcal{Z}}{V} = -\frac{\ln \mathcal{Z}'}{b^d V} = b^{-d} f(b^2 t, bv, b^{1+d/2} h)$$

Setting $b^2 t = 1$, one obtains

$$f(t, v, h) = t^{d/2} f(v/t^{1/2}, h/t^{1/2+d/4})$$

i.e. $2 - \alpha = d/2$, $w = 1/2$ and $\Delta = 1/2 + d/4$.

(e) For $t' = t$ and $L' = L$, we have $z = b^{d/2}$ and $y = b^{2+d/2}$. In this case, one obtains [3]

$$\begin{cases} K' = b^{-2} K v' = v \\ h' = b^{d/2} h \end{cases}$$

Here one can see that K represents an irrelevant operator, h is relevant, and v is marginal with the fixed point $h^* = K^* = 0$ and v^* is arbitrary.