I COLLECTIVE BEHAVIOUR - PARTICLES TO FIELDS

IA INTRODUCTION

microscopic physics $\xrightarrow{\text{Stat. Mech.}}$ Macroscopic (Equilibrium)

Microstates $\{\mu\}$ degrees of freedom ; e.g. $\{\underline{q}, \underline{p}\}$; $\{\sigma_i\}$ $\{n_i\}$ governed by $H[\mu]$

Macrostates M few state functions e.g. P V T S E Laws of Therm.

Stat. Mech. $p_M(\mu)$ e.g. canonical T ; $p_T(\mu) = \dfrac{e^{-H[\mu]/kT}}{Z}$; $Z(T) = \sum_\mu e^{-\beta H[\mu]}$

Non-interacting systems

$$Z = Z_i \ldots Z_j$$

Interacting systems :

(1) New phases of matter ; gas → liquid → solid

superfluids in quantum systems

liquid crystals , magnets , superconductors , . . .

(2) Collective modes : phonons , magnons , . . .

Look for Universal Properties (material independent)

(like central limit theorem)

Collective modes have universal limits $(k, \omega) \to 0$ (Hydrodynamic limit)

Our goal will be to clarify and classify phases and phase transitions in matter

Tools : Classical Fields , Symmetries , RG

IB Phonons and Elasticity

Find $E(T)$ or $C(T)$ of a solid as $T \to 0$

c.f 1st principles vs phenomenological

1st principles Ionic positions $\underline{r}_i$

Classical potential energy $\Phi(\underline{r}_1, \ldots \underline{r}_N)$

Find equilibrium coordinates $\underline{r}_i^{\,0}$ from $\dfrac{\partial \Phi}{\partial r_i} = 0$ which form a regular lattice

Now introduce small deformations $\underline{r}_i = \underline{r}_i^{\,0} + \underline{u}_i$

Have a p.e. of deformation $\Phi = \Phi_0 + \dfrac{1}{2} \dfrac{\partial^2 \Phi}{\partial r_{i\alpha} \partial r_{j\beta}} u_i^{\alpha} u_j^{\beta} + O(u^3)$

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Translational symmetry
$$K_{ij}^{\alpha\beta} \equiv \frac{\partial^2 \Phi}{\partial r_i^{\alpha} r_j^{\beta}}$$
$$= K^{\alpha\beta}(\underline{r}_i^{\circ} - \underline{r}_j^{\circ})$$

Matrix can be partially diagonalised in fourier space

$$u^{\alpha}(q) = \sum_i e^{i\underline{q}\cdot\underline{r}_i^{\circ}} u_i^{\alpha}$$

$$\Phi = \Phi_0 + \sum_{\underline{q}} K^{\alpha\beta}(q)\, u^{\alpha}(q)\, u^{\beta}(q^*) + O(u^3)$$

Hamiltonian
$$H = \sum_{\underline{q}\alpha}\left[\frac{m}{2m} |\dot{u}^{\alpha}(q)|^2 + \frac{1}{2} K^{\alpha\beta}(q)\, u^{\alpha}(q)\, u^{\beta}(q)^* \right] + ..$$

For simplicity, assume $K^{\alpha\beta} = K(q)\,\delta^{\alpha\beta}$

So
$$H = \sum_{\underline{q}\alpha}\left[\frac{m}{2} |\dot{u}^{\alpha}(q)|^2 + \frac{1}{2} K(q) |u^{\alpha}(q)|^2 \right] + ..$$

Quantise set of Harmonic Oscillators

$$H = \sum_{q\,\alpha} \hbar\omega(q)\left(n^{\alpha}(q) + 1/2\right) \quad , \quad \omega(q) = \sqrt{\frac{K(q)}{m}}\,, \quad n^{\alpha}(q) = 0, 1, 2 ..$$

Apply statistical mechanics

Find $\mathcal{Z}$, $E = \langle H \rangle = \sum_{q\,\alpha} \hbar\omega(q)\left(\langle n^{\alpha}(q)\rangle + 1/2\right)$

$$\langle n^{\alpha}(q)\rangle = \frac{1}{e^{\beta\hbar\omega(q)} - 1}$$

Consider $K(q)$ for 1d chain.

$$H = \sum_n \frac{K_1}{2}(u_{n+1} - u_n)^2 + \frac{K_2}{2}\sum_n (u_{n+1} - u_n)^2 + \dots \quad \text{Hookean springs}$$

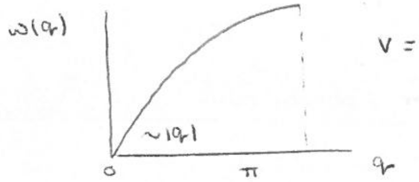
$$u(q) = \sum_n e^{iqn} u(n)$$

$$u_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} u(q)$$

$$H = \sum_n \frac{K_1}{2} \int \frac{dq_1}{2\pi} \int \frac{dq_2}{2\pi} e^{-iq_1 n - iq_2 n} \dots$$

$$= \int \frac{dq}{2\pi} \frac{K_1}{2} (2 - 2\cos q) |u(q)|^2 + \dots$$

So $K(q) = 2K_1(\cos q - 1)$



$$v = \sqrt{\frac{K_1}{m}}$$

Energy
$$E(T) = E_0 + N\int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{\hbar\omega(q)}{e^{\beta\hbar\omega(q)} - 1} \overset{T\to 0}{\approx} E_0 + N\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{\hbar v|q|}{e^{\hbar v|q|/kT} - 1}$$

$$= E_0 + N\hbar v \left(\frac{kT}{\hbar v}\right)^2 \underbrace{\zeta_2}_{\text{a number}} \qquad C(T) = \frac{dE}{dT} \propto Nk\left(\frac{kT}{\hbar v}\right) \quad \text{linear in } T$$

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Note.

(1) Further range interactions do not change result

$$K(q) = K_1(2-2\cos q) + K_2(2-2\cos 2q) + \dots$$

$$\xrightarrow[q\to 0]{} \bar{K} q^2$$

(2) $C(T) \propto T^1$ } power is universal and material independent

(3) Universal result follows from $q \to 0$, $\lambda \gg 1$ modes

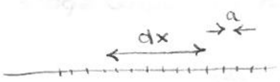
Phenomenological Calculation

At low T energy dominated by the long-wavelength modes

$$\lambda \sim \hbar v / kT \gg a$$

Coarse graining.

Average over interval $a \ll dx \ll \lambda$



Define $u_n \to u(x)$ a slowly varying function

Now find $\Phi(u(x))$ from basic principles

(1) locality : $\Phi[u(x)] = \int dx\, \phi[u(x), \frac{du}{dx}, \dots]$

potential energy density — a local quantity that in principle can depend on higher derivatives of u

(2) Symmetry Since $\Phi[u(x)+c] = \Phi[u(x)]$

So ϕ cannot depend on u, but on $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$

(3) Stability => No 1st order terms (2nd order terms must be positive definite)

Most general Φ, $$\Phi[u] = \int dx \left[\frac{K}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{L}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \dots + M\left(\frac{\partial u}{\partial x}\right)^3 + \dots \right]$$

Fourier space $$\Phi[u(q)] = \int \frac{dq}{2\pi} \left[\frac{K}{2} q^2 |u(q)|^2 + \frac{L}{2} q^4 |u(q)|^2 + \dots \right]$$

Kinetic Energy $= \int dx \left[\frac{\rho}{2} \dot{u}(x)^2 \right]$ $\rho = m/a$

Long Wavelength Hamiltonian

$$H = \int dx \left[\frac{K}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 \right] \qquad \omega(q) = \sqrt{\frac{K}{\rho}}\, |q|$$

Symmetry, locality etc have lead to the correct dispersion relation and the elastic energy of a spring.

Higher dimensions

$\underline{u}_n \to \underline{u}(\underline{x})$

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Symmetries of isotropic solid

Φ invariant under translation and rotations $\quad u^\alpha(x) \longrightarrow u^\alpha(x) + c^\alpha + \underbrace{R^{\alpha\beta}}_{\text{rotation matrix}} u^\beta$

$\therefore$ Φ must depend on derivatives of u, $\frac{\partial u^\alpha}{\partial x^\beta}$,

or more conveniently, a strain tensor $\quad u_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha}\right)$

Most general (quadratic) form

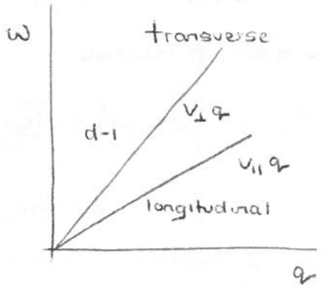
$$\Phi = \int d^d x \left[\frac{2\mu}{2} u_{\alpha\beta} u_{\alpha\beta} + \frac{\lambda}{2} u_{\alpha\alpha} u_{\beta\beta} \right]$$

μ, λ - Lamé coefficients

Invariance under rotation seen in fourier space

$$u^\alpha(q) = \int dx\, e^{i\underline{q}\cdot\underline{x}} u^\alpha(x)$$

$$\Phi = \int \frac{d^d q}{(2\pi)^d} \left[\frac{\mu+\lambda}{2} (\underline{q}\cdot\underline{u})^2 + \frac{\lambda}{2} q^2 |\underline{u}|^2 \right]$$



Normal modes : Longitudinal and transverse

$$\underbrace{q \parallel u}_{K(q) = (2\lambda+\mu) q^2} \qquad \underbrace{q \perp u}_{K(q) = \lambda q^2}$$

$$E(T) = E_0 + N \int \frac{d^d q}{(2\pi)^d} \left[\frac{\hbar v_\parallel |q|}{e^{\beta\hbar v_\parallel |q|} - 1} + \frac{(d-1)\hbar v_\perp |q|}{e^{\beta\hbar v_\perp |q|} - 1} \right]$$

$$= E_0 + N A(v_\parallel, v_\perp, \rho)\, T^{d+1}$$

$$C(T) \propto T^d$$

T dependence is universal

This defines the philosophy of approach for the rest of the course

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Other examples of exponents

- particle dynamics $x \sim t^{1/2}$ diffusion
  $x \sim t$ balistic
  $x \sim t^2$ constant a

Phenomenological Theories

ake few Inputs

Symmetry
Range of interaction
d

$\xrightarrow{\text{Coarse - grain}}$
Non-linear,
perturbation
RG

Output
functional dependencies,
exponents
c.f expts.

Ⓒ PHASE TRANSITIONS

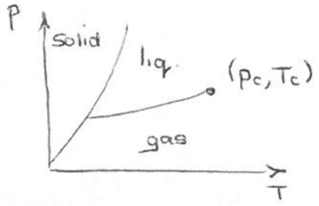
Natural outcome of particle interactions

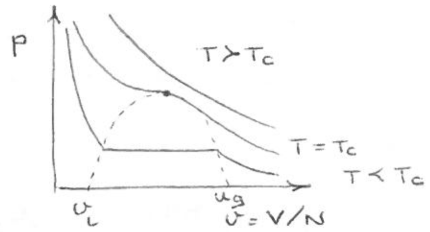
Relies on a thermodynamic limit $(N \to \infty)$ for full transition

$$Z_N = \sum_{\mu} e^{-\beta H}$$

for finite system there can be no singularities

Liquid-Gas condensation





Note

1) Coexistence line terminates at critical point (p_c, T_c)

2) For $T < T_c$; $\rho_L = 1/v_L$, $\rho_g = 1/v_g$ coexist

3) Gas → Liquid can occur without phase transition

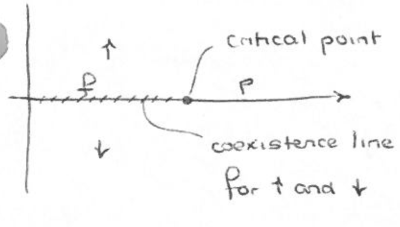
Observations close to critical point

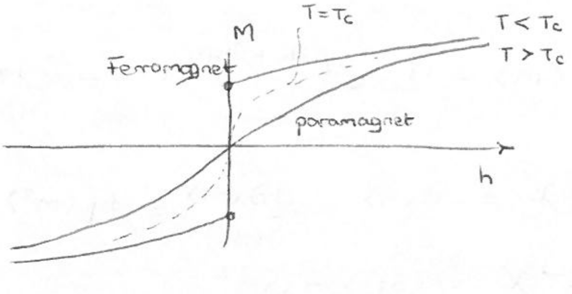
4) As $T \to T_c^-$, $\rho_L \to \rho_g$

5) As $T \to T_c^+$, compressibility $k_T = -\frac{1}{v}\frac{\partial v}{\partial p}\Big|_T$ diverges

6) Critical opalescence, "Milkiness" near transition ⇒ fluctuations at length scales of light which is much larger than interparticle spacing – validates coarse graining.

Ferromagnet / Paramagnet





Similar to liquid-gas but "rotated"

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⑩ CRITICAL BEHAVIOUR

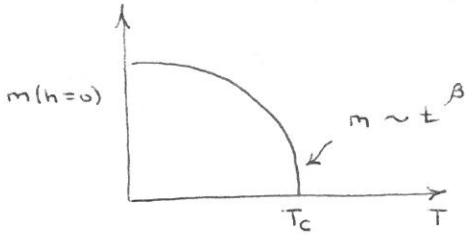
Non-analyticities close to critical point are described by a set of critical exponents

1) Order Parameter.

Distinguishes the phases on a coexistence line (vanishes at critical point)

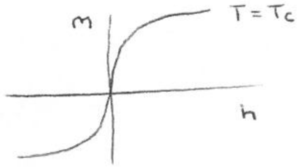
For magnet it is the magnetisation

$$m\,(h=0) = \lim_{h\to 0} \frac{M\,(h,T)}{V}$$



$m \sim t^{\beta}$ where $t = \frac{T_c - T}{T_c}$ is the reduced temperature and β is the critical exponent

On the critical isotherm, m again vanishes

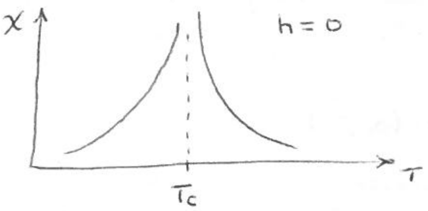


$$m\,(T=T_c) \sim h^{1/\delta}$$

2) Response Function

χ_T, C

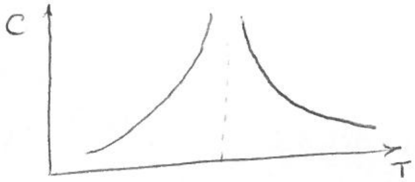
Susceptibility $\chi_T = \frac{\partial m}{\partial h}$



$h = 0$ $\qquad \chi_{\pm}\big|_{h=0} \sim |t|^{-\gamma_{\pm}}$ (frequently $\gamma_+ = \gamma_- = \gamma$)

For liquid-gas, the analagous quantity is the compressibility K_T

Heat Capacity



$C_{\pm} \sim |t|^{-\alpha_{\pm}}$ (Again frequently $\alpha_+ = \alpha_- = \alpha$)

3) Correlation Functions

Diverging response functions => long-range correlations

e.g. Susceptibility

$$Z(h) = \sum_{\mu} e^{-\beta H_0 + \beta h M}$$

$$\langle m \rangle = \frac{\sum_{\mu} e^{-\beta H_0 + \beta h M} M}{Z} = \frac{\partial \ln Z}{\partial(\beta h)}$$

$$\chi = \frac{\partial \langle m \rangle}{\partial h} = \frac{\beta\,\partial \langle m \rangle}{\partial (\beta h)} = \beta\left(\langle m^2 \rangle - \langle m \rangle^2\right)$$

$$kT\chi = \langle m^2 \rangle - \langle m \rangle^2$$

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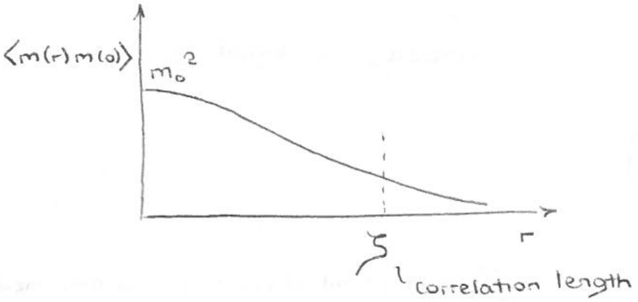
Could write $M = \int d^3r\, m(r)$

So $kT\chi = \int d^3r\, d^3r' \left[\langle m(r) m(r') \rangle - \langle m(r) \rangle \langle m(r') \rangle \right]$

$$\equiv \underbrace{\langle m(r) m(r') \rangle_c}_{\text{cumulant}} \equiv \underbrace{G(r-r')}_{\text{by translational symmetry}}$$

$$kT\chi = V \int d^3r \, \langle m(r) m(0) \rangle_c$$

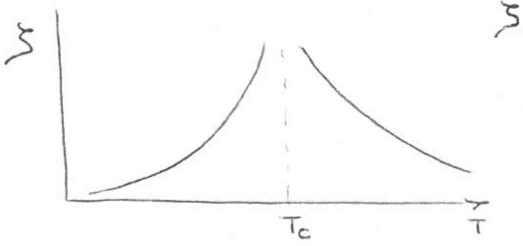


So expect

$$kT\chi \lesssim V m_0^2 \xi^3$$

if $\frac{\chi}{V} \to \infty$ at T_c, expect $\xi \to \infty$ at critical point

$\xi_\pm \sim |t|^{-\nu_\pm}$ – ν is related to γ by a bound



II LANDAU-GINZBURG MODEL

IIA Introduction – focus on magnet; coarse-grained magnetisation field

works well if $\underline{m}$ is sufficiently slowly varying

Generalise to d-dimensions $\underline{x} \equiv (x_1, x_2, \dots x_d)$

and for n spin components $\underline{m} \equiv (m_1, m_2, \dots m_n)$

$n = 3$ isotropic
$n = 2$ planar, superfluid
$n = 1$ uniaxial, liquid-gas

Construct coarse grained Hamiltonian on the basis of

1) locality – $\beta H = \int d^3x \, \Psi[m(\underline{x}), \nabla m, \dots]$ i.e. interactions are sufficiently short-ranged.

2) For $\underline{h} = 0$; have full rotational symmetry

$$\beta H[\underline{m}] = \beta H[\mathbb{R}_n \underline{m}(\underline{x})]$$

3) Translation and Rotational symmetry in $\underline{x}$

$$\text{So} \quad \beta H[\underline{m}(\underline{x})] = \int d^d\underline{x} \left[\frac{t}{2}\, \underline{m}^2(\underline{x}) + a\, \underline{m}^4(\underline{x}) + \dots + \frac{K}{2} (\nabla \underline{m})^2 + \frac{L}{2} (\nabla^2 \underline{m})^2 + \underline{m}^2 (\nabla \underline{m})^2 + \dots - \underline{h}\cdot\underline{m} \right]$$

defines the Landau-Ginzburg Hamiltonian

generally add small $\underline{h}$

Partition Function

$$\mathcal{Z}[h] = \int \mathcal{D}\underline{m}(\underline{x}) \exp\left[-\beta H[\underline{m}(\underline{x})] \right]$$

functional integral

That this limit can be taken is assumed from the physical discreteness of the system

$$\int \mathcal{D}\underline{m}(\underline{x})\, f[\underline{m}(\underline{x}), \nabla \underline{m}, \dots] \equiv \lim_{\substack{a \to 0 \\ N \to \infty}} \prod_{i=1}^{N} d\underline{m}_i\, f\left(\underline{m}_i, \frac{\underline{m}_{i+1} - \underline{m}_i}{a}, \dots\right)$$

Note: $t, a, K \dots$ are functions of original microscopic parameters and of T

IIB Saddle-Point Approximation (Mean-field Theory)

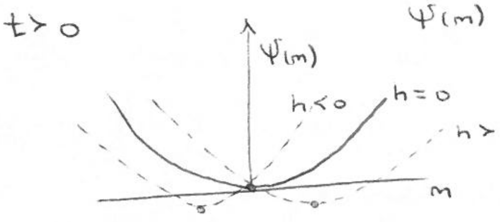
$$\mathcal{Z}[h] = \int \mathcal{D}\underline{m}(\underline{x})\, e^{-\beta H[\underline{m}]}$$

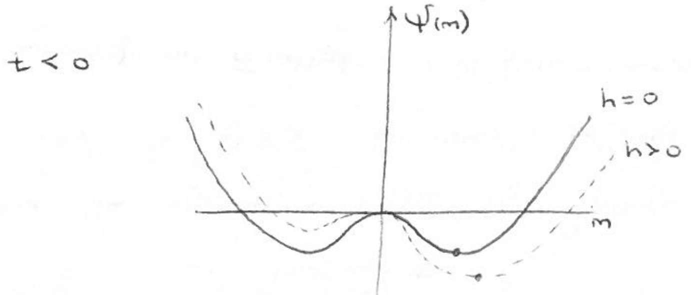
$$\sim \text{max. of integrand} = e^{-\beta F}$$

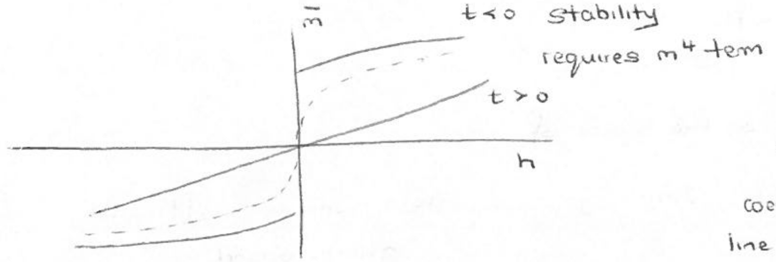
$$\beta F = \min_{\underline{m}} \left[\beta H[\underline{m}] \right]$$

(For $K > 0$, maximum occurs for uniform $\underline{m}(\underline{x}) = \underline{m}$) $\equiv \bar{m}\,\hat{\underline{h}}$

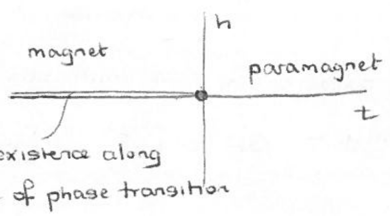
$$\beta F = \min_{\underline{m}} \left[\underbrace{\frac{t}{2} m^2 + u m^4 - h.m}_{\psi(m)} \right] . V$$







stability requires m^4 term



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Must have $t(T,\ldots) = a(T-T_c) + O(T-T_c)^2$

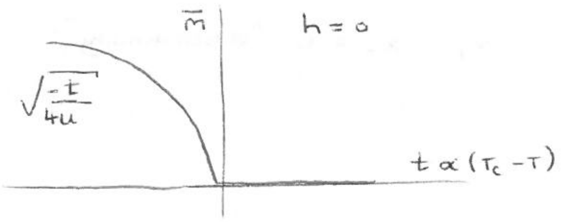
$u(T,\ldots) = u_0 + a_1(T-T_c) + \ldots$ } Taylor expansion of coefficients about T_c

$a, \overset{K_0}{u_0} > 0$

$K(T,\ldots) = K_0 + K_1(T-T_c) + \ldots$

Expect to analytic functions - they derive from a course grained model where within each "grain" singularities can not occur

$\bar{m}$ from $\frac{\partial \Psi}{\partial m} = 0 = t\bar{m} + 4u\bar{m}^3 - h = 0$



$\Rightarrow \beta = 1/2$

At $T = T_c$ $(t=0)$ $\bar{m} = \left(\frac{h}{4u}\right)^{1/3} \sim h^{1/6}$

$\Rightarrow \delta = 3$

a) Susceptibility – magnetic Response function

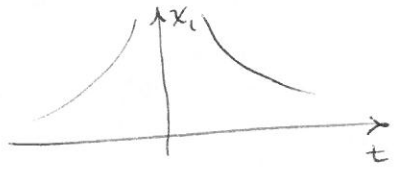
(really vector quantity) $\chi_L = \frac{\partial \bar{m}}{\partial h}$, $\chi_L^{-1} = \frac{\partial h}{\partial \bar{m}} = t + 12u\bar{m}^2$

longitudinal

$$= \begin{cases} t & t > 0 \\ -2t & t < 0 \end{cases}$$

Susceptibility measures variance of magnetisation and so must be positive

$$\chi_L^{\pm} = A_{\pm}|t|^{-\gamma_{\pm}} \qquad \gamma_+ = \gamma_- = 1$$



Exponents are parameter independent, but amplitudes are non-universal.

However $A_+/A_- = 2$ which is universal

2b) Heat capacity – thermal Response function

$$C \propto \frac{\text{var}(E)}{V} = \frac{1}{V}\frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$\left.\frac{\beta F}{V}\right|_{h=0} = \min_m [\Psi(m)]\Big|_{h=0} = \begin{cases} 0 & t > 0 \\ -\frac{t^2}{16u} & t < 0 \end{cases}$$

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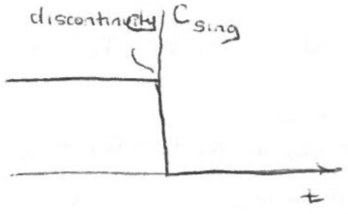
$$\frac{\partial}{\partial \beta} = -kT^2 \frac{\partial}{\partial T} \approx -akT_c^2 \frac{\partial}{\partial t}$$

$$C_{sing} \; \alpha - \frac{\partial^2 (\beta F)}{\partial t^2} = \begin{cases} 0 & t > 0 \\ 1/8u & t < 0 \end{cases}$$



$$C_\pm = A_\pm |t|^{-\alpha_\pm}$$

$\alpha_+ = \alpha_- = 0$ "discontinuity"

But notice, at the level of coarse graining the analytical constants, C(T, ...) were crossed out. This background can make the heat capacity non-zero

IIc Continuous Symmetry Breaking and Goldstone Modes

For h = 0, βH has full rotational symmetry – but ground state is feromagnetic for $T < T_c$ and spontaneously breaks rotational symmetry and Long-range order develops.

A uniform rotation costs no energy but requires motion of whole system

A slowly varying rotation costs very little energy – corresponds to low energy excitation

Breaking of a continuous symmetry => low energy excitations (Goldstone modes)

e.g. Magnet (Spin Waves)
Solid (Phonons)

n = 1 magnet is a discrete symmetry – no low energy excitations

n = 2 (Superfluid) $m_1 + im_2 = \psi(x) = |\psi| e^{i\theta}$

$$\beta H = \int d^d\underline{x} \left[\frac{K}{2} |\nabla \psi|^2 + \frac{t}{2} |\psi|^2 + u |\psi|^4 \right]$$

Applying saddle-point method – get minimum on the circumference of a circle.
i.e. minimum fixes amplitude but not phase. $\psi(x) = \bar{\psi} e^{i\theta}$

Distortions at fixed $\bar{\psi}$

$$\beta H[\theta(x)] = \beta H_0 + \frac{K\bar{\psi}^2}{2} \int d^d\underline{x} \, (\nabla \theta)^2$$

$$\theta(x) = \sum_q e^{iqx} \theta_q$$

$$\beta H[\theta_q] = \beta H_0 + \frac{K\bar{\psi}^2}{2} \sum_q q^2 \theta_q^2$$

$$P[\theta(x)] \propto \exp\left[-\bar{K} \int d^d\underline{x} \, (\nabla\theta)^2 \right] \quad , \quad \bar{K} = \frac{K\bar{\psi}^2}{2}$$

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$$\underline{P}[\theta_q] \;\propto\; \exp\left[-\frac{\bar{K}}{2}\sum_q q^2|\theta_q|^2\right] \equiv \prod_q e^{-\frac{\bar{K}}{2}q^2|\theta_q|^2}$$

So $\{\theta_q\}$ are independent, Gaussian random variables.

$$\langle\theta_q\rangle = 0 \;, \quad \langle|\theta_q|^2\rangle = \frac{1}{\bar{K} q^2} \;, \quad \langle\theta_q\theta_{q'}\rangle = \frac{\delta_{q+q',0}}{\bar{K} q^2}$$ since θ_q's uncorrelated

Real-Space Correlation functions

$$\langle\theta(x)\rangle = 0 \;, \quad \langle\theta(x)\theta(x')\rangle = \sum_{qq'} e^{i(q.x+q'x')}\langle\theta_q\theta_{q'}\rangle$$

$$= \sum_q e^{iq.(x-x')}\frac{1}{\bar{K}q^2}$$

$$= \int\frac{d^dq}{(2\pi)^d}\frac{e^{iq.(x-x')}}{\bar{K}q^2}$$

$$= -\frac{C_d(x-x')}{\bar{K}}$$

where $$C_d(x) = -\int\frac{d^dq}{(2\pi)^d}\frac{e^{iq.x}}{q^2}$$

$$\nabla^2 C_d(x) = \int\frac{d^dq}{(2\pi)^d}e^{iq.x} = \delta^d(x)$$

Coulomb potential for δ function charge distribution

Use Gauss' law

$$\int d^dx\, \nabla^2 C_d(x) = \oint dS.\nabla C$$

$$1 = \frac{dC}{dx}S_d x^{d-1}$$ S_d = d-dim. solid angle = $\frac{2\pi^{d/2}}{(d/2-1)!}$

$$\frac{dC}{dx} = \frac{1}{S_d}x^{1-d}$$

$$C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + c$$

$$\langle[\theta(x)-\theta(x')]^2\rangle = 2[\langle\theta(x)^2\rangle^2 - \langle\theta(x)\theta(x')\rangle]$$

$$= \frac{2}{\bar{K}}\frac{x^{2-d}-a^{2-d}}{S_d(2-d)}$$ where 'a' is some minimum lattice spacing

Long-distance correlations change significantly at $d=2$

$$\langle[\theta(x)-\theta(x')]^2\rangle \underset{|x-x'|\ \text{large}}{=} \begin{cases} \text{constant} & d>2 \\ \text{logarithmic divergence} & d=2 \\ \text{diverges} & d<2 \end{cases}$$

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At level of mean-field theory LRO exists. However in low dimensions, fluctuations destroy the long-range order.

Recall $\Psi(x) = \bar{\Psi}\, e^{i\theta(x)}$

$$\langle \underline{m}(x) . \underline{m}(0) \rangle \propto \langle \Psi(x)\Psi(0)\rangle = \bar{\Psi}^2 \langle e^{i[\theta(x)-\theta(x')]}\rangle$$

For Gaussian variables

$$\langle e^{\alpha\theta}\rangle = e^{\frac{\alpha^2}{2}\langle\theta^2\rangle}$$

So $\langle\Psi(x)\Psi(0)\rangle = \bar{\Psi}^2 \exp\left[-\frac{1}{2}\langle(\theta(x)-\theta(x'))^2\rangle\right]$

$$= \begin{cases} \bar{\Psi}^2 e^{-a^{2-d}/\bar{K}S_d(2-d)} & d>2 \\ \bar{\Psi}^2 e^{-x^{2-d}/\bar{K}S_d(2-d)} \to \infty & d<2 \text{ and in } d=2 \end{cases}$$

Mermin-Wagner Theorem

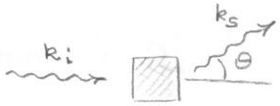
For systems with continuous symmetry there is no LRO for $d \leq 2$

$d=2$ is the lower critical dimension

Actually, lower critical dimension for discrete symmetry is $d=1$

IID FLUCTUATIONS AND SCATTERING

- Probe fluctuations by scattering



$k_s = k_i + q$

- Amplitude $A(q) \propto \sum_\alpha \langle k_s \otimes f | U_\alpha | k_i \otimes i \rangle$

$$\propto \underbrace{\sigma(q)}_{\text{Form factor (local)}} \int d^dx\, e^{iq.x} \rho(x)$$

└ Density of scatterers

- Intensity $S(q) \propto \langle |A(q)|^2 \rangle$

average which if ergodic has a time average = to thermal average

$$\propto \langle |\rho(q)|^2\rangle$$

$$\rho = \begin{cases} \text{atomic density} & \text{(light)} \\ \text{charge density} & \text{(electron)} \\ \text{magnetic density} & \text{(neutron)} \end{cases}$$

Mostly interested in small q, long wavelength correlations

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Fluctuations from the L.G. model

$$\mathcal{P}[\underline{m}(\underline{x})] \propto \exp\left[-\int d^dx\left[\frac{K}{2}(\nabla m)^2+\frac{t}{2}m^2+um^4\right]\ \right]$$

Most probable state $\quad \underline{m}(\underline{x}) = \bar{m}\,\hat{\underline{e}}_1 \qquad \bar{m} = \begin{cases} 0 & t>0 \\ \sqrt{\dfrac{-t}{4u}} & t<0 \end{cases}$

If no fluctuations, probes → δ fn correlations.

Include fluctuations as $\quad \underline{m}(\underline{x}) = [\bar{m} + \underbrace{\phi_L(x)}_{\text{longitudinal}}]\hat{\underline{e}}_1 + \sum_{\alpha=2}^{n} \underbrace{\phi_t^\alpha(x)}_{\text{transverse}}\,\hat{\underline{e}}^\alpha$

Expand to second order

$$\begin{cases} (\nabla m)^2 = (\nabla\phi_L)^2 + (\nabla\overset{n-1\text{ component vector}}{\phi_t})^2 \\ \underline{m}^2 = \bar{m}^2 + 2\bar{m}\phi_L + \phi_L^2 + \phi_t^2 \\ \underline{m}^4 = \bar{m}^4 + 4\bar{m}^3\phi_L + 6\bar{m}^2\phi_L^2 + 2\bar{m}^2\phi_t^2 + O(\phi^3,\phi^4) \end{cases}$$

$$\beta H = V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \text{linear terms in } \phi_L \text{ which vanish}$$
$$+ \int d^dx\left[\frac{K}{2}(\nabla\phi_L)^2 + \frac{t+12u\bar{m}^2}{2}\phi_L^2\right] + \int d^dx\left[\frac{K}{2}(\nabla\phi_t)^2 + \frac{t+4u\bar{m}^2}{2}\phi_t^2\right] + O(\phi^3)$$

On dimensional grounds there are two length scales ; ξ_L, ξ_t

$$\frac{K}{\xi_L^2} \triangleq t + 12u\bar{m}^2 = \frac{\partial^2\Psi}{\partial\phi_L^2} = \begin{cases} t & t>0 \\ -2t & t<0 \end{cases}$$

$$\frac{K}{\xi_t^2} \triangleq t + 4u\bar{m}^2 = \frac{\partial^2\Psi}{\partial\phi_t^2} = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$

$$\mathcal{P}[\phi_L(x),\phi_t(x)] \propto \exp\left\{-\frac{K}{2}\int d^dx\left[(\nabla\phi_L)^2 + \xi_L^{-2}\phi_L^2\right]\right\}\exp\left\{-\frac{K}{2}\int d^dx\left[(\nabla\phi_t)^2 + \xi_t^{-2}\phi_t^2\right]\right\}$$

Note (1) For $t>0$, ϕ_t, ϕ_L are interchangable

(2) For $t<0$, No restoring force for transverse fluctuations (Goldstone Modes)

(3) At this order, ϕ_L and ϕ_t are independent modes. Diagonalised by fourier transform

$$\mathcal{P}[\phi_L(q),\phi_t(q)] \propto \prod_q \exp\left[-\frac{K}{2}(q^2+\xi_L^{-2})|\phi_L(q)|^2\right]\exp\left[-\frac{K}{2}(q^2+\xi_t^{-2})|\phi_t(q)|^2\right]$$

Since ϕ's are Gaussian random variables we can immediately write

* $$\langle\phi_\alpha(q)\phi_\beta(q')\rangle = \delta_{\alpha\beta}\,\delta_{q+q',0}\,\frac{1}{K(q^2+\xi_\alpha^{-2})}$$

* $m(q) = \bar{m}\,\delta(q) + \phi(q)$

$A(q) \propto \mathrm{Im}(q)$

$S(q) \propto \langle|m(q)|^2\rangle$

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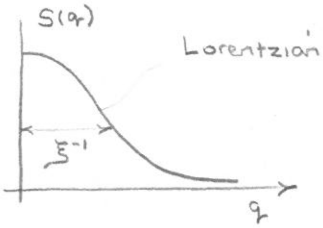
$t > 0$

$S(q)$?

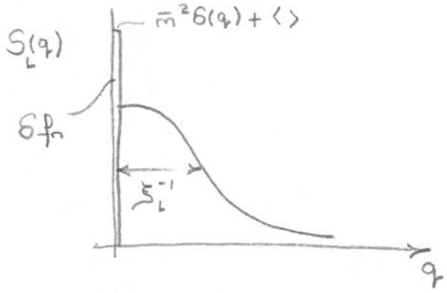


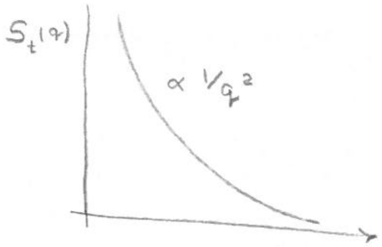
$\xi^{-1} = \sqrt{\frac{t}{K}}$ — narrows as transition approached

$t < 0$

Now depends on L or t modes

Can be selected by polarised neutrons





At $t = 0$, $S(q) \propto 1/q^2$ — origin of critical opalescence

In fact $S(q)\Big|_{T=T_c} \propto \frac{1}{q^{2-\eta}}$ $\eta = 0$ in the Gaussian approximation

II E CORRELATION FUNCTIONS AND SUSCEPTIBILITIES

$$G_c(x,x')_{\alpha\beta} = \langle (m(x)_\alpha - \bar{m}_\alpha)(m(x')_\beta - \bar{m}_\beta)\rangle$$

$$= \langle \phi_\alpha(x)\phi_\beta(x')\rangle$$

$$= \sum_{q,q'} e^{iq.x + iq'.x'} \langle \phi_\alpha(q)\phi_\beta(q')\rangle$$

$$= \delta_{\alpha\beta} \sum_q e^{iq(x-x')} \frac{1}{K(q^2 + \xi_\alpha^{-2})}$$

$$= \delta_{\alpha\beta} \int \frac{d^dq}{(2\pi)^d} \frac{e^{iq(x-x')}}{K(q^2 + \xi_\alpha^{-2})}$$

$$= \delta_{\alpha\beta} \frac{I_d}{K}(x-x')$$

where $I_d(x) = \int \frac{d^dq}{(2\pi)^d} \frac{e^{iqx}}{q^2 + \xi_\alpha^{-2}}$

$$\nabla^2 I_d(x) = -\int \frac{d^dq}{(2\pi)^d} \frac{(q^2 + \xi^{-2} - \xi^{-2})}{q^2 + \xi_\alpha^{-2}} e^{iqx}$$

$$= -\delta(x) + \frac{I_d}{\xi^2}$$

can show using Gauss' law

For spherically symmetric $I_d(x)$, $\nabla^2 I(x) = \frac{d^2I}{dx^2} + \frac{d-1}{x}\frac{dI}{dx} = -\delta(x) + \frac{I(x)}{\xi^2}$

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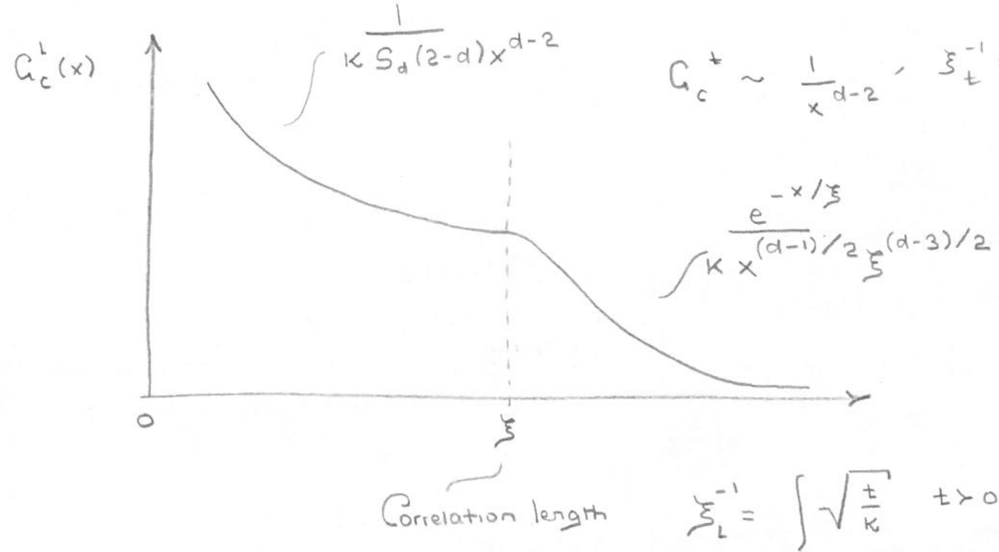
Try $I_d(x) = \frac{e^{-x/\xi}}{x^p}$

$$I_d' = -\left(\frac{p}{x^{p+1}} + \frac{1}{\xi x^p}\right)e^{-x/\xi}, \quad I_d'' = \left(\frac{p(p+1)}{x^{p+2}} + \frac{2p}{\xi x^{p+1}} + \frac{1}{\xi^2 x^p}\right)e^{-x/\xi}$$

for $x \neq 0$, $\frac{p(p+1)}{x^{p+2}} + \frac{2p}{\xi x^{p+1}} + \frac{1}{\xi^2 x^p} - \frac{p(d-1)}{x^{p+2}} - \frac{d-1}{\xi x^{p+1}} = \frac{1}{\xi^2 x^p}$

For $x \ll \xi$ $\quad \frac{1}{x^{p+2}} \gg \frac{1}{x^{p+1}\xi}$, $\quad p+1 = d-1$
$p = d-2$ (c.f coulomb potential in d dim.)

$x \gg \xi$ $\quad \frac{1}{x^{p+2}} \ll \frac{1}{x^{p+1}\xi}$, $\quad 2p = d-1$
$p = \frac{d-1}{2}$



$G_c^t \sim \frac{1}{x^{d-2}}$, $\xi_t^{-1} = 0$

$$\xi_L^{-1} = \begin{cases} \sqrt{\frac{t}{K}} & t > 0 \\ \sqrt{\frac{-2t}{K}} & t < 0 \end{cases}$$

$\xi_{L_\pm} = A_\pm |t|^{-\nu_\pm}$, $\nu_\pm = 1/2$, $A_+/A_- = \sqrt{2}$
As before

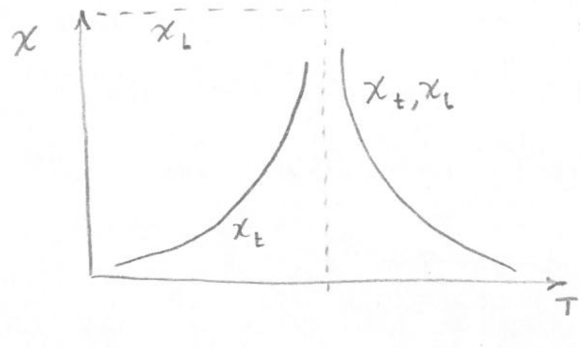
Susceptibility

$$\chi_L \propto \int d^dx\, G_c^L(x) \propto \int_0^{\xi_L} d^dx \frac{1}{x^{d-2}} \propto \xi_L^2 = B_\pm |t|^{-1}$$

$\gamma_+ = \gamma_- = 1$ As before
$B_+/B_- = (A_+/A_-)^2 = 2$

$$\chi_t \propto \int d^dx\, G_c^t(x) \propto \int^L d^dx \frac{1}{x^{d-2}} \propto L^2 \to \infty$$

Transverse Susceptibility measures response to a magnetic field perpendicular to the direction of magnetisation. But there is no restoring force in this direction.



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$$G_c(T=T_c, x) \propto \frac{1}{x^{d-2}}$$

In general $\propto \frac{1}{x^{d-2+\eta}}$ where $\eta = 0$ in Gaussian approximation

IIF COMPARISON WITH EXPERIMENT

	α	β	γ	ν
MFT	0	1/2	1	1/2
Ferromagnet Fe/Ni ($n=3$)	-0.1	0.3	1.3	
Superfluid, He^4 ($n=2$)	0		1.3	0.7
Liquid / Gas Co$_2$ ($n=1$)	0.1	0.4	1.2	0.6
Ferroelectric Superconductor	0	1/2	1	1/2

Actual exponents depend on both n and d

MF exponents are too universal

IIG FLUCTUATION CORRECTIONS TO SADDLE POINT

- 1 integral

$$I = \int dm\, e^{-Nf(m)} \simeq \int dm\, e^{-N[f(\bar{m}) + (m-\bar{m})^2 \underset{>0}{f''(m)}/2]}$$

$$\simeq e^{-Nf(\bar{m})} \sqrt{\frac{2\pi}{Nf''(\bar{m})}} = e^{-N[f(\bar{m}) + O(\frac{\ln N}{N})]}$$

- Functional Integral $\mathcal{Z} = \int \mathcal{D}\underline{m}(x) \exp\left[-\int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4\right]\right]$

Maximum of integrand $\bar{m}(x) = \bar{m}\,\hat{e}_1$ where $\bar{m} = \begin{cases} 0 & t > 0 \\ \sqrt{-t/4u} & t < 0 \end{cases}$

Fluctuations $\bar{m}(x) = [\bar{m} + \phi_L(x)]\hat{e}_1 + \sum \phi_t^\alpha(x)\,\hat{e}_\alpha$

$$\mathcal{Z} \simeq e^{-V(t\bar{m}^2/2 + u\bar{m}^4)} \int \mathcal{D}\phi_L \mathcal{D}\phi_t\, e^{-\frac{K}{2}\int d^dx\left[(\nabla\phi_L)^2 + \frac{\phi_L^2}{\xi_L^2}\right]} e^{-\frac{K}{2}\int d^dx\left[(\nabla\phi_t)^2 + \frac{\phi_t^2}{\xi_t^2}\right]}$$

Recall $\langle \Phi_\alpha(x)\Phi_\beta(x')\rangle = \frac{\delta_{\alpha\beta}}{K} I_d(x')\,\delta(x'-(x-x'))$ because of decoupling

Mathematical digression on Gaussian integrals

- 1 integral

$$\int_{-\infty}^{\infty} d\phi \exp\left[-\frac{K}{2}\phi^2 + a\phi\right] = \sqrt{\frac{2\pi}{K}}\, e^{a^2/2K}$$

$$\frac{d}{da}: \int_{-\infty}^{\infty} d\phi\; \phi \exp\left[-\frac{K}{2}\phi^2 + a\phi\right] = \sqrt{\frac{2\pi}{K}}\,\frac{a}{K}\, e^{a^2/2K} \quad ; \quad \langle\Phi\rangle = a/K$$

$$\frac{d^2}{da^2}: \int_{-\infty}^{\infty} d\phi\; \phi^2 \exp\left[-\frac{K}{2}\phi^2 + a\phi\right] = \left[\left(\frac{a}{K}\right)^2 + \frac{1}{K}\right] e^{a^2/2K}\sqrt{\frac{2\pi}{K}} \quad ; \quad \langle\phi^2\rangle_c = 1/K$$

$\langle\Phi^3\rangle_c \ldots = 0$ — property of Gaussians

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- Many Variables

$$I = \int \prod_{i=1}^{N} d\phi_i \; e^{-\sum_{ij}\frac{K_{ij}}{2}\phi_i\phi_j + \sum_i a_i\Phi_i}$$

Diagonalise K $\quad K_{ij}\bar{\Phi}_q^{\,j} = K_q\bar{\Phi}_q^{\,i}$

Change variables : $\quad I = \int \prod_q d\phi_q \, e^{-\sum_q\left(\frac{K_q}{2}\bar{\Phi}_q^2 - a_q\cdot\bar{\Phi}_q\right)}$

$$= \prod_q \left\{\sqrt{\frac{2\pi}{K_q}}\; e^{a_q^2/2K_q}\right\} = \frac{(2\pi)^{N/2}}{\sqrt{\prod_q K_q}}\; e^{\sum_q a_q K_q^{-1} a_q/2}$$

$$= \frac{(2\pi)^{N/2}}{\sqrt{\det K}}\; e^{a_i K_{ij}^{-1} a_j/2}$$

$\frac{d}{da_i}$: $\quad \langle\Phi_i\rangle = K_{ij}^{-1}a_j$

$\frac{d^2}{da_i da_j}$: $\quad \langle\Phi_i\Phi_j\rangle_c = K_{ij}^{-1}$, Similarly higher order cumulants vanish

If $\mathcal{P}[\phi_i] \propto e^{-K_{ij}\phi_i\phi_j/2}$; $\quad \langle e^{a_i\phi_i}\rangle = e^{a_iK_{ij}^{-1}a_j/2}$

$$= e^{a_ia_j\langle\Phi_i\Phi_j\rangle/2}$$

$$= e^{\langle(\sum a_i\Phi_i)^2\rangle/2}$$

- Functional Integrals

$$Z = \int \mathcal{D}\phi(x)\exp\left\{-\frac{1}{2}\int d^dx\,d^dy\;\Phi(x)\Phi(y)\,K(x,y) + \int d^dx\; a(x)\,\Phi(x)\right\}$$

$$= \frac{(2\pi)^{\text{"}N/2\text{"}}}{\sqrt{\text{"}\det K\text{"}}}\exp\left\{\frac{1}{2}\int d^dx\,d^dy\; a(x)a(y)\,K^{-1}(x,y)\right\}$$

where $\int d^dy\; K(x,y)K^{-1}(y,z) = \delta^d(x-z)$ $\quad$ Integral equation defines inverse kernel

$$\langle\Phi(x)\Phi(y)\rangle = K^{-1}(x,y)$$

For LG model

$$\frac{K}{2}\int d^dx\left[(\nabla\Phi)^2 + \frac{\Phi^2}{\xi^2}\right] = \frac{K}{2}\int d^dx\,d^dy\;\Phi(x)\overset{\delta(x-y)}{\left\{-\nabla^2 + \frac{1}{\xi^2}\right\}}\Phi(y)$$

(by integration by parts)

So $\quad K(x,y) = K\delta(x-y)\left[-\nabla^2 + \frac{1}{\xi^2}\right]$

$$\int d^dy\left\{K\,\delta^d(x-y)\left[-\nabla^2+\frac{1}{\xi^2}\right]K^{-1}(y,z)\right. = \delta^d(x-z)$$

$$K\left(-\nabla^2 + \frac{1}{\xi^2}\right)K^{-1}(x,z) = \delta^d(x-z)$$

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$$K^{-1}(x,z) = \frac{I_d}{K}(x-z) \qquad \text{where} \qquad \nabla^2 I_d = \frac{I_d}{\xi^2} - \delta^d(x)$$

$$\equiv \langle \phi(x)\phi(z) \rangle$$

as before.

K is diagonalised by fourier transforms

$$\bar{\Phi}_q = \int d^dx\, e^{-iqx}\phi(x)$$

$$K(q) = K(q^2+\xi^{-2})$$

$$\det K(q) = \prod K(q)$$

$$\ln\det K(q) = \sum_q \ln\left[K(q^2+\xi^{-2})\right]$$

$$= V\int \frac{d^dq}{(2\pi)^d} \ln\left[K(q^2+\xi^{-2})\right]$$

So recall the problem

$$Z \simeq e^{-V(t\bar{m}^2/2 + u\bar{m}^4)} \int \mathcal{D}\phi_L \mathcal{D}\phi_t\, e^{-\frac{K}{2}\int d^dx\left[(\nabla\phi_L)^2 + \frac{\phi_L^2}{\xi_L^2}\right]} e^{-\frac{K}{2}\int d^dx\left[(\nabla\phi_t)^2 + \frac{\phi_t^2}{\xi_t^2}\right]}$$

$$\ln Z = -V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) - \frac{1}{2}V\int \frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi_L^{-2})\right] - \frac{(n-1)V}{2}\int\frac{d^dq}{(2\pi)^d}\ln\left[K(q^2+\xi_t^{-2})\right]$$

$$+ \text{"}\frac{N}{2}\ln 2\pi\text{"}$$

B. free energy $f = -\frac{\ln Z}{V}$

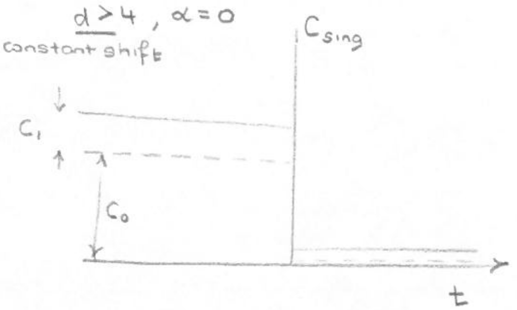
- All contributions are order V. So how much do the corrections affect the critical behaviour?

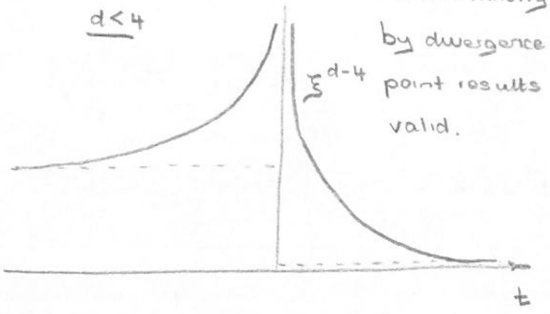
$$C_{sing} \sim -\frac{\partial^2 f}{\partial t^2} = C_0 + C_1 = \begin{cases} 0 + \frac{n}{2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2+t)^2} & t>0 \\ \frac{1}{8u} + 2\int\frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2-2t)^2} & t<0 \end{cases}$$

from saddle point corrections

$$\text{recall} \quad \xi_{l,t}^{-2} \sim \begin{cases} t,\, t & t>0 \\ -2t,\, 0 & t<0 \end{cases}$$

$$C_1 \sim \frac{1}{K^2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+\xi^{-2})^2} \sim \begin{cases} \frac{1}{K^2}a^{4-d} & d>4 \\ \frac{1}{K^2}\xi^{4-d} & d<4 \end{cases}$$





discontinuity replaced by divergence — saddle point results are not valid.

Result holds for susceptibility, magnetisation, etc.

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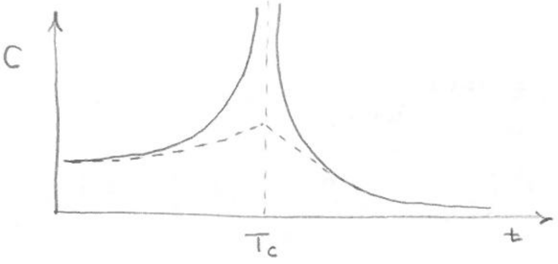
However we cannot abstract α because what we have discovered is that the first order correction dominates the zeroth order in the vicinity of the transition but we do not trust the convergence of the perturbation theory.

II H Ginzburg Criterion

Why are saddle point exponents valid for superconductor?

Resolution



Suppose experiments average over a range Δt – might discover dotted curve.

What resolution to see divergence?

- To see fluctuation corrections must get close enough to T_c so that $C_1 \gtrsim C_0 \simeq \Delta C$

$$C_1 \sim \frac{1}{K^2}\xi^{4-d} \sim \frac{1}{K^2}\Big(\underbrace{\sqrt{K}}_{\xi_0 \equiv \text{characteristic length}}\, t^{-1/2}\Big)^{4-d}, \quad t = \frac{T-T_c}{T}$$

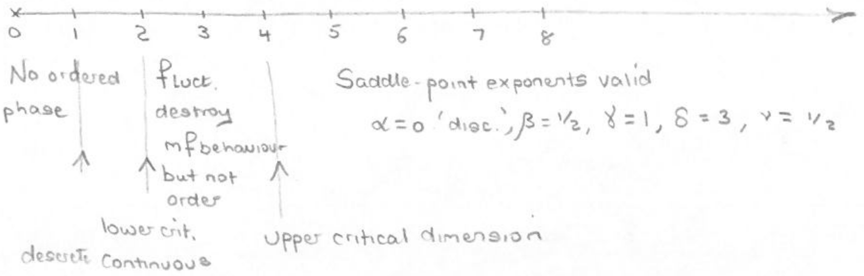
$$C_1 \sim \frac{1}{\xi_0^4}\,\xi_0^{4-d}\, t^{-\frac{4-d}{2}} \gg \Delta C$$

$$t \ll t_G \sim \left(\frac{1}{\Delta C\, \xi_0^d}\right)^{2/4-d}$$

In principle, ΔC and ξ_0 are available from experiment and can check self-consistently

- $\Delta C \sim 1\,k$ per particle
- $\xi_0 \sim$ Range of interactions in units of particle spacing
- For superconductors, $\xi_0 \sim$ size of Cooper pair $\sim 10^3\,\mathring{A}$; $t_G \sim 10^{-18}$, out of reach of device
- For superfluid, $\xi_0 \sim$ thermal wavelength $\sim 2\text{-}3\,\mathring{A}$; $t_G \sim 10^{-1} - 10^{-2}$, Accessible

Summary for L.G.



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III THE SCALING HYPOTHESIS

IIIA Assumption of Homogeneity

Recall that many properties have singular behaviour at the critical point. Are the exponents related?

Saddle-point free energy $f = \min\left[\frac{t}{2}m^2 + um^4 - hm\right] \sim \begin{cases} -\frac{t^2}{u} & h=0,\ t<0 \\ -\frac{h^{4/3}}{u^{1/2}} & t=0 \end{cases}$

Homogenous form $f(t,h) = t^2 g_f(h/t^\Delta)$
(assertion)

$$\lim_{x \to 0} g(x) \sim -\frac{1}{u} \ \text{(constant)} \qquad f(t,h=0) \sim -\frac{t^2}{u}$$

$$\lim_{x \to \infty} g(x) \sim x^{4/3} \qquad f(t=0,h) \sim t^2\left(\frac{h}{t^\Delta}\right)^{4/3} \sim h^{4/3} \quad \text{if } \frac{4}{3}\Delta = 2,\ \Delta = \frac{3}{2}$$

'Gap exponent'

Assumption The correct singular free energy also has homogeneous form

$$f_{\text{sing.}}(t,h) = t^{2-\alpha} g_f(h/t^\Delta) \quad \text{with } \alpha, \Delta \text{ unspecified}$$

Homogeneity implies the same exponents for $t>0$ and $t<0$ ($h=0$) (Ma. p113)

- The point is that we are not confined to $h=0$ but can continue function f everywhere around single branch cut

1) Magnetisation

$$m(t,h) = -\frac{\partial f}{\partial h} = -t^{2-\alpha} g_f'(h/t^\Delta)\frac{1}{t^\Delta} = t^{2-\alpha-\Delta} g_m(h/t^\Delta)$$

$$m(t,h=0) = t^{2-\alpha-\Delta} g_m(0) \sim t^\beta \qquad \beta = 2-\alpha-\Delta$$

$$m(t=0,h) = t^{2-\alpha-\Delta}\underbrace{\left(\frac{h}{t^\Delta}\right)^p}_{\text{By assumption}} \qquad p\Delta = 2-\alpha-\Delta$$

$$\sim h^{(2-\alpha-\Delta)/\Delta} \sim h^{1/\delta} \qquad \delta = \frac{\Delta}{2-\alpha-\Delta} = \frac{\Delta}{\beta}$$

Singular behaviour of free energy defines singular behaviour of magnetisation.

2) Response functions

- Susceptibility $\chi(t) = \frac{\partial m}{\partial h} = t^{2-\alpha-2\Delta} g_\chi(h/t^\Delta)$

$$\chi(t,h=0) \sim t^{-(2\Delta+\alpha-2)} \sim t^{-\gamma} \qquad \gamma = 2\Delta + \alpha - 2$$

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Heat Capacity

$$C \sim -\frac{\partial^2 f}{\partial t^2} \qquad \frac{\partial f}{\partial t} = (2-\alpha)\, t^{1-\alpha} g_f\left(h/t^{\Delta}\right) + t^{2-\alpha} g_f'\left(h/t^{\Delta}\right)\left(-\frac{\Delta h}{t^{\Delta+1}}\right)$$

$$= t^{1-\alpha} g_E\left(h/t^{\Delta}\right)$$

$$g_E\left(h/t^{\Delta}\right) = (2-\alpha)\, g_f\left(h/t^{\Delta}\right) - \frac{\Delta h}{t^{\Delta}}\, g_f'\left(h/t^{\Delta}\right)$$

$$\frac{\partial^2 f}{\partial t^2} \sim C(t,h) = t^{-\alpha} g_c\left(h/t^{\Delta}\right)$$ α is heat capacity exponent

Consequences

(1) For all $Q(t,h)$, exponents above and below T_c are the same – can compare with experiments

(2) All quantities come with the same gap exponent Δ

(3) Almost all exponents can be obtained from two independent ones, e.g. α, Δ

Exponent Identities

$$\alpha + 2\beta + \gamma = \alpha + 2(2-\alpha+\Delta) + 2\Delta + \alpha - 2 = 2 \qquad \text{(Rushbrooke)}$$

$$\delta - 1 = \frac{\Delta}{2-\alpha-\Delta} - 1 = \frac{2\Delta+\alpha-2}{2-\alpha-\Delta} = \frac{\alpha}{\beta} \qquad \text{(Widom)}$$

eck against table of exponents

		α	β	γ	δ	ν	η	
d = 3	n = 1	0.12	0.31	1.25	~5	0.64	0.05	
	n = 2	0.00	0.33	1.33	~5	0.60	0.00	
	n = 3	– 0.14	0.35	1.4	~5	0.7	0.04	
d = 2	n = 1	0	1/8	7/4	15	1	1/4	(exact)

d = 3, n = 1 $\qquad \alpha + 2\beta + \gamma = 1.99$

etc.

But ν, η are not determined

III B Hyperscaling and Correlation Length.

(1) Start the other way around and assume homogeneity of ξ

$$\xi(t,h) = t^{-\nu} g_{\varepsilon}\left(h/t^{\Delta}\right)$$

$$\xi(0,h) \sim h^{-\gamma/\Delta}$$ t dependence must vanish at $t = 0$

(2) Close to critical point ξ is the only important length scaling and solely responsible for singular behaviour of the free energy

$\ln Z$ is dimensionless and extensive $\sim$ (i.e. $\propto L^d$)

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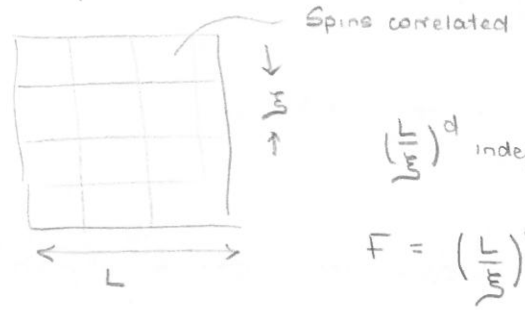
$$\ln Z = \left(\frac{L}{\xi}\right)^d \times \left(\text{Non-singular of dimensionless arguments}\right) + \underbrace{\left(\frac{L}{a}\right)^d}_{\text{Lattice spacing}} \times \underbrace{(\text{ditto})}_{\text{Harmless term}}$$

Geometric interpretation



$\left(\frac{L}{\xi}\right)^d$ independent boxes

$$F = \left(\frac{L}{\xi}\right)^d \times \text{Free energy of each box}$$

$$f_{\text{sing}}(t,h) \sim \frac{\ln Z}{L^d} \sim \xi(t,h)^{-d}$$ which combining with earlier result

$$f_{\text{sing}}(t,h) = t^{d\nu} g_f\left(h/t^{\Delta}\right)$$

Consequences

(1) Homogeneity of f follows from that of ξ

(2) Hyperscaling identity $2-\alpha = d\nu$

$d=2,\ n=1 \qquad 2-\alpha = 2-0 = 2\times 1$ ✓

* Does not work for mean-field exponents

$\alpha = 0$
$\nu = 1/2$ for $d>4$ $\quad 2-0 \neq d/2$ $\qquad$ Reason is that function $\times\left(\frac{L}{a}\right)^d$ is divergent

At $d=4$ and below the $\left(\frac{L}{\xi}\right)^d$ becomes dominant

However no scaling argument for η yet.

III C Critical Correlation Functions and Self-similarity

At critical point, $t=h=0$

$$G_{mm}(x) = \langle m(x)\,m(0)\rangle_c \sim \frac{1}{|x|^{d-2+\eta}}$$

$$S(q) \sim \frac{1}{q^{2-\eta}}$$ used to identify η

Energy density $$G_{EE}(x) = \langle E(x)\,E(0)\rangle_c \sim \frac{1}{|x|^{2(d-2)+\underbrace{\eta'}_{\text{correction to mf.}}}}$$

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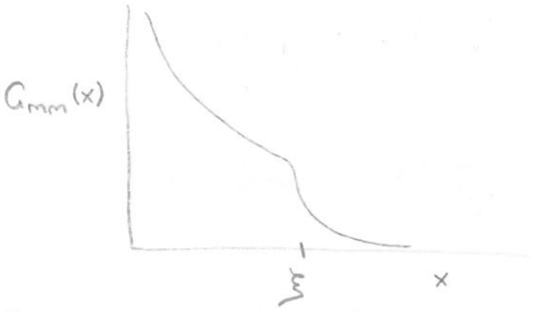
Away from $t=h=0$ correlations are cut-off by ξ

Recall



$$\chi \sim \int d^dx \, \langle m(x)m(0)\rangle_c \sim \int_0^{\xi} \frac{d^dx}{x^{d-2+\eta}} \sim \xi^{2-\eta} \sim t^{-\nu(2-\eta)}$$

So $\gamma = \nu(2-\eta)$ (Fisher)

$d=3$, $n=2$ $1.33 \simeq 0.6 \times (2-0.00)$

$d=2$, $n=1$ $7/4 = 1.(2-1/4)$

$$C \sim \int d^dx \, \langle E(x)E(0)\rangle \sim \int_0^{\xi} \frac{d^dx}{x^{2(d-2)+\eta'}} \sim \xi^{4-d-\eta'}$$

So $\alpha = \nu(4-d-\eta')$

η' is not a popular exponent since it cannot be related simply to experiments.

In principle, if correlation functions at T_c are known, all behaviour away from T_c can be determined.

- At the critical point, there is no length scale and the correlations are self-similar. Power law has no particular length scale.
- Exact self-similarities; e.g. fractals, Kantor sets, etc. 
- Statistical self-similarity e.g. Coastline, moutain landscape

Magnetisation profile at $t=h=0$ is statistically self-similar

$G_{mm}(\lambda x) \sim \lambda^p G_{mm}(x)$ – works only at T_c with power law correlations not away from T_c with exponential decay

- The critical point has an additional symmetry (self-similarity)

(Except possibly for $d=2$ we can't construct directly βH with such symmetry — which would therefore etermine G's and hence η, η' and all exponents)

- Away from $t=h=0$ self-similar correlations are cut-off at ξ.

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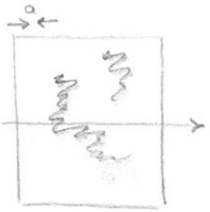
III D Kadanoff Renormalisation Group (Conceptual)

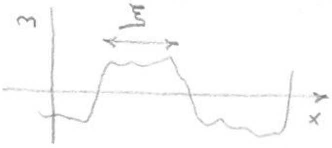
Assume ξ is the only important length scale.

Renormalisation (eliminates short scale fluctuations)

(1) Start with a configuration $\underline{m}(x)$ generated with a weight $W[m] \equiv \exp[-\beta H[m]]$

There is also an implicity short distance length 'a'





Coarse grain, by reducing resolution to ba $(b > 1)$ with $\bar{m} = \int_{\substack{\text{cell}\\ \text{of size } b}} d^d x' \, m(x') \cdot \frac{1}{b^d}$ $(a=1)$

(2) Rescaling: distances $x_{new} = \frac{x_{old}}{b}$ so that the apparent resolution returns to original

(3) Renormalising: reduce contrast by a factor ζ (old picture was compressed enlarging the apparent contrast)

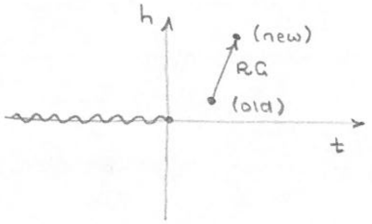
So that
$$m_{new}(x) = \underbrace{\frac{1}{\zeta}}_{\text{As yet unspecified}} \frac{1}{b^d} \int_{\substack{\text{cell}\\ \text{centred at } bx}} d^d x' \, m(x')$$

$$\begin{array}{ccc} m_{old} & & m_{new} \\ W[m_{old}] & \xrightarrow{RG} & W_b[m_{new}] \end{array}$$

Kadanoff (1) Close to criticality W_b and W are similar (statistical self-similarity)

If the original model is at $(t = h = 0)$, there is no characteristic length scale and the new configuration has a critical Hamiltonian. $W_b = \exp[-\beta H_b[m_{new}]]$

However if originally off criticality, then renormalised Hamiltonian, then (t_b, h_b) is further away from the original, because



$\xi_{new} = \frac{\xi_{old}}{b}$ is smaller. But ξ increases close to T_c.

(Here we have assumed that the new Hamiltonian is characterised by a t and h — this will be demonstrated true later)

(β which is close to T_c for RG can be rescaled out of the problem if required. Otherwise simply treat as a constant)

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Kadanoff (2) The mapping is analytic since it only involves eliminating short-distance fluctuations

This assumption must be checked later

$$\begin{cases} t(b;t,h) = A(b)t + B(b)h + O(t^2; h^2, th) \\ h(b;t,h) = C(b)t + D(b)h + O(t^2, h^2, th) \end{cases}$$

Cannot have constant because of known behaviour at $t = h = 0$

But by Symmetry $C(b) = 0$ since h would spontaneously break symmetry

also $B(b) = 0$

Commutivity $A(b_1 \times b_2) = A(b_1)A(b_2)$, $A(1)$

$$\Rightarrow A(b) = b^{y_t}$$

and also $D(b) = b^{y_h}$

So to lowest order $\begin{cases} t(b) = b^{y_t}\, t \\ h(b) = b^{y_h}\, h \end{cases}$

Consequences

(1) Free Energy $m_{old} \xrightarrow{RG} m_{new}(x)$

Information lost so non-invertable

$$W[m] \longrightarrow W_b[m_{new}]$$

$$\mathcal{Z} = \int \mathcal{D}m_{old}(x)\, W[m] = \int \mathcal{D}m_{new}\, W_b[m_{new}] = \mathcal{Z}'_b$$

So $\ln\mathcal{Z} = \ln\mathcal{Z}'_b$ probabilities must be conserved by mapping

$$f(t,h) = \frac{\ln\mathcal{Z}}{V} = \frac{\ln\mathcal{Z}_b}{V_b\, b^d} = b^{-d} f(t_b, h_b) \quad \text{but functions } f \text{ are the same}$$

$$= b^{-d} f(b^{y_t} t,\, b^{y_h} h)$$

This is another way of writing Homogeneity assumption

Choose b: $b^{y_t} t$ = Constant, say 1

$$b = t^{-1/y_t}$$

$$\Rightarrow f(t,h) = t^{d/y_t} f(1, h/t^{y_h/y_t})$$

But recall $f(t,h) = t^{2-\alpha} g_f(h/t^{\Delta})$ where we now discover $2-\alpha = d/y_t$

$\Delta = y_h/y_t$

So if y_t, y_h known we can generate all critical exponents

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(2) Correlation Length $$\xi_{new} = \frac{\xi_{old}}{b} \rightarrow \xi(t,h) = b\,\xi(b^{y_t}t,\, b^{y_h}h)$$

Again setting $b^{y_t}t = 1 \Rightarrow \xi(t,h) = t^{-1/y_t}\, g(h/t^{y_t/y_h})$, $\nu = 1/y_t$

(3) Magnetisation
$$m = \frac{1}{V}\frac{\partial \ln Z(t,h)}{\partial h} = \frac{1}{b^d V_b}\,\frac{\partial \ln Z_b(t',h')}{b^{-y_h}\partial h'} , \qquad \begin{array}{l} t' = t_b \\ h' = h_b \end{array}$$
$$= b^{y_h-d}\, m(b^{y_t}t,\, b^{y_h}h)$$
$$= t^{\frac{d-y_h}{y_t}}\, g_m(h/t^{y_t/y_h}) \qquad , \quad \beta = \frac{d-y_h}{y_t}$$

So generally
$$\begin{cases} X(t,h) = b^{-y_x} X(b^{y_t}t,\, b^{y_h}h) \\ F(t,h) = b^{-y_f} F(b^{y_t}t,\, b^{y_h}h) \end{cases}$$

If X and F are conjugate, $y_x + y_F = d$, e.g. mh

III E Kadanoff's RG (Formal)

(1) Start with most general $W[m] = e^{-\beta H[m]}$ consistent with symmetries

$$\text{e.g. } \beta H = \int d^dx \left[\frac{t}{2}m^2 + um^4 + vm^6 + \dots \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \dots\right]$$

(βH) is described by a point in the parameter space $S \equiv \{t, u, v, \dots K, L, \dots\}$

(2) Apply the RG procedure 1) Coarse grain by b

2) Rescale $x' = x/b$

3) Renormalise $m' = m$

i.e use the mapping $m'(x) = \frac{1}{b^d \zeta}\int_{\substack{\text{cell}\\ \text{around } bx}} d^dx\, m(x)$ to get $W'[m'] \equiv \exp[-\beta H'[m']]$

$(\beta H)'$ is described by a new point $S' \equiv \{t', u', v', \dots K', L' \dots\}$

Mapping
$$\begin{cases} t' = t(b;\, t, u, \dots) \\ u' = u(b;\, t, u, \dots) \\ \quad\vdots \end{cases} \qquad S' = R_b S$$

(3) Find Fixed Points such that $S^* = R_b S$ — The Hamiltonian with these parameters are self-similar. At fixed point $\xi^* = \frac{\xi^*}{b} \Rightarrow \xi^* = \underset{\text{uncorrelated}}{0}$ or $\underset{\text{Scale invariant fluctuations}}{\infty}$

(4) Linearise around S^* :

$$\begin{cases} S_\alpha = S_\alpha^* + \delta S_\alpha \\ S_\alpha' = (R_b S)_\alpha = S_\alpha^* + \underbrace{\left.\frac{\partial S_\alpha}{\partial S_\beta}\right|_{S^*}}_{R^L_{\alpha\beta}(b)} \delta S_\beta + \text{h.o.t} \end{cases}$$

* Diagonalise $R^L_{\alpha\beta}(b)$ to get

eigendirections $\mathcal{O}_L$ and eigenvalues $\lambda_i(b)$

Due to Semi-group property $R^L(b_1) R^L(b_2) = R^L(b_1 b_2)$

Not reversible

$$\text{So } \lambda_i(b_1)\lambda_i(b_2) = \lambda_i(b_1 b_2)$$

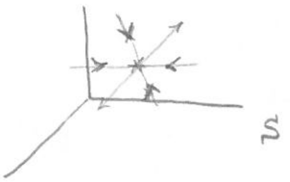
$$\Rightarrow \lambda_i = b^{y_i}$$

So in vicinity of fixed point

$$(\beta H) = (\beta H)^* + \sum_i g_i \underset{\sim}{\mathcal{O}_i} \quad \text{Scaling Directions}$$

$$(\beta H)' = (\beta H)^* + \sum_i g_i b^{y_i} \mathcal{O}_i$$

* Terminology



(1) if $y_i > 0$ then deviation increases : Relevant Direction

(2) if $y_i < 0$ deviation gets smaller : Irrelevant Direction

(3) if $y_i = 0$ marginal - need to go to higher order

* The subspace spanned by irrelevant directions is called the basin of attraction and any point on that surface has infinite correlation length , $\xi = \infty$

Proof $\xi' = \dfrac{\xi}{b}$, but $\xi^* = \infty$ $\therefore$ $\xi = \infty$ on surface .

The surface defines the phase transition . Imagine varying β at differt parameter values — always hit surface

* Relevant directions give critical exponents

$$\xi(g_1, g_2, \ldots) = b\,\xi(b^{y_1} g_1, b^{y_2} g_2, \ldots)$$

if $y_1, y_2 > 0$ and $y_3, < 0$ so importance dependence comes from parameters that don't renomalise to zero $\sim (g_1)^{-1/y_1} g(g_2/g_1^{y_2/y_1})$

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III F The Gaussian Model (Direct Solution and then RG)

* $$Z = \int \mathcal{D}m(x) \exp\left\{-\int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 - \underline{h}.\underline{m}\right]\right\}$$

Z exists if $K > 0$ and $t \geq 0$ - The disordered side of the transition

* Fourier Transform $\quad m(q) = \int d^dx\, e^{iq.x} m(x) \quad , \quad m(x) = \int \frac{d^dq}{(2\pi)^d} e^{-iqx} m(q)$

$$\int \mathcal{D}m(x) \rightarrow \int \prod_q dm_q \equiv \int \mathcal{D}m(q) \quad , \text{Jacobian} = 1$$

$$\beta H = \int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 - \underline{h}.\underline{m}\right] = \underbrace{\int \frac{d^dq}{(2\pi)^d} \frac{Kq^2+t}{2}}_{\sum_q \frac{Kq^2+t}{2V} |m_q|^2 \text{ when discretised}} |m(q)|^2 - \underline{h}.\underline{m}(q=0)$$

$$Z = \int \prod_q dm_q \exp\left[-\sum_q \frac{Kq^2+t}{2V}|m_q|^2 + \underline{h}.\underline{m}(q=0)\right]$$

$$= \prod_{q\neq 0}\left(\sqrt{\frac{2\pi V}{Kq^2+t}}\right)^n \left(\sqrt{\frac{2\pi V}{t}}\right)^n e^{-Vh^2/2t} \qquad n\text{-component vectors}$$

$$= \exp\left[-\frac{n}{2}\sum_q \ln(Kq^2+t) - \frac{Vh^2}{2t} + \text{Const.}\right]$$

$$f(t,h) = -\frac{\ln Z}{V} = \frac{n}{2}\int \frac{d^dq}{(2\pi)^d} \ln(Kq^2+t) + \frac{h^2}{2t} + \text{Const.}$$

Allowed values of q form a Brillouin Zone

However we are interested in long wavelength singular contributions so assume BZ is spherical

Interest in $q < \bar{q} = \sqrt{\frac{t}{K}} \sim \xi^{-1} \propto t^{1/2}$

So $f(t,h) \sim \left(t/K\right)^{d/2}$ - Singular contribution

$$f(t,h) = f_{reg.} + t^{d/2}\left(A + \frac{h^2}{2t^{1+d/2}}\right)$$

$$f_{sing}(t,h) = t^{d/2} g_f\left(h/t^{1/2+d/4}\right)$$

$2-\alpha = d/2$ or $\alpha = 2 - d/2 \quad (\nu = 1/2)$

$\Delta = \frac{1}{2} + \frac{d}{4} \qquad (\gamma_+ = 2\Delta \mp 2 + \alpha = 1)$

$\{$c.f. $\frac{\partial^2 f}{\partial h^2} \sim \frac{1}{t}$

III G The Gaussian Model via RG

$$Z = \int \mathcal{D}m(x) \exp\left\{-\int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 - \underline{h}.\underline{m}\right]\right\}$$

$$= \int \mathcal{D}m(q) \exp\left\{-\int_0^\Lambda \frac{d^dq}{(2\pi)^d}\left(\frac{Kq^2+t}{2}\right)|m_q|^2 + \underline{h}.\underline{m}(q=0)\right\}$$

i) Coarse Grain

Eliminate fluctuations at scales $a < x < ba \iff$ eliminate modes for $\frac{\Lambda}{b} < q < \Lambda$

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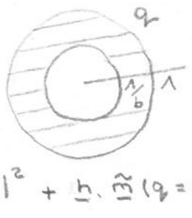

$$\mathcal{Z} = \underbrace{\exp\left[-\frac{n}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\ln\left(t+Kq^2\right)\right]}_{\mathcal{Z}_>}$$

$$\times \int \mathcal{D}\tilde{m}(q)\exp\left[-\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\left(\frac{Kq^2+t}{2}\right)|\tilde{m}(q)|^2 + \underline{h}\cdot\underline{\tilde{m}}(q=0)\right]$$

ii) $x' = x/b \ , \quad q' = bq$

$$\mathcal{Z} = \mathcal{Z}_> \int \mathcal{D}\tilde{m}(q')\exp\left\{-\int_0^{\Lambda}\frac{d^dq'}{(2\pi)^d}\, b^{-d}\left(\frac{Kb^{-2}q'^2+t}{2}\right)|\tilde{m}(q')|^2 + \underline{h}\cdot\underline{\tilde{m}}(q'=0)\right\}$$

iii) $m'(x) = \dfrac{\tilde{m}}{\zeta} \ , \quad m'(q') = \dfrac{\tilde{m}(q')}{z}$

$$\mathcal{Z} = \mathcal{Z}_> \int \mathcal{D}m'(q')\exp\left\{-\int_0^{\Lambda}\frac{d^dq'}{(2\pi)^d}\, z^2b^{-d}\left(\frac{Kb^{-2}q'^2+t}{2}\right)|m'(q')|^2 + z\underline{h}\cdot\underline{m}'(q'=0)\right\}$$

The constant Jacobian part can be neglected in favour of the singular part — does not affect shape of prob. dist.

Original parameter space $\mu \equiv \{K, t, h\} \longrightarrow \mu' \equiv \{K', t', h'\}$

LG model generates new parameters

$$\begin{cases} K' = Kb^{-d-2}z^2 \\ t' = tb^{-d}z^2 \\ h' = hz \end{cases}$$

* Singular point $t = h = 0$

Singular distribution has to be fixed if only the new K = old K

$K' = K \ , \quad z = b^{1+d/2}$

$$\begin{cases} t' = b^2 t & y_t = 2 \\ h' = b^{1+d/2}h & y_h = 1 + d/2 \end{cases}$$

For $t' = t$ fixed point K becomes weaker, and spins become uncorrelated → high T phase

$$f_{sing}(t,h) = b^{-d} f_{sing}\left(b^2t, b^{1+d/2}h\right) \qquad b^2t = 1$$

$$= t^{d/2}\, g_f\left(h/t^{1/2+d/4}\right)$$

recovers homogeneity $\quad 2-\alpha = d/2 \ , \ \Delta = y_h/y_t = \frac{1}{2} + \frac{d}{4}$

$$\nu = 1/y_t = 1/2$$

For this case the RG works

* Fixed Hamiltonian $t = h = 0$

$$(\beta H)^* = \frac{K}{2}\int d^dx\,(\nabla m)^2 \qquad \text{a scale invariant quantity}$$

Dimensional analysis $\quad x = bx' , \ m(x) = \zeta m'$

$$(\beta H)^* = \frac{K}{2}\, b^{d-2}\zeta^2 \int d^dx'\,(\nabla m')^2 \ , \qquad \zeta = b^{1-d/2}$$

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Small perturbations

$$(\beta H)^* + u_n \int d^d x \; m^n(x) \longrightarrow (\beta H)^* + u_n b^d \zeta^n \int d^d x' \; m'(x')^n$$

$$u_n \longrightarrow u_n' = b^d \, b^{n - nd/2} = b^{y_m} u_m$$

$$y_m = n - d\left(\frac{n}{2} - 1\right)$$

$$y_1 = y_h = 1 + \frac{d}{2}$$

$$y_2 = y_t = 2$$

LG → $y_4 = 4 - d$, so u_4 is relevant for $d < 4$, irrelevant for $d > 4$

In $d > 4$, small m^4 perturbation has no effect on fixed point.

$y_6 = 6 - 2d$ u_6 is relevant for $d < 3$, irrelevant for $d > 3$

IV PERTURBATIVE RG

IV A Expectation values in unperturbed Hamiltonian

Regard Landau-Ginzburg model as a perturbation to the Gaussian model

$$Z = \int \mathcal{D}m(x) \exp\Big[- \underbrace{\int d^d x \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 \right]}_{(\beta H_0)} - \underbrace{u \int d^d x \, m^4}_{U} \Big]$$

$$(\beta H)_0 = \int \frac{d^d q}{(2\pi)^d} \left(\frac{Kq^2 + t}{2} \right) |m(q)|^2 = \sum_q \left(\frac{Kq^2 + t}{2V} \right) |m_q|^2$$

$$U = u \int d^d x \, m^4 = u \int d^d x \int \frac{d^d q_1 \, d^d q_2 \, d^d q_3 \, d^d q_4}{(2\pi)^{4d}} e^{i(q_1 + q_2 + q_3 + q_4) \cdot x} \, m(q_1) \cdot m(q_2) \, m(q_3) \cdot m(q_4)$$

$$= u \int \frac{d^d q_1 \, d^d q_2 \, d^d q_3}{(2\pi)^{3d}} \; m_i(q_1) \cdot m_i(q_2) \, m_j(q_3) \, m_j(-q_1 - q_2 - q_3)$$

* Perturbation Theory requires 'bare' expectation values

$$\langle \Theta \rangle_0 = \frac{\int \mathcal{D}\underline{m}(x) \, e^{-\beta H_0} \, \Theta}{\int \mathcal{D}\underline{m}(x) \, e^{-\beta H_0}}$$

e.g. $\langle m^\alpha_q \, m^\beta_{q'} \rangle = \delta^{\alpha\beta} \, \delta_{q+q',0} \, \dfrac{V}{Kq^2 + t}$

$$\langle m_\alpha(q) \, m_\beta(q') \rangle = \delta_{\alpha\beta} \, (2\pi)^d \, \frac{\delta^d(q + q')}{Kq^2 + t}$$ (Bare propagator)

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* For product of many m's use :

$$\left\langle e^{a_i m_i}\right\rangle = e^{\frac{1}{2}\cdot a_i a_j \langle m_i m_j\rangle_0}$$

generates all products

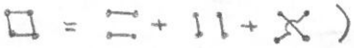
$$1 + a_i\langle m_i\rangle_0 + \frac{1}{2}a_i a_j\langle m_i m_j\rangle_0 + \frac{1}{6}a_i a_j a_k\langle m_i m_j m_k\rangle_0 + \frac{1}{24}a_i a_j a_k a_l\langle m_i m_j m_k m_l\rangle_0 + \dots$$

$$= 1 + \frac{1}{2}a_i a_j\langle m_i m_j\rangle_0 + \frac{1}{8}a_i a_j a_k a_l\langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \dots$$

which must match for all a_i

$$\left\langle \prod_{\alpha=1}^{L} m_\alpha\right\rangle_0 = \begin{cases} 0 & L \text{ odd} \\ \sum \text{all 2-point contractions} & L \text{ even}\end{cases} \qquad \text{Wick's Theorem}$$

$$\langle m_i m_j m_k m_l\rangle_0 = \langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 + \langle m_i m_l\rangle_0\langle m_j m_k\rangle_0$$

(Linked cluster theorem )

IV B Expectation Values in Perturbation Theory

$$\langle\Theta\rangle = \frac{\int \mathcal{D}\underline{m}(x)\, e^{-\beta H_0 - U}\,\Theta}{\int \mathcal{D}\underline{m}(x)\, e^{-\beta H_0 - U}}$$

$$= \frac{\int \mathcal{D}\underline{m}(x)\, e^{-\beta H_0}\left[1 - U + U^2/2 + \dots\right]\Theta}{Z_0\int \mathcal{D}\underline{m}(x)\, \frac{e^{-\beta H_0}}{Z_0}\left[1 - U + U^2/2 + \dots\right]}$$

$$= \frac{Z_0\left[\langle\Theta\rangle_0 - \langle\Theta U\rangle_0 + \langle\Theta U^2\rangle_0/2 - \dots\right]}{Z_0\left[1 - \langle U\rangle_0 + \langle U^2\rangle_0/2 - \dots\right]}$$

$$= \left[\langle\Theta\rangle_0 - \langle\Theta U\rangle_0 + \langle\Theta U^2\rangle_0/2\right]\left(1 + \langle U\rangle_0 + \langle U\rangle_0^2 - \langle U^2\rangle_0/2 - \dots\right)$$

$$= \langle\Theta\rangle_0 - \left(\langle\Theta U\rangle_0 - \langle\Theta\rangle_0\langle U\rangle_0\right) + \frac{1}{2}\left(\langle\Theta U^2\rangle - 2\langle\Theta U\rangle_0\langle U\rangle_0 + 2\langle\Theta\rangle_0\langle U\rangle_0^2 - \langle\Theta\rangle_0\langle U^2\rangle_0\right) + O(U^3)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle\Theta U^n\rangle_0^c$$

and this defines the contraction c.

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For the LG model:

$$\beta H_0 = \int d^dx \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 \right] = \int \frac{d^dq}{(2\pi)^d} \frac{(t+Kq^2)}{2} |m(q)|^2$$

$$U = u\int d^dx\, m^4 = \int \frac{d^dq_1 d^dq_2 d^dq_3 d^dq_4}{(2\pi)^{4d}} m_i(q_1) . m_i(q_2) m_j(q_3) m_j(q_4) \; (2\pi)^d \delta(q_1+q_2+q_3+q_4)$$

e.g. $$\langle m_\alpha(q) m_\beta(q')\rangle = \underbrace{\langle m_\alpha(q) m_\beta(q')\rangle_0}_{\frac{\delta_{\alpha\beta}(2\pi)^d\delta^d(q+q')}{Kq^2+t}} - u\int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}} \Big[\langle m_\alpha(q) m_\beta(q') m_i(q_1) m_i(q_2) m_j(q_3) m_j(-q_1-q_2-q_3)\rangle_0$$

$$- \langle m_\alpha(q) m_\beta(q')\rangle_0 \langle m_i(q_1) m_i(q_2) m_j(q_3) m_j(-q_1-q_2-q_3)\rangle_0 \Big]$$

$$+ O(u^2)$$

$\langle(\theta U^n)\rangle^c$ ~ only contractions between θ and U are included as illustrated here

Two types of contractions

1/ (α,β) connected to (i,i) or (j,j)

$$\langle m_\alpha(q)\, m_\beta(q')\, m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0$$

4   1   1

$$= 4\, \frac{\delta_{\alpha i}(2\pi)^d\delta^d(q+q_1)}{t+Kq^2} . \frac{\delta_{\beta i}(2\pi)^d\delta^d(q'+q_2)}{t+Kq'^2} . \frac{\delta_{jj}(2\pi)^d\delta^d(q_3+q_4)}{t+Kq_3^2}$$

$$\sum_{ij}\int d^dq_1 d^dq_2 d^dq_3 \Rightarrow 4n\delta_{\alpha\beta} \frac{(2\pi)^d\delta^d(q+q')}{(t+Kq^2)^2} \int \frac{d^dq_3}{(2\pi)^d} \frac{1}{t+Kq_3^2}$$

2/ (α,β) connected to $(i\neq j)$

$$\langle m_\alpha(q)\, m_\beta(q')\, m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0$$

4   2

$$= 4\times 2\; \frac{\delta_{\alpha i}(2\pi)^d\delta^d(q+q_1)}{t+Kq^2} . \frac{\delta_{\beta j}(2\pi)^d\delta^d(q'+q_3)}{t+Kq'^2} . \frac{\delta_{ij}(2\pi)^d\delta^d(q_2+q_4)}{t+Kq_2^2}$$

$$\sum\int \Rightarrow 8\,\delta_{\alpha\beta} \frac{(2\pi)^d\delta^d(q+q')}{(t+Kq^2)^2} \int \frac{d^dq_2}{(2\pi)^d} \frac{1}{t+Kq_2^2}$$

$$\langle m_\alpha(q) m_\beta(q')\rangle = \delta_{\alpha\beta}(2\pi)^d\delta^d(q+q') \left[\frac{1}{k+Kq^2} - \frac{4(n+2)}{(t+Kq^2)^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{t+Kk^2} + O(u^2) \right]$$

IV C Representing Perturbations by Diagrams

$$\langle\theta\rangle = \sum \frac{(-1)^n}{n!} \langle\theta U^n\rangle_0^c$$

To calculate $\langle\theta\rangle = \left\langle \prod_{i=1}^{L} m_{\alpha_i}(q_i) \right\rangle$ to pth order in $U = u\int d^dx\, m^4$

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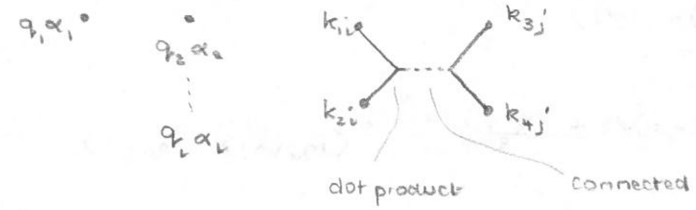
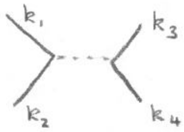
(1) Draw L external points for (q_i, α_i) and p vertices of 4 internal points



(2) Wicks Theorem :

join the L + 4p points together by drawing pairwise lines in all topologically distinct ways

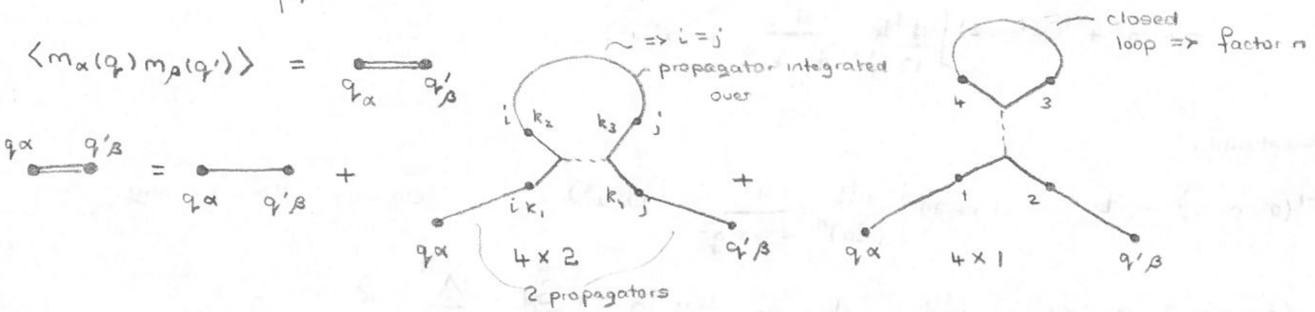
(3) Each line connecting  contributes $\delta_{\alpha\beta} \dfrac{(2\pi)^d \delta^d(q+q')}{t+\kappa q^2}$

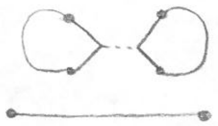
Each  contributes $-u(2\pi)^d \delta^d(k_1+k_2+k_3+k_4)$

(4) Sum over all indicies and integrate over all internal momenta

(5) Overall factor of $\frac{1}{p!}$ × # topologically equivalent graphs

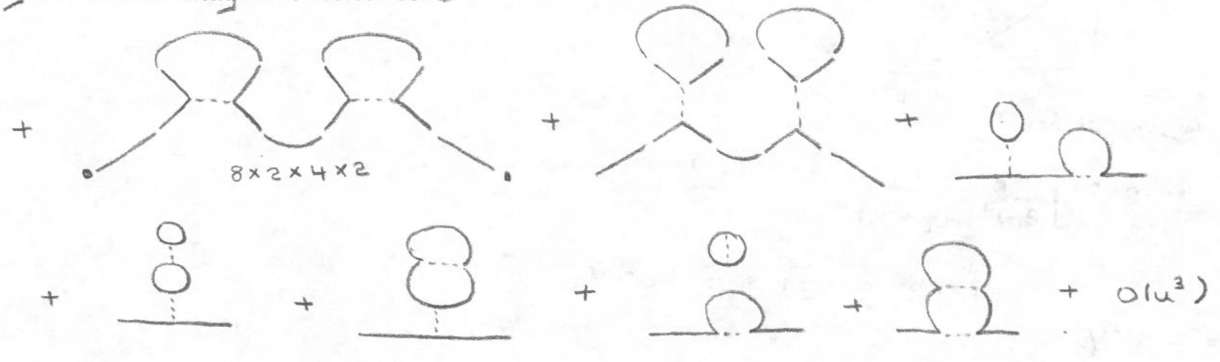
e.g. $\langle m_\alpha(q) m_\beta(q') \rangle$ =





does not contribute

(6) Only connected diagrams contribute



$$\langle m_\alpha(q) m_\beta(q') \rangle = \delta_{\alpha\beta}(2\pi)^d \delta^d(q+q') \Bigg[\frac{1}{t+\kappa q^2} - \frac{u\,4(n+2)}{(t+\kappa q^2)^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{t+\kappa k^2}$$

$$\overset{(-1)^2}{+} \frac{8\times2\times4\times2}{2!} \frac{1}{(t+\kappa k^2)^3} \left(\int \frac{d^dk}{(2\pi)^d} \frac{1}{t+\kappa k^2} \right)^2 + (c_1 n + c_2 n^2) \times \text{same factor except last diagram which mixes integral} \Bigg]$$

$$= \delta_{\alpha\beta}(2\pi)^d \delta^d(q+q') \chi(q)$$

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$\chi(q) \propto |m(q)|^2$ the quantity observed in scattering $\propto S(q)$

$$\langle m_\alpha(q) m_\beta(q')\rangle = \int d^dx\, d^dx'\, e^{iqx+iq'x} \underbrace{\langle m_\alpha(x) m_\beta(x')\rangle}_{G(x-x')}$$

$$= \int d^d\frac{(x+x')}{2}\, d^d(x-x')\, e^{i(q+q')(x+x')/2 + i\frac{(q-q')}{2}(x-x')} \langle m_\alpha(x-x')\, m_\beta(0)\rangle$$

$$= (2\pi)^d\, \delta(q+q') \int d^dr\, e^{iqr} \underbrace{\langle m_\alpha(r)\, m_\beta(0)\rangle}_{\text{isotropic in high T phase} \Rightarrow \delta_{\alpha\beta}}$$

$$= (2\pi)^d\, \delta(q+q')\, \delta_{\alpha\beta}\, \chi(q)$$

$$\chi = \int d^dr \langle m(r)m(0)\rangle_c = \lim_{q\to 0} \chi(q) \quad \text{for the disordered phase}$$

$\chi(q)$ is the momentum dependent susceptibility

$$\chi(q) = \frac{1}{t+Kq^2} - \frac{4u(n+2)}{(t+Kq^2)^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{t+Kq^2} + O(u^2)$$

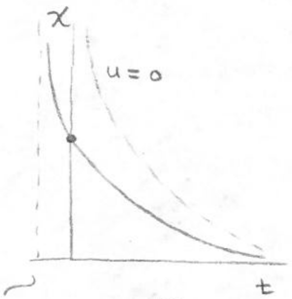
$$\approx \frac{1}{t+Kq^2 + 4u(n+2)\int \frac{d^dk}{(2\pi)^d} \frac{1}{t+Kq^2}} + O(u^2)$$

Inverse Susceptibility

$$\chi^{-1}(q=0,t) = t + 4u(n+2)\int \frac{d^dk}{(2\pi)^d} \frac{1}{t+Kq^2} + O(u^2)$$

working in $d > 4$ anyway.

$$\chi^{-1}(t=0) = 4u(n+2)\int \frac{d^dk}{(2\pi)^d} \frac{1}{Kq^2} = 4\frac{u}{K}(n+2) \underbrace{\frac{S_d}{(2\pi)^d}}_{K_d} \frac{\Lambda^{d-2}}{d-2} \neq 0$$



divergence at new T_c

$$t_c = -4u(n+2)\int \frac{d^dk}{(2\pi)^d} \frac{1}{\underbrace{t_c}_{\text{can be neglected since } t_c \sim u} + Kk^2} + O(u^2)$$

So $t_c = -\chi^{-1}(t=0)$

Fluctuations shifts position of critical point. But T_c is not universal so its not so interesting.

But how does the susceptibility dwerge?

$$\chi^{-1}(t) - \underbrace{\chi^{-1}(t_c)}_{0} = (t-t_c) + 4(n+2)\int \frac{d^dk}{(2\pi)^d}\left(\frac{1}{t+Kk^2} - \frac{1}{t_c+Kk^2}\right) + O(u^2)$$

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$$= (t-t_c)\left[1 - 4u(n+2)\int \frac{d^d k}{(2\pi)^d} \frac{1}{\kappa k^2(\kappa k^2 + t - t_c)} + O(u^2)\right]$$

$$= (t-t_c)\left[1 - \frac{4u(n+2)}{\kappa^2}\int_0^\Lambda \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 + \frac{t-t_c}{\kappa})} + O(u^2)\right]$$

How does integral behave? $[k]^{d-4}$

$k \to 0$ is okay for $d > 2$

$k \to \infty$ $\int \sim k^{d-4}$ diverges for $d > 4$

$$\int \propto \begin{cases} \Lambda^{d-4} & d > 4 \\ \left(\sqrt{\frac{t-t_c}{\kappa}}\right)^{d-4} & 2 < d < 4 \end{cases}$$

So for $d > 4$ $\chi^{-1} = (t-t_c)\left[1 - \text{const} \times u\right]$ $\gamma = 1$

$d < 4$ $\chi^{-1} = (t-t_c)\left[1 - \frac{cu}{\kappa^2}\left(\frac{t-t_c}{\kappa}\right)^{\frac{d-4}{2}} + \cdots\right]$ $\gamma \neq 1$

divergent correction → perturbation theory is failing

In fact u always appears as $\frac{u(t-t_c)^{\frac{d-4}{2}}}{\kappa^{d/2}}$ and always has a divergent effect in $d < 4$

R[illegible]ll

$$\begin{cases} (\beta H)_0 = \int d^d x \left[\frac{t}{2}m^2 + \frac{\kappa}{2}(\nabla m)^2\right] = \int_0^\Lambda \frac{d^d q}{(2\pi)^d}\left(\frac{t + \kappa q^2}{2}\right)|m(q)|^2 \\ U = u\int d^d x\, m^4 = \rangle\cdots\langle \end{cases}$$

from which we found

$$\chi^{-1} = (t-t_c)\left[1 - \frac{4u(n+2)}{\kappa^2}\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 + \frac{t-t_c}{\kappa})} + O(u^2)\right]$$

In general

$$\frac{\langle\Theta\rangle}{\langle\Theta\rangle_0} = 1 + c_1 u + c_2 u^2 + \cdots f\left(\frac{u}{\kappa^2}\xi^{4-d}, \frac{u}{\kappa^2}a^{4-d}\right)$$

diverges near T_c and perturbation theory fails in $d < 4$

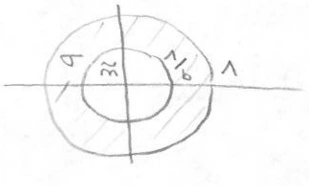
IV E WILSON'S PERTURBATIVE RG

To get round the problem. Really trying to find a small parameter about which to do p.t.

$$Z = \int \mathcal{D}m(q)\, e^{-\beta H_0 - U}$$

(1) [illegible]oarse grain

$$m(q) = \begin{cases} \tilde{m}(q) & 0 < q < \Lambda/b \\ \sigma(q) & \frac{\Lambda}{q} < q < \Lambda \end{cases}$$



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$$Z = \int \mathcal{D}\tilde m(q)\, \mathcal{D}\sigma(q) \exp\left[- \int_0^{\Lambda} \frac{d^dq}{(2\pi)^d} \left(\frac{t+Kq^2}{2}\right)\left(|\tilde m(q)|^2 + |\sigma(q)|^2\right) - U[\tilde m, \sigma] \right]$$

Integrate over σ

$$= \int \mathcal{D}\tilde m(q) e^{-\beta H_0[\tilde m]}\; e^{-\frac{Vn}{2}\int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} \ln(t+Kq^2)} \left\langle e^{-U[\tilde m,\sigma]}\right\rangle_\sigma$$

Where $$\langle \Theta \rangle_\sigma = \frac{\int \mathcal{D}\sigma(q) e^{-\beta H_0[\sigma]}\, \Theta}{\int \mathcal{D}\sigma(q) e^{-\beta H_0[\sigma]}}$$

Then $Z = \int \mathcal{D}\tilde m(q)\, e^{-\beta \tilde H[\tilde m]}$

$$\beta H[\tilde m] = V \delta f_b^0 + \int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde m(q)|^2 - \ln\left\langle e^{-U[\tilde m,\sigma]}\right\rangle_\sigma$$

Constant
of integration

Evaluate perturbatively $\ln\langle e^{-U}\rangle_\sigma = -\langle U\rangle_\sigma + \frac{1}{2}\left(\langle U^2\rangle_\sigma - \langle U\rangle_\sigma^2\right) + \dots + \frac{(-1)^L}{L!}$ Lth cumulant of U + . .

We will calculate first two terms

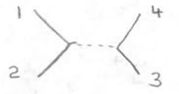
$$U = u\int \frac{d^dq_1\, d^dq_2\, d^dq_3\, d^dq_4}{(2\pi)^{4d}} (2\pi)^d \delta(q_1+q_2+q_3+q_4)$$

$$\times\; [\tilde m(q_1)+\sigma(q_1)]\cdot[\tilde m(q_2)+\sigma(q_2)]\;[\tilde m(q_3)+\sigma(q_3)]\cdot[\tilde m(q_4)+\sigma(q_4)]$$

Note definition of $\tilde m$: zero for $q > \Lambda/b$
σ : zero for $q < \Lambda/b$

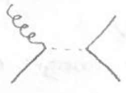
denote $\tilde m$ —— ; σ ∿∿∿∿ diagramatically

[1] $\tilde m(q_1)\cdot\tilde m(q_2)\,\tilde m(q_3)\cdot\tilde m(q_4)$



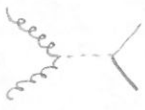
× 1

[2] $\sigma(q_1)\cdot\tilde m(q_2)\,\tilde m(q_3)\cdot\tilde m(q_4)$



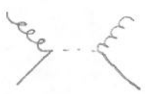
× 4

[3] $\sigma(q_1)\cdot\sigma(q_2)\;\tilde m(q_3)\cdot\tilde m(q_4)$



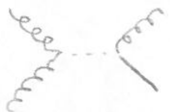
× 2

[4] $\sigma(q_1)\cdot\tilde m(q_2)\;\sigma(q_3)\cdot\tilde m(q_4)$



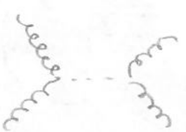
× 4

[5] $\sigma(q_1)\cdot\sigma(q_2)\;\sigma(q_3)\cdot\tilde m(q_4)$



× 4

[6] $\sigma(q_1)\cdot\sigma(q_2)\;\sigma(q_3)\cdot\sigma(q_4)$



× 1

$\langle\;\rangle_\sigma$ => Contract wavy lines

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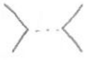
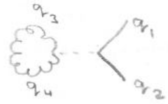
$\langle [1] \rangle_\sigma$ 

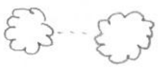
$\langle [2] \rangle_\sigma$ 0 Average of σ is zero

$\langle [3] \rangle_\sigma$ 

$= u \times 2 \int \frac{d^d q_1 \cdots d^d q_4}{(2\pi)^{4d}} (2\pi)^d \delta(q_1 + \cdots q_4) \delta_{\alpha\alpha} \frac{(2\pi)^d \delta(q_3+q_4)}{t + Kq_3^2} \tilde{m}(q_1) \cdot \tilde{m}(q_2)$

$\langle [4] \rangle_\sigma$  $= \quad 4$

$\langle [5] \rangle_\sigma$ 0

$\langle [6] \rangle_\sigma$   $= uV\delta f_b^{1}$

$$\langle [3] \rangle_\sigma = 2un \int_{\Lambda/b}^{\Lambda} \frac{d^d q_3}{(2\pi)^d} \frac{1}{t + Kq_3^2} \int \frac{d^d q}{(2\pi)^d} |\tilde{m}(q)|^2$$

$$\langle [4] \rangle_\sigma = \frac{4}{2n} \langle [3] \rangle_\sigma$$ if you calculate it !

$$\beta \tilde{H}[\tilde{m}] = V(\delta f_b^{0} + u\delta f_b^{1}) + \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \left(\frac{\tilde{t} + \tilde{K}q^2}{2} \right) |\tilde{m}(q)|^2 - \tilde{u}$$ 

$$\tilde{t} = t + 4(n+2)u \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{t + Kk^2}$$

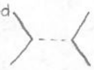
$$\tilde{K} = K$$

$$\tilde{u} = u$$

Now correct to leading order

ii) Rescale $q = b^{-1} q'$

iii) Renormalise $\tilde{m}(q) = z\, m'(q')$

$$(\beta \tilde{H})'[m'] = V(\delta f_b^{0} + u\delta f_b^{1}) + \int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d} b^{-d} \left(\frac{\tilde{t} + Kb^{-2} q^2}{2} \right) z^2 |m'(q')|^2 - uz^4 b^{-3d}$$ 

$$(t, K, u) \longrightarrow (t', K', u')$$

At this order no new terms are generated

Recursion relations

$$\begin{cases} K' = b^{-d-2} z^2 K \\ t' = b^{-d} z^2 \tilde{t} \\ u' = b^{-3d} z^4 u \end{cases}$$

like Gaussian model

Choose z st. $K' = K$

$$\Rightarrow z = b^{1+d/2}$$

$$\begin{cases} t'_b = b^2\left[t + 4u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{1}{t+Kk^2}\right] \\ u'_b = b^{4-d}u \end{cases}$$

* Construct Differential Recursion Relations

Set $b = 1 + \delta L \simeq e^{\delta L}$

$$t'_b \equiv t(b) = t(1+\delta L) = t + \delta L\frac{dt}{dL} + \cdots$$

$$u'_b \equiv u(b) = u + \delta L\frac{du}{dL}$$

$$\begin{cases} t + \delta L\frac{dt}{dL} = (1+2\delta L)\left(t + 4u(n+2)\frac{1}{t+K\Lambda^2}\underbrace{\frac{S_d}{(2\pi)^d}}_{K_d}\Lambda^{d-1}\Lambda\delta L\right) \\ u + \delta L\frac{du}{dL} = (1+(4-d)\delta L)u \end{cases}$$

$$\begin{cases} \frac{dt}{dl} = 2t + \frac{4u(n+2)K_d}{t+K\Lambda^2}\Lambda^d \\ \frac{du}{dl} = (4-d)u \end{cases} \qquad u = u_0e^{(4-d)L} = ub^{4-d} \qquad b = e^L$$

* Fixed point

$$\frac{dt}{dl} = \frac{du}{dl} = 0 \quad , \quad (u^* = 0,\ t^* = 0)$$

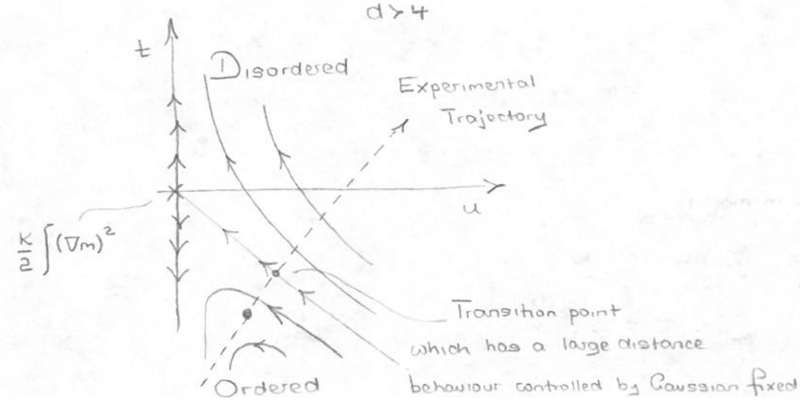
* linearise

$$\begin{cases} t = t^* + \delta t \\ u = u^* + \delta u \end{cases} \qquad \frac{d}{dt}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix} = \begin{pmatrix} 2 & \frac{4(n+2)K_d}{K\Lambda^{2-d}} \\ 0 & 4-d \end{pmatrix}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix}$$

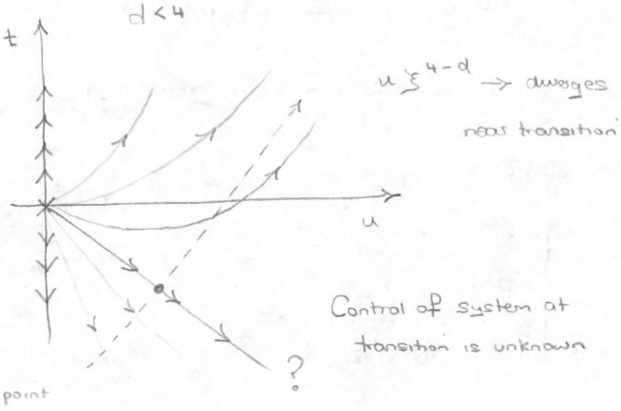
Matrix of Stability

Eigenvalues : 2, 4 − d (off-diagonal term does not change Gaussian result)

RG flows : plotting recursion relations



Transition point which has a large distance behaviour controlled by Gaussian fixed point



$u\,\xi^{4-d} \to$ diverges near transition

Control of system at transition is unknown

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- Within mft, transition at t = 0

Fluctuations make even some points with $t < 0$ disordered

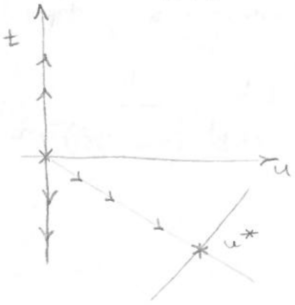
- In principle can also calculate higher order curvature terms

Include terms of order u^2. Expect
$$\begin{cases} \dfrac{dt}{dl} = \left.\dfrac{dt}{dl}\right|_{O(u)} - Au^2 \\[2ex] \dfrac{du}{dl} = \left.\dfrac{du}{dl}\right|_{O(u)} - Bu^2 \end{cases}$$
with $A, B > 0$ because series is alternating

Such a term gives a fixed point at $u^* = \dfrac{4-d}{B}$

If u^* is close to the origin then the u^3 term can be assumed small so high order terms are not important.

This forces an expansion in $\varepsilon = 4 - d$

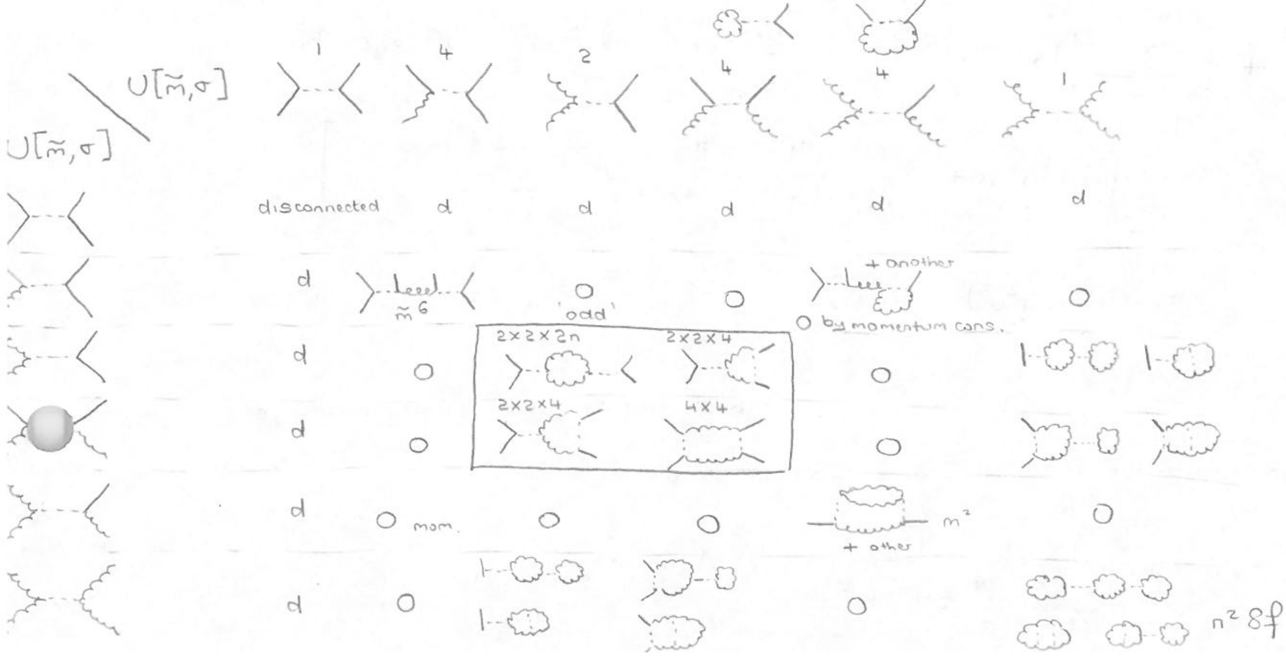


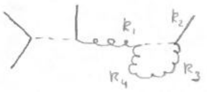
Recall

$$\mathcal{Z} = \int \mathcal{D}\tilde{m}(q)\,\mathcal{D}\sigma(q)\, e^{-\beta H_0[\tilde{m}(q)]}\, e^{-\beta H_0[\sigma]}\, e^{-U[\tilde{m},\sigma]} = \int \mathcal{D}\tilde{m}(q)\, e^{-\widetilde{\beta H}[\tilde{m}]}$$

$$\widetilde{\beta H}[\tilde{m}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde{m}(q)|^2 - \underbrace{\ln\langle e^{-U}\rangle_\sigma}_{\langle U\rangle_\sigma + \frac{1}{2}\left(\langle U^2\rangle_\sigma - \langle U\rangle_\sigma^2\right) + O(U^3)}$$

$$\langle U\rangle_\sigma = u \;\rangle\!\cdots\!\langle\; + uV\delta f_b^1 + 2(n+2)u\int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}\frac{1}{t+Kk^2}\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}|\tilde{m}(q)|^2$$





$k_1 + k_2 + k_3 + k_4 = 0$

or $k_3 + k_4 = 0$

$\therefore k_1 + k_2 = 0$

$\Rightarrow 0$

Important factors are boxed

$\tilde{m}^6$ is a new power generated in Hamiltonian

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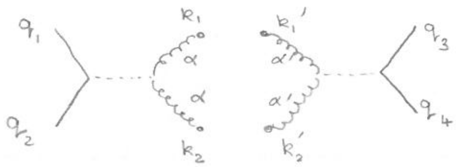
Typical Contribution to $\tilde{m}^4$



$$= \frac{u^2}{2!} \times (2\times 2\times 2)n \int \frac{d^dq_1 \ldots d^dq_4}{(2\pi)^{8d}} (2\pi)^{2d} \delta^d(q_1+q_2+k_1+k_2)\,\delta^d(q_3+q_4+k_1'+k_2')$$

$$\times\ \tilde{m}(q_1).\tilde{m}(q_2)\,\tilde{m}(q_3).\tilde{m}(q_4)\ \frac{(2\pi)^d\delta^d(k_1+k_1')\delta_{\alpha\alpha'}}{t+Kk_1^2}\,.\,\frac{(2\pi)^d\delta^d(k_2+k_2')\ \delta_{\alpha\alpha'}}{t+Kk_2^2}$$

$$= \frac{u^2}{2}.8n \int_0^{\Lambda/b} \frac{d^dq_1 \ldots d^dq_4}{(2\pi)^{4d}} (2\pi)^d \delta^d(q_1+q_2+q_3+q_4)\,\tilde{m}(q_1).\tilde{m}(q_2)\,\tilde{m}(q_3).\tilde{m}(q_4)$$

$$\times \int_{\Lambda/b}^{\Lambda} \frac{d^dk_1}{(2\pi)^d}\ \underset{\nearrow}{\frac{1}{t+Kk_1^2}}\ .\ \frac{1}{t+K(q_1+q_2+k_1)^2}$$

This multiplier is q dependent

There is a constraint on this term that $\Lambda/b < k_2 < \Lambda$ but you have to think about small q_1+q_2 when you can expand, and the constraint goes away.

Expanding, find

$$.\ \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\ \frac{1}{(t+Kk^2)^2}\left[1 - \underbrace{\frac{2Kk.(q_1+q_2)+K(q_1+q_2)^2}{t+Kk^2}+\ldots}_{f(q_1+q_2)}\right]$$

$f(q_1+q_2)$ ↳ correspond to gradient terms in real space $m^2(\nabla m)^2$ etc.

In general we will neglect higher order gradient terms and terms of $\tilde{m}^6$ generated by the RG

$$\widetilde{\beta H}[\tilde{m}] = V\left(\delta f_b^0 + u\delta f_b^1 - \frac{u^2}{2}\delta f_b^2\right) + \int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^d} |\tilde{m}(q)|^2 \left[\frac{t+Kq^2}{2}\right.$$


etc

$$\left. + 2(n+2)u \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{t+Kk^2} - \frac{u^2}{2} A(t,K,\Lambda,q)\right]$$

$$+ \int_0^{\Lambda/b} \frac{d^dq_1 \ldots d^dq_4}{(2\pi)^{4d}} (2\pi)^d \delta^d(q_1+q_2+q_3+q_4)\,\tilde{m}(q_1).\tilde{m}(q_2)\,\tilde{m}(q_3).\tilde{m}(q_4)$$

$$\times \left[u - 4u^2(n+8)\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \left(\frac{1}{t+Kk^2}\right)^2 + \frac{u^2}{2} f(q_1+q_2)\right]$$

$$+\ O(\tilde{m}^6 q^2 u^2) + O(u^3)$$

new interactions consistent with symmetry

$\widetilde{\beta H}$ is characterised by rescaled coarse grained parameters

$$\tilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{t+Kk^2} - u^2 A(q=0)$$

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$$\tilde{K} = K - u^2 A''(q=0)$$

$$\tilde{u} = u - 4(n+8)u^2 \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(t+Kk^2)^2}$$

Rescaling $q = q'/b$

iii) Renormalise $\tilde{m} = z m'$

$$\Rightarrow \quad t' = b^{-d} z^2 \tilde{t}$$

$$K' = b^{-d-2} z^2 \tilde{K}$$

$$u' = b^{-3d} z^4 \tilde{u}$$

Choose z s.t. $K' = K \Rightarrow z^2 = \dfrac{b^{d+2}}{1 - \frac{u^2}{K}A''} = b^{d+2}(1 + \overset{O(\varepsilon^2)}{O(u^2)})$

$$z = b^{1+d/2+O(\varepsilon^2)}$$

Construct differential recursion relations as before, $b = 1 + \delta l$

$$\begin{cases} \dfrac{dt}{dl} = 2t + \dfrac{4u(n+2)}{t+K\Lambda^2} K_d \Lambda^d - u^2 A \\ \dfrac{du}{dl} = \underbrace{(4-d)}_{\varepsilon} u - \dfrac{4(n+8)u^2}{(t+K\Lambda^2)^2} K_d \Lambda^d \end{cases}$$

Second order tem stops divergence

find $u, t \sim O(\varepsilon)$ so $u^2 A$ tem is not important

* Fixed Points

$$\begin{cases} \dfrac{du}{dl} = 0 \\ \dfrac{dt}{dl} = 0 \end{cases} \qquad \begin{cases} u^* = 0. \\ t^* = 0 \end{cases} \qquad \begin{cases} u^* = \dfrac{(t^* + K\Lambda^2)^2}{4(n+8)K_d\Lambda^d} \varepsilon = \dfrac{K^2}{4(n+8)K_4}\varepsilon + O(\varepsilon^2) \\ t^* = -\dfrac{2(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)} u^* = -\dfrac{n+2}{2(n+8)} K\Lambda^2\varepsilon + O(\varepsilon^2) \end{cases}$$

Gaussian Fixed Point

O(n)

* Linearise near fixed point

$$\frac{d}{dl}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 - \dfrac{4(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^* - u^{*2}A' & \dfrac{4(n+2)K_d\Lambda^d}{t^*+K\Lambda^2} - 2Au^* \\ \dfrac{8(n+8)u^{*2}K_d\Lambda^d}{(t^*+K\Lambda^2)^2} & \varepsilon - \dfrac{8(n+8)K_d\Lambda^d u^*}{(t^*+K\Lambda^2)^2} \end{pmatrix} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix}$$

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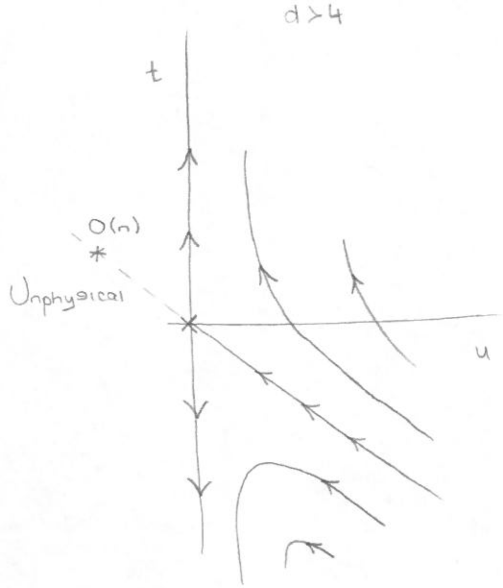
Gaussian $$\frac{d}{dl}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2 & 4(n+2)K_d\Lambda^{d-2}/K \\ 0 & \varepsilon\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} \qquad \begin{cases} y_t = 2 \\ y_u = \varepsilon\end{cases}$$

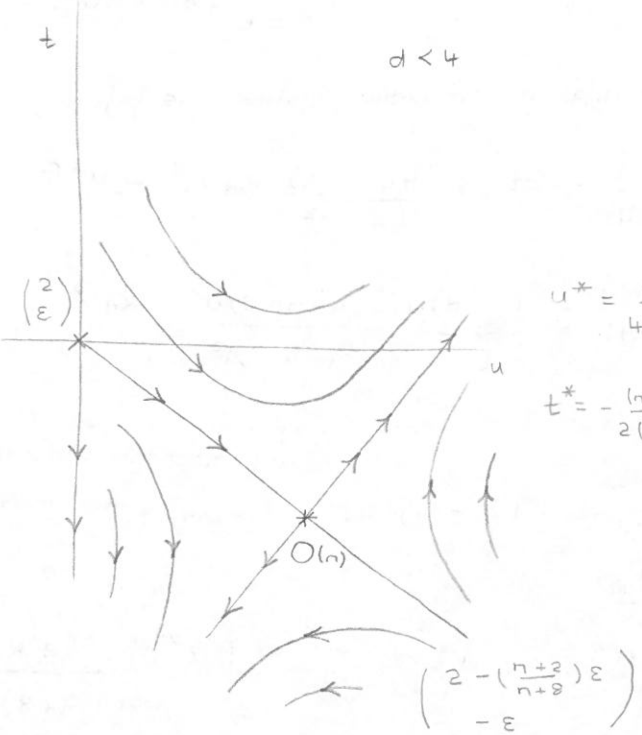
O(n) $$\frac{d}{dl}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2 - \frac{4(n+2)K_4\Lambda^4}{K^2\Lambda^4}\frac{K^2\varepsilon}{4(n+8)K_4} + O(\varepsilon^2) & \text{Whatever} \\ O(\varepsilon^2) & \varepsilon - \frac{8(n+8)K_4\Lambda^4}{K^2\Lambda^4}\cdot\frac{K^2}{4(n+8)K_4}\varepsilon\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}$$

$$\begin{cases} y_t = 2 - \left(\frac{n+2}{n+8}\right)\varepsilon + O(\varepsilon^2) \\ y_u = -\varepsilon + O(\varepsilon^2)\end{cases}$$

— Stability of fixed point independent of material parameters

So as expected, y_t, y_u are independent of K, Λ, etc and are therefore universal





$$u^* = \frac{K^2}{4(n+8)K_4} + O(\varepsilon^2)$$

$$t^* = -\frac{(n+2)K_4\Lambda^2}{2(n+8)}\varepsilon + O(\varepsilon^2)$$

Exponents

$$\xi \sim (\delta t)^{-\nu}$$

$$\nu = 1/y_t = \frac{1}{2}\left[1 + \frac{(n+2)}{(n+8)}\varepsilon + \dots\right]$$

for $n = 1$, in $d = 3$, $\nu \simeq \frac{1}{2} + \frac{1}{12} \simeq 0.58$

(c.f $\nu = 0.63$ Best estimate)

Recall Perturbative RG

$$\beta H = \int d^dx\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4\right]$$

RG (i) Coarse grain (perturbative in u) ii) $x' = bx$; (iii) $m' = zm$

Set $K' = K$ by choosing $z = b^{1+d/2+O(\varepsilon^2)}$; $b = e^l$

$$\frac{dt}{dl} = 2t + \frac{4(n+2)K_d\Lambda^d u}{t + K\Lambda^2} - Au^2$$  + 

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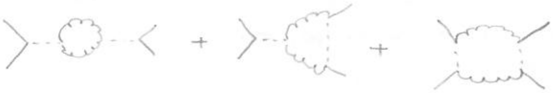
$$\frac{du}{dl} = \underbrace{(4-d)}_{\varepsilon}u - 4\frac{(n+8)K_d\Lambda^d}{(t+K\Lambda^2)^2}u^2$$



We have $y_t = 2 - \left(\frac{n+2}{n+8}\right)\varepsilon + O(\varepsilon^2)$

Need another y_h for $(\beta H)^* - h\int d^dx\, m(x)$

$$-h\,\tilde{m}(q=0) \sim \text{never in integration range}$$

$$-h\,z\,m'(q=0)$$

$$h' = zh = b^{1+d/2}h$$

So $y_h = 1 + d/2 + O(\varepsilon^2)$

Remarks ① For descrete RG $g_1' = b^{y_t}g_1$; differential RG $\frac{dg_1}{dl} = y_t g_1$

② $(\beta H) = (\beta H)^* + g_1\mathcal{O}_1 + g_2\mathcal{O}_2 + h\mathcal{O}_h$ perturbation from fixed Hamiltonian

eigendirections

$$(\beta H)' = (\beta H)^* + g_1 b^{y_t}\mathcal{O}_1 + g_2 b^{-\varepsilon}\mathcal{O}_2 + h b^{y_h}\mathcal{O}_h$$

Shrinks
(irrelevant direction)

③ Exponents

$$f(g_1 = \delta t, h) = b^{-d} f(b^{y_t}\delta t, b^{y_h}h) \quad \text{under rescaling}$$

$$\xi(\delta t, h) = b^{-1}\xi(b^{y_t}\delta t, b^{y_h}h)$$

So $\xi \sim (\delta t)^{-\nu}$, $\nu = 1/y_t = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\varepsilon + O(\varepsilon^2)$

by setting $b^{y_t}\delta t = 1$

$$C \sim (\delta t)^{-\alpha} \qquad \alpha = 2 - \frac{d}{y_t} = 2 - \frac{4-\varepsilon}{2}\left(1 + \frac{n+2}{2(n+8)}\varepsilon\right) = \frac{4-n}{2(n+8)}\varepsilon + O(\varepsilon^2)$$

$$m \sim (\delta t)^{\beta} \sim \frac{\partial f}{\partial h}, \quad \beta = \frac{d-y_h}{y_t} = \left(\frac{4-\varepsilon}{2} - 1\right)\frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\varepsilon\right) = \frac{1}{2} - \frac{3}{2(n+8)}\varepsilon + O(\varepsilon^2)$$

$$\chi \sim (\delta t)^{-\gamma} \sim \frac{\partial m}{\partial h} \sim \frac{\partial^2 f}{\partial h^2} \quad \gamma = \frac{2y_h - d}{y_t} = 2\,\frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\varepsilon\right) = 1 + \frac{n+2}{2(n+8)}\varepsilon + O(\varepsilon^2)$$

β for $n=1$ $d=3$ $(\varepsilon=1)$ $\beta = 1/3$ (Best estimate 0.312)

$d=2$ $(\varepsilon=2)$ $\beta = 1/6$ (Exact value $= 1/8$)

α for $d=3$

$n =$	1	2	3	4
$O(\varepsilon)$	0.17	0.10	0.05	0
Expt	0.10	-0.02	-0.08	/

gives the correct trend but incorrect value of the crossover

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IV F General Discussion on the ε - expansion

① Irrelevance of other interactions

In general, we must include all terms in the Hamiltonian consistent with symmetry

$$\beta H = \int d^dx \left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots + um^4 + vm^2(\nabla m)^2 + \ldots u_6 m^6 + \ldots u_8 m^8 + \right]$$

RG (ii) $x = bx'$

(iii) $m = \zeta m'$

$$t' = b^d \zeta^2 \tilde{t}$$

$$K' = b^{d-2}\zeta^2 \tilde{K} = b^{d-2}\zeta^2 K\left[1 + O(u^2)\right]$$

$$L' = b^{d-4}\zeta^2 \tilde{L}$$

$$u' = b^d \zeta^4 \tilde{u}$$

$$v' = b^{d-2}\zeta^4 \tilde{v}$$

$$u_6' = b^d \zeta^6 \tilde{u}_6$$

Choose ζ s.t. $K' = K$, $\zeta = b^{1-d/2}$

$$t' = b^2 \tilde{t}$$

$$K' = K$$

$$L' = b^{-2}\tilde{L}$$

$$u' = b^{4-d}\tilde{u}$$

$$v' = b^{2-d}\tilde{v}$$

$$u_6' = b^{6-2d}\tilde{u}_6$$

$$u_8' = b^{8-3d}\tilde{u}_8$$

Then set $b = e^L$

$$\frac{dt}{dL} = 2t + O(u, v, u_6, u_8, \ldots)$$

$\Rightarrow$ second order

$$\frac{dL}{dL} = -2L + O(u^2, uv, v^2, \ldots)$$

$$\frac{du}{dL} = \varepsilon u + O(u^2, uv, v^2, \ldots)$$

$$\frac{dv}{dL} = (-2+\varepsilon)v + O(u^2, uv, v^2, \ldots)$$ ← As seen before

$$\frac{du_6}{dL} = (-2+2\varepsilon)u_6 + O(u^3, u^2u_6, \ldots)$$

* Fixed Points

Always have Gaussian fixed point $t^* = u^* = v^* = u_6^* = \ldots = 0$

Eigenvalues are just the Naive dimensions, $2, -2, \varepsilon, -2+\varepsilon, -2+2\varepsilon, \ldots$

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* $O(n)$ fixed point

$$t^* = O(\varepsilon) \quad , \quad u^* = O(\varepsilon)$$

$$L^* = O(\varepsilon^2) \quad , \quad v^* = O(\varepsilon^2) \quad , \quad u_6^* = O(\varepsilon^3)$$

So shift of fixed point out of t, u plane is small

Eigenvalues

As $\varepsilon \to 0$, eigendirections / values of $O(n)$ merge with Gaussian

$$\left(2 - \frac{n+2}{n+8}\varepsilon \, , \; -2 + O(\varepsilon) \, , \; -\varepsilon \, , \; -2 + O(\varepsilon) \, , \; -2 + O(\varepsilon) \, , \; \ldots \right)$$

The eigenvalues not considered earlier remain negative always – and are therefore irrelevant

So only one 1 relevant direction and $p-1$ dimensions of basin of attraction.

param.

② Higher Order Terms

Calculated by field theoretic methods that avoid the cut-off, Λ.

E.g. for $n=1$

$$2\nu = 1 + \varepsilon/6 + 0.04\varepsilon^2 - 0.16\varepsilon^3 + 0.77\varepsilon^4 - 0.2\varepsilon^5 + 0.67\varepsilon^6 - 2.5\varepsilon^7 + 10.3\varepsilon^8 + \ldots$$

$d=3$ 1.26 = 1, 1.167, 1.207, 1.191, 1.268, 1.068, 1.738, −0.76, 9.54, ...

Best estimate

The series is not asymptotically convergent, and alternating

But it is Borel summable

Borel Summation
$$\int_0^\infty dx\, x^n e^{-x} = n!$$

$$f(\varepsilon) = \sum_n a_n \varepsilon^n = \sum_n a_n \varepsilon^n \frac{1}{n!} \int_0^\infty dx\, e^{-x} x^n$$

$$= \int_0^\infty dx\, e^{-x} \sum_n a_n \frac{(\varepsilon x)^n}{n!}$$

If ratio test ⇒ convergence, then Borel summable.

Such tricks can be used to determine $f(\varepsilon)$

③ $1/n$ Expansion

$$u^* = \frac{(4-d)\,(t^* + K\Lambda^2)^2}{4(n+8)K_d\Lambda^d} \xrightarrow[n\to\infty]{} O(1/n)$$

Can expand around $n \to \infty$.

In fact the results we obtained become exact as $n \to \infty$

$$y_t = 2 - \frac{n+2}{n+8}\varepsilon \longrightarrow 2-(4-d) = d-2$$

$$\nu = \frac{1}{d-2} + O(1/n)$$

This can be checked by saddle point method in n

④ Universality

· Critical exponents are pure numbers that depend on d, n and range of interactions, but no material parameters enter.

· We have solved the problem of getting exponents perturbatively.

· K.G. Wilson. Rev. Mod. Phys. 55, 583 (1983) (Nobel Prize)

V Position Space RG.

VA Discrete Models

ε-expansions become difficult to get reliable results for low $d \sim 2, 3$;

Easier to visualise

Ising Model

$\sigma_i = \pm 1$ on sites i of a regular lattice

For N spins, 2^N microstates

$$H = -J \underbrace{\sum_{\langle ij \rangle}}_{n.n.} \sigma_i \sigma_j - \underbrace{\hat{h}}_{\text{magnetic field}} \sum_i \sigma_i$$

favours ferromagnetism of neighbouring spins

$$\cdot\ Z = \sum_{\{\sigma\}} e^{K\sum_{\langle ij \rangle}\sigma_i\sigma_j + h\sum_i \sigma_i}$$ where $K = \beta J$, $h = \beta\hat{h}$ are dimenless

Most general interaction between two spins

$$B(\sigma_1, \sigma_2) = g + \frac{h}{z}(\sigma_1 + \sigma_2) + K\sigma_1\sigma_2 \qquad \sigma^2 = 1$$

For $h=0$, ground state is either $\{\sigma_i = +1\}$ or $\{\sigma_i = -1\}$ in $n=1$ universality class

Potts Model

related to permutation symmetry of q objects

$s = 1, 2, \ldots, q$

$$\beta H = K \underbrace{\sum_{\langle ij \rangle} \delta_{s_i, s_j}}_{\text{symmetrical}} + \left(h \sum_i \delta_{s_i, 1} + \ldots\right)$$

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$q = 2$ is equivalent to Ising model

$q = 3$ Kr on graphite

$q = 4$  tetrahedron (New Universality)

Spin S-models

$S_i = -S, -S+1, \ldots, +S$ $(2S+1)$ states per site

$$\beta H = \sum_{\langle ij\rangle}\left[K S_i S_j + K_2 (S_i S_j)^2 + \ldots + K_{2S}(S_i S_j)^{2S}\right] + h\sum_i S_i$$

2S independent couplings

$S = 1/2$ Ising

$S = 1$ Blume-Emery-Griffiths model $\overset{\text{non-magnetic}}{0}, \underset{\text{magnetic}}{\pm 1}$ dilute magnets

All S models break only up/down symmetry and fall into $n=1$ universality class

O(n) models

$\underline{S}_i = (S_i^1, S_i^2, \ldots S_i^n)$ unit vector $\sum_\alpha (S_i^\alpha)^2 = 1$

$$\beta H = K\underbrace{\sum_{\langle ij\rangle} \underline{S}_i\cdot\underline{S}_j}_{f(\underline{S}_i\cdot\underline{S}_j)} + \underline{h}\cdot\sum_i \underline{S}_i$$

$n = 1$ Ising model

$n = 2$ x-y model

$n = 3$ Heisenberg model

* Tools to Study model

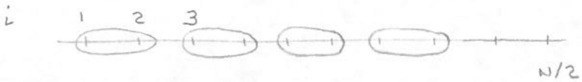
i) PSRG (V)

ii) Series expansions (about Low T, high T) (VI)

iii) Exact Solution (VI)

iv) Monte Carlo computer simulations

VB Exact Treatment in d=1



b = 2 RG $$Z = \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]} \overset{RG}{=} \sum_{\{\sigma_i'\}}^{N/2} e^{-(\beta H)'[\sigma_i']}$$

Choice of σ_i' e.g. $\frac{\sigma_i + \sigma_{i+1}}{2}$ not convenient since $\sigma_i' = +1, -1$, or 0

Choose odd numbered spin to break tie. i.e. $\sigma_i = \sigma_{2i-1}$ In fact $\overset{\equiv S_i}{\sigma_{2i}}$ is eliminated

This procedure is called Decimation

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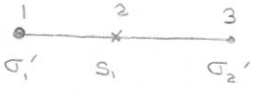
$$\mathcal{Z} = \sum_{\{\sigma_i'\}}^{N/2} \sum_{\{s_i\}}^{N/2} e^{\sum_i^{N/2} \left[B(\sigma_i', s_i) + B(s_i, \sigma_{i+1}') \right]}$$

$$= \sum_{\{\sigma_i'\}} \underbrace{\prod_i^{N/2} \sum_{s_i=-1}^{1} e^{B(\sigma_i', s_i) + B(s_i, \sigma_{i+1}')}}_{e^{\sum_i^{N/2} B'(\sigma_i', \sigma_{i+1}')}}$$ reversing order of sum and product



$$\sum_{s_1 = \pm} e^{B(\sigma_1', s_1) + B(s_1, \sigma_2')} = e^{B'(\sigma_1', \sigma_2')}$$

$$B(\sigma_1, \sigma_2) = g + \frac{h}{2}(\sigma_1 + \sigma_2) + K\sigma_1\sigma_2$$

$$B'(\sigma_1', \sigma_2') = g' + \frac{h'}{2}(\sigma_1' + \sigma_2') + K\sigma_1'\sigma_2'$$ No new terms can be generated since B is most general

Let $x = e^K$, $y = e^h$, $z = e^g$ and primes

σ_1', σ_2'	$e^{B'(\sigma_1', \sigma_2')} = \sum_{s_i = \pm 1} e^{2g + h/2(\sigma_1' + \sigma_2') + hs_1 + K s_1(\sigma_1' + \sigma_2')}$
+ +	$z' y' x' = z^2 y\,[x^2 y + x^{-2} y^{-1}]$
− −	$z' y'^{-1} x' = z^2 y^{-1}\,[x^2 y^{-1} + x^{-2} y]$
+ −	$z' x'^{-1} = z^2\,[y + y^{-1}]$
− +	as + −

3 equations for 3 unknowns

$$z'^4 = z^8 [x^2 y + x^2 y^{-1}]\,[x^2 y^{-1} + x^{-2} y]\,[y + y^{-1}]^2$$

$$y'^2 = y^2 \frac{x^2 y + x^{-2} y^{-1}}{x^2 y^{-1} + x^{-2} y}$$

$$x'^4 = \frac{(x^2 y + x^{-2} y^{-1})(x^2 y^{-1} + x^{-2} y)}{(y + y^{-1})^2}$$

So $$\begin{cases} g' = 2g + \delta g(K, h) & - g \text{ does not contribute to } h', K' \\ h' = h + \delta h(K, h) \\ K' \equiv K'(K, h) \end{cases}$$

Fixed Points

$h = 0$ is a fixed subspace

For $h = 0$ $$e^{4K'} = (e^{2K} + e^{-2K})^2 / 4$$

$$K' = \frac{1}{2} \ln \operatorname{Cosh} K \quad , \quad (\tanh K' = \tanh^2 K)$$

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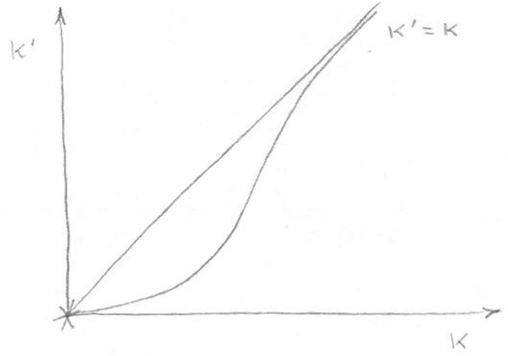


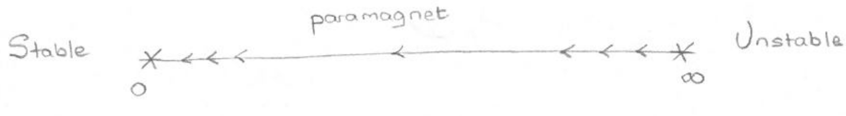
Small K

$$K' = \frac{1}{2}\ln\left(1+\frac{1}{2}(2K)^2\right) = K^2$$

large K

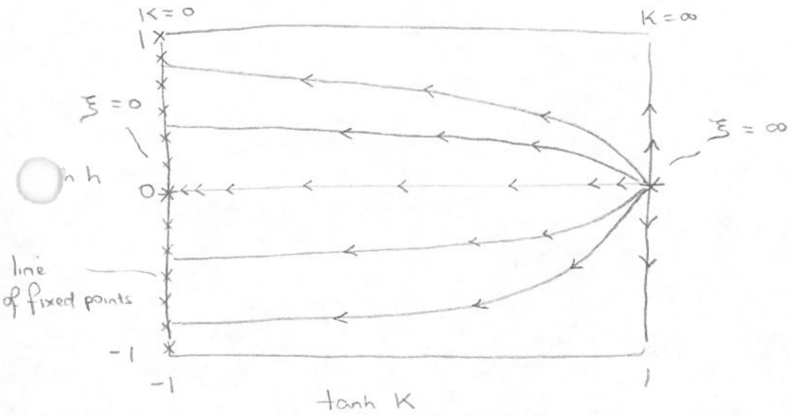
$$K' = \frac{1}{2}\ln\left(\frac{e^{2K}}{2}\right) = K - \frac{\ln 2}{2}$$

$$\left.\begin{array}{l} K^* = 0 \\ K^* = \infty \end{array}\right\} \text{Fixed points}$$



1d Ising magnet does not order

ii) Flows for $h \neq 0$



iii) Region around $K=\infty$, $h=0$ can be examined by linearising RG recursions.

$$\begin{cases} x'^4 \simeq \dfrac{x^4}{4} \\ y'^2 \simeq y^4 \end{cases} \Rightarrow \begin{cases} e^{-K'} = \sqrt{2}\, e^{-K} \\ h' = 2h \end{cases}$$

scaling variables

$$\underbrace{\xi' = \frac{\xi}{2}}_{\text{because of decimation}} \qquad \xi(e^{-K}, h) = 2\,\xi(\sqrt{2}\,e^{-K}, 2h)$$

$$\overset{l \text{ times}}{=} 2^l\,\xi(2^{l/2}e^{-K}, 2^l h)$$

Choose $2^{l/2}e^{-K} \sim O(1)$

$$\xi(e^{-K}, h) = e^{2K} g(he^{2K})$$

So as unstable fixed point is approached correlation length scales as exponential of K

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Free Energy $f \sim \xi^{-d}$ $\qquad f(e^{-K}, h) = e^{-2K} g_f(h e^{2K})$

Susceptibility $\chi \sim \dfrac{\partial^2 f}{\partial h^2} \sim e^{2K} \sim \xi$

$\langle S_x S_0 \rangle \sim \dfrac{e^{-x/\xi}}{x^{d-2+\eta}} \qquad \chi \sim \int dx \, \dfrac{e^{-x/\xi}}{x^{-1+\eta}} \sim \xi^{2-\eta} \Rightarrow \eta = 1$

So $\eta = 1 \Rightarrow \sim e^{-x/\xi}$ c.f. exact Transfer matrix result.

* Transfer Matrix

$$Z = \sum_{S_1, \dots S_N} e^{B(S_1,S_2) + B(S_2,S_3) + \dots + B(S_N, S_1)}$$

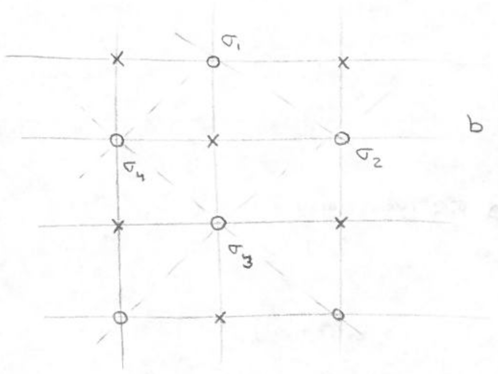
$\langle S_1 | T | S_2 \rangle = e^{B(S_1, S_2)}$ matrix

So $Z = \mathrm{Tr}\,(T^N)$

RG by factor b becomes trivial

$$Z = \mathrm{Tr}\,(T'^{N/b}) \quad , \quad T' = T^b$$

Decimation in Higher d



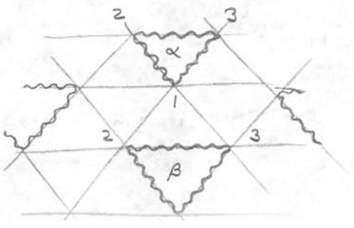
$b = \sqrt{2}$

$$e^{-\beta H'[\sigma_1, \sigma_4]} = \sum_{S=\pm 1} e^{KS(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = 2\,\mathrm{Cosh}\,K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)$$

$$= e^{g + K'(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1) + \underbrace{K'(\sigma_1\sigma_3 + \sigma_2\sigma_4) + K_4\sigma_1\sigma_2\sigma_3\sigma_4}_{\text{New terms are generated}}}$$

Forced into approximation.

VC Niemeijer - van Leeuwen (NvL) Cumulant RG



3 spins per cell, 2 bonds connecting cells

- Triangular Lattice

$$-\beta H = K \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

- Divide lattices into cells α

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Spins in cell α $\quad \sigma_\alpha^1, \sigma_\alpha^2, \sigma_\alpha^3$ $\quad$ (Note lattice structure conserved)

Coarse grained cell spin

$$\sigma_\alpha' = \mathrm{Sign}(\sigma_\alpha^1 + \sigma_\alpha^2 + \sigma_\alpha^3) \qquad \text{(Majority rule)}$$

$$\mathcal{Z} = \sum_{\{\sigma_i\}} \exp\left[-\beta H[\sigma_i]\right]$$

$$= \sum_{\{\sigma_\alpha'\}} \exp\left[-(\beta H)'[\sigma_\alpha']\right]$$

$$e^{-(\beta H')[\sigma']} = \sum_{\{\sigma_i|\sigma_\alpha'\}}{}' \; e^{-\beta H[\sigma]}$$

e.g. if $\sigma_\alpha = 1$, all config. $+++$, $++-$ must be included in sum

NvL proposed an ah hoc perturbation

$$\beta H = (\beta H_0) + U$$

No small parameter — just a choice to make progress

$$-(\beta H)_0 = K \sum_\alpha (\sigma_\alpha^1 \sigma_\alpha^2 + \sigma_\alpha^2 \sigma_\alpha^3 + \sigma_\alpha^3 \sigma_\alpha^1) \qquad \text{Intracell interaction}$$

$$-U = K \sum_{\langle\alpha\beta\rangle} (\sigma_\alpha^1 \sigma_\beta^2 + \sigma_\alpha^1 \sigma_\beta^3) \qquad \text{Intercell interaction}$$

$$e^{-(\beta H)'[\sigma']} = \sum_{\{\sigma_i|\sigma_\alpha'\}}{}' \; e^{-\beta H_0}\left[1 - U + \frac{U^2}{2} + \ldots\right]$$

$$= \mathcal{Z}_0[\sigma']\left[1 - \langle U\rangle_0[\sigma'] + \frac{1}{2}\langle U^2\rangle_0[\sigma'] - \ldots\right]$$

$$-\beta H'[\sigma'] = \ln \mathcal{Z}_0 - \langle U\rangle_0[\sigma'] + \frac{1}{2}\left(\langle U^2\rangle_0 - \langle U\rangle_0^2\right) - \ldots$$

$$\mathcal{Z}_0[\sigma'] = \prod_\alpha \Bigg\{ \sum_{\substack{\{\sigma_\alpha^u|\sigma_\alpha'\} \\ u=1,2,3}} e^{K(\sigma_\alpha^1\sigma_\alpha^2 + \sigma_\alpha^2\sigma_\alpha^3 + \sigma_\alpha^3\sigma_\alpha^1)}$$

σ_α'	σ_α^1	σ_α^2	σ_α^3	$e^{-\beta H_0}$
+	+	+	+	e^{3K}
+	−	+	+	e^{-K}
+	+	−	+	e^{-K}
+	+	+	−	e^{-K}
−	−	−	−	e^{3K}
−	+	−	−	e^{-K}
−	−	+	−	e^{-K}
−	−	−	+	e^{-K}

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So $\mathcal{Z}_0[\sigma'] = \left(e^{3K} + 3e^{-K}\right)^{N/3}$ independent of σ'

$$-\langle U\rangle_{\sigma|\sigma'} = K\sum_{\langle\alpha\beta\rangle}\left(\langle\sigma_\alpha'\rangle_0\langle\sigma_\beta^2\rangle_0 + \langle\sigma_\alpha'\rangle_0\langle\sigma_\beta^3\rangle_0\right)$$ because of indepence of H_0

$$= 2K\sum_{\langle\alpha\beta\rangle}\langle\sigma_\alpha'\rangle_0\langle\sigma_\beta'\rangle_0$$ by symmetry

$$\langle\sigma_\alpha'\rangle = \begin{cases}\dfrac{e^{3K} - e^{-K} + 2e^{-K}}{e^{3K} + 3e^{-K}} & \sigma_\alpha' = +1\\[2ex] \dfrac{-e^{3K} + e^{-K} - 2e^{-K}}{e^{3K} + 3e^{-K}} & \sigma_\alpha' = -1\end{cases} = \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)\sigma_\alpha'$$

$$-(\beta H)'[\sigma'] = \frac{N}{3}\ln\left(e^{3K} + 3e^{-K}\right) + 2K\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)^2\sum_{\langle\alpha\beta\rangle}\sigma_\alpha'\sigma_\beta' + \text{h.o.t.}$$

$$K' = 2K\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)^2 \quad , \quad 1 \rightarrow 1 \text{ parameter}$$

* Fixed points & their stability

i) $K^* = 0$ $(\xi = 0)$ K small $K' \simeq 2K\left(\frac{2}{4}\right)^2 \simeq \frac{K}{2}$ stable

ii) $K^* \rightarrow \infty$ K large $K' \simeq 2K$ stable

$$\frac{e^{3K^*} + e^{-K^*}}{e^{3K^*} + 3e^{-K^*}} = \frac{1}{\sqrt{2}} \qquad \sqrt{2}\,e^{4K^*} + \sqrt{2} = e^{4K^*} + 3$$

$$K^* = \frac{1}{4}\ln\left(\frac{3 - \sqrt{2}}{\sqrt{2} - 1}\right) \simeq 0.3356 \qquad (K_c = 0.2747 \text{ exact})$$

linearise $$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2\left(\frac{e^{3K^*} + e^{-K^*}}{e^{3K^*} + 3e^{-K^*}}\right) + 4K^*\left(\quad\right)$$

$$\simeq 1.642 = b^{y_t} = (\sqrt{3})^{y_t}$$

b

by tradition denoted y_t for 1 relevant direction

$$y_t = \frac{\ln 1.642}{\ln\sqrt{3}}$$

$$\simeq 0.883 \qquad (y_t = 1 \text{ exact})$$

$$(y_t = 2 \text{ m.f.t.})$$

* Magnetic field

$$-U' = h\sum_i\sigma_i = h\sum_\alpha\left(\sum_{u=1,2,3}\sigma_\alpha^u\right)$$

$$-(\beta H)' = \ln\mathcal{Z}_0 - \langle U\rangle_0 - \langle U'\rangle_0$$

$$-\langle U'\rangle_0 = h\sum_\alpha 3\langle\sigma_\alpha'\rangle_0 = \overbrace{3h\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)}^{h'}\sum_\alpha\sigma_\alpha'$$

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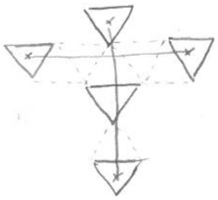
$$h' = 3h\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)$$

$$\frac{\partial h'}{\partial h} = 3\left(\frac{e^{3K^*} + e^{-K^*}}{e^{3K^*} + 3e^{-K^*}}\right) = \frac{3}{\sqrt{2}} = b^{y_h} \quad , \quad y_h = \frac{\ln 3/\sqrt{2}}{\ln\sqrt{3}} \simeq 1.37 \quad \left(\text{c.f. } y_h = 1.875 \text{ exact} \atop y_h = 2 \text{ m.f.t}\right)$$

As y_h, y_t are universal we can calculate all critical exponents.

It gives correct sign of change away from m.f.t.

$$-(\beta H)' = -\langle U\rangle_0 + \underbrace{\tfrac{1}{2}\left(\langle U^2\rangle_0 - \langle U\rangle_0^2\right)}_{\text{generates LR inteactions}}$$

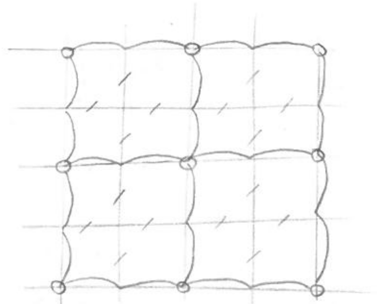


Requires 3 parameter RG, K_1, K_2, K_3

$y_t \simeq 1.053$ } fortuitously good.

VD Migdall - Kadanoff Decimation

b = 2 RG



Move bonds $\not-$ to $\frown$

1/ Bond moving $\tilde{K} = 2K$

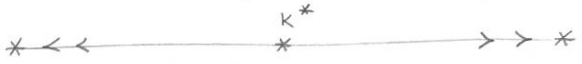
2/ Decimation

$K' = f_{1d\,RG}(2K)$

$$K' = \frac{1}{2}\ln\text{Cosh.}(2\times 2K)$$

i) $K^* = 0$, K small $K' \simeq 4K^2$ stable

ii) $K^* = \infty$, K large $K' \simeq 2K$ stable



iii) $K^* = \frac{1}{2}\ln\text{Cosh}\,4K^*$, $K^* \simeq 0.305$ ($K_c = 0.441$ exact)

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} \Rightarrow y_t \simeq 0.7472$$

($y_t = 1$ exact, as Δ since universal for all 2d)

* Easily generalised to higher dimensions

In d-dim, and for scaling b, require b^{d-1} strengthening

bond moving $\tilde{K} = b^{d-1}K$

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2/ Decimation

$$K' = f_{1dRG}\,(b^{d-1}K)$$

e.g. $d=3$, $b=2$

$$K' = \frac{1}{2}\ln\cosh(2^2\times 2K)$$

Some structure of flows, $K^* = 0.065$ $(K_c = 0.221$ Best estimate$)$

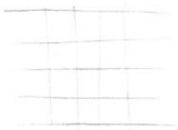
$y_t \simeq 0.934$ $(y_t = 1.59$ Best estimate$)$

Approximation gets worse in higher d

VI SERIES EXPANSION

VI A low T expansion by including excitations around the ground state

e.g. the Ising model has a ground state will all spins ↑, and excitations are islands of opposite spins



d-dimensional cube

$$Z = \sum_{\{\sigma_i\}} e^{K\sum_{\langle ij\rangle}\sigma_i\sigma_j} = e^{dNK}\Big[1 + \underbrace{N e^{-2K.2d}}_{\text{1 spin flip}} + \underbrace{dN e^{-2K(4d-2)}}_{\text{2 adjacent spin-flips}}$$

$$+ \underbrace{\frac{N(N-1-2d)}{2} e^{-8Kd}}_{\text{2 isolated spin flips}} + \cdots\Big]$$

$$\frac{\ln Z}{N} = dK + e^{-4dK} + d e^{-4(2d-1)K}$$

$$- \frac{2d+1}{2} e^{-8Kd} + \frac{N}{2} e^{-8Kd} + \cdots - \frac{N}{2} e^{-8Kd} + \cdots$$

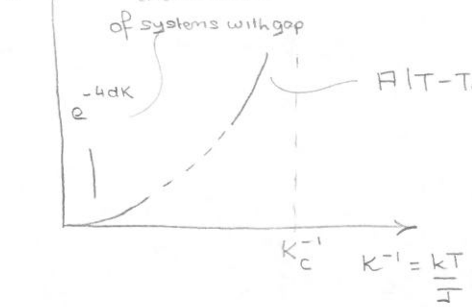
$\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2$

Only connected contributions are left → extensive

- Energy $$E = -\frac{\partial \ln Z}{\partial\beta} = -J\frac{\partial \ln Z}{\partial K} = -JN\Big[d - 4de^{-4dK} - 4d(2d-1)Ke^{-4(2d-1)K}$$ $$+ 4(2d+1)e^{-8dK} + \cdots\Big]$$

- Heat Capacity

$$C = \frac{dE}{dT} \propto -\frac{\partial^2 \ln Z}{\partial K^2} = 16d^2 e^{-4dK} + 16d(2d-1)^2 e^{-4(2d-1)K} - 32d(2d+1)e^{-8dK}$$



- Need to find α, T_c from Series around $T=0$

$$C = \sum_{l=0}^{\infty} a_l u^L \qquad (\text{e.g. } u = e^{-2K})$$

$$\alpha\left(1-\frac{u}{u_c}\right)^{-\alpha} = 1 + \alpha\frac{u}{u_c} + \frac{\alpha(\alpha+1)}{2}\left(\frac{u}{u_c}\right) + \cdots$$

$$+ \frac{\alpha(\alpha+1)\ldots\alpha+L-1)}{L!}\left(\frac{u}{u_c}\right)^L$$

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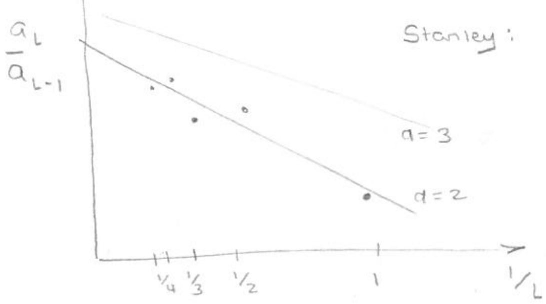
If series are identical, the ratio of consecutive terms, for example should become equal

- Equate the ratio of successive terms in the two expressions for C

$$\frac{a_L}{a_{L-1}} = \frac{\alpha + L - 1}{L u_c} = u_c^{-1}\left[1 - \frac{1-\alpha}{L}\right]$$

Suggests plot of a/a_{L-1} vs $1/L$ should asymptote to a straight line as $1/L \to 0$

with intercept u_c^{-1}, slope $-\frac{1-\alpha}{u_c}$

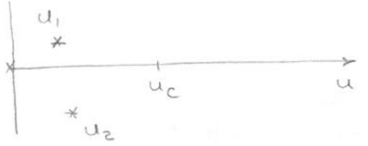


Stanley: $d = 2$, $\alpha = 0$, $d = 3$, $\alpha \simeq 0.12$ Both very good estimates

Remarks

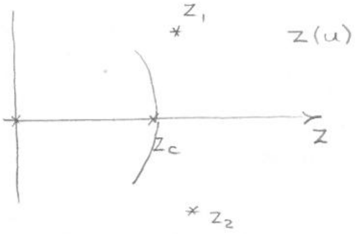
1/ Adding a regular background e.g. $C_{analytic} = \sum_{\gamma=0}^{M} b_\gamma u^\gamma$ does not change divergence or any singularity, but renders the first M terms in the expansion useless. A posteriori, such terms must be small

2/ Sometimes series is alternating, usually a sign of nearby singularities in the imaginary plane



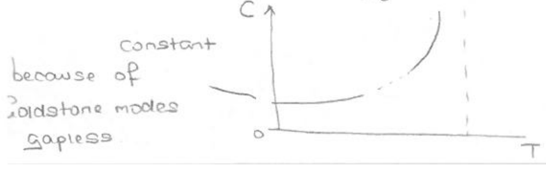
radius of convergence doesn't in general get to u_c

Resolution is to map other singularities further out (Padé method)



3/ For continuous spins the low energy excitations are extended spin waves

e.g. $\mathcal{Z} = \int_0^{2\pi} \prod_i d\theta_i \, e^{K \sum \cos(\theta_i - \theta_j)} \simeq e^{NKd} \int \prod_i d\theta_i \, e^{-K/2 \sum (\theta_i - \theta_j)^2}$



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VI B High T Expansions

Expand $e^{-\beta H}$ in powers of β

$$\mathcal{Z}(\beta) = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left[1 - \beta H + \frac{\beta^2}{2} H^2 - \dots\right]$$

$$\ln \mathcal{Z} = C - \beta \langle H \rangle_0 + \frac{\beta^2}{2} \langle H^2 \rangle^c - \dots$$

$\langle\ \rangle_0$ are with respect to non interacting spins

For Ising spins it more elegant to do the following high T expansion

$$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{K \sum \sigma_i \sigma_j} = \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} e^{K \sigma_i \sigma_j}$$

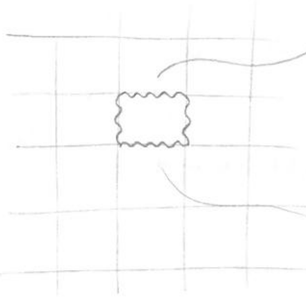
$$= \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} \left[\frac{e^K + e^{-K}}{2} + \sigma_i \sigma_j \frac{e^K - e^{-K}}{2} \right]$$

$$= \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} \mathrm{Cosh}\, K \left[1 + \tanh K \ \sigma_i \sigma_j \right]$$

Use $t = \tanh K$ as expansion parameter

$$\lim_{T \to \infty} \tanh K \simeq K \to 0$$

$$\mathcal{Z} = \underset{\text{No. of bonds}}{(\mathrm{Cosh}\, K)^{N_b}} \underbrace{\sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} (1 + t \sigma_i \sigma_j)}_{\text{represent the } 2^{N_b} \text{ terms graphically}}$$



presence of $t\sigma_i\sigma_j$ in expansion

first graph to appear in expansion

$$\text{Sum over } \sigma_i : \begin{cases} \text{For a site chosen } \overset{\to \text{ or even}}{0} \text{ times by bond} & \sum_{\sigma_i} 1 = 2 \\ \text{For a site chosen } \overset{\to \text{ or odd}}{1} \text{ time by bond} & \sum_{\sigma_i} \sigma_i = 0 \end{cases}$$

$$\mathcal{Z} = 2^N (\mathrm{Cosh}\, K)^{N_b} \sum_{\substack{\text{graphs with} \\ \text{even no. of bonds} \\ \text{per site}}} t^{\#\text{ bonds in graph}}$$

graphs with even no. of bonds per site } — All possible closed loops

For a d-dimensional hypercubic lattice

$$\mathcal{Z} = 2^N (\mathrm{Cosh}\, K)^{dN} \left[1 + N \frac{d(d-1)}{2} t^4 + \frac{dN(2d-2)(2d-3)}{2} t^6 + O(t^8) \right]$$

$$\frac{\ln \mathcal{Z}}{N} = \ln 2 + d \ln \mathrm{Cosh} K + \frac{d(d-1)}{2} t^4 + d(d-1)(2d-3) t^6$$

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* Use high T expansion for 1/ solution of 1d Ising model

2/ Duality

3/ High dimensions . MF

4/ Solve the 2d model exactly

VI C 1d Ising model

- Open boundary conditions

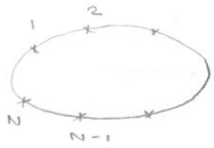
$$Z = 2^N (\text{Cosh } K)^{N-1} . 1$$ These are no closed loops

$$\langle \sigma_m \sigma_n \rangle = \frac{\sum_{\{\sigma_i\}} \sigma_m \sigma_n \prod_{\langle ij \rangle} e^{K\sigma_i\sigma_j}}{\sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} e^{K\sigma_i\sigma_j}} = \frac{\sum_{\substack{\text{graphs} \\ \text{for which } m,n \\ \text{act as end points}}}}{\sum_{\text{closed graphs}}}$$

$$= t^{|n-m|} = e^{-|n-m|/\xi}$$

$$\xi = - \frac{1}{\ln \tanh K}$$

- Closed chain



$$Z = 2^N (\text{Cosh } K)^N [1 + t^N]$$

$$\langle \sigma_m \sigma_n \rangle = \frac{t^{|n-m|} + t^{N-|n-m|}}{1 + t^N}$$ Can traverse around loop in either direction

VI D Duality

Compare high and low T expansions for square lattice

Low T

$$Z = e^{2KN}\left[1 + Ne^{-8K} + 2Ne^{-12K} + \dots\right]$$

$$= e^{2KN} \sum_{\substack{\text{islands of} \\ \ominus \text{ excitation}}} e^{-2K \times \text{Perimeter of island}}$$

$$\frac{\ln Z}{N} = 2K + e^{-8K} + 2e^{-12K} + \dots$$

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High T $$\mathcal{Z} = 2^N(\cosh K)^{2N}\left[1 + Nt^4 + 2Nt^6 + \cdots\right]$$

$$= 2^N(\cosh K)^{2N}\sum_{\text{Closed graphs}} t^{\#\text{ bonds in graph}}$$

$$\frac{\ln \mathcal{Z}}{N} = \ln 2\cosh^2 K + t^4 + 2t^6$$

* These are identical series

$$\frac{\ln \mathcal{Z}}{N} = 2K + g(e^{-2K}) = \ln 2\cos^2 K + g(\tanh K)$$

Both analytic => must be singular (because Ising model has a T_c)

So singularity must be mapped to itself $\quad e^{-2K_c} = \tanh K_c$

$$K_c = \ln\frac{\sqrt{2}+1}{2}$$

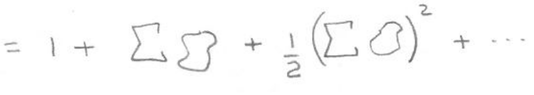
VI E High T Series in high dimensions

- Hypercubic lattice in d-dimensions

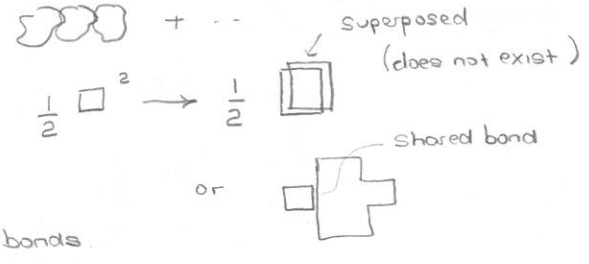
$$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{K\sum_{\langle ij\rangle}\sigma_i\sigma_j} = \sum_{\{\sigma_i\}}\prod_{\langle ij\rangle}\cosh K\,(1 + \overset{\tanh K}{t}\sigma_i\sigma_j) = \underbrace{2^N\cosh^{dN}K}_{\text{Analytic — no singularities so we will ignore this part}}\sum \text{all graphs with even \# bonds per site} \times t^{\#\text{ bonds}}$$

- Recalling cumulant expansions, examine

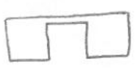
$\mathcal{Z}' = \exp\left[\sum \text{closed loops}\right] \equiv$ 



Note that $\mathcal{Z}'$ includes configurations not in $\mathcal{Z}$, e.g.



All such configurations are associated with shared bonds which are not allowed in the sum. $\mathcal{Z}$

In the same spirit, ignore self-intersections in 1 loop graphs e.g.  is included in $\mathcal{Z}'$ but not in $\mathcal{Z}$

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So
$$Z = \sum \text{all non-intersecting loops} \quad \Big\} \simeq ?$$
$$Z' = \exp\left(\sum \text{all loops}\right)$$

The loops can be thought of as polymers with some chemical potential

$$\ln Z' = \underbrace{N}_{\substack{\text{Walks start}\\ \text{at one of } N \text{ points}}} \sum_L \underbrace{\frac{t^L}{L}}_{\substack{\text{over counting}\\ \text{from } L \text{ possible starting}\\ \text{points on trajectory}}} \times \left(\begin{array}{c}\#\text{ random walks from 0 to 0}\\ \text{in } L \text{ steps}\end{array}\right)$$

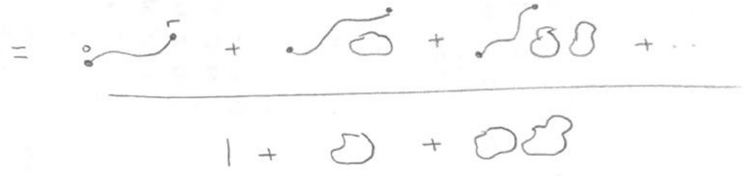
— e.g. $L = 4$ → 4 on square lattice in $d = 2$

Introduce a set of matrices $W(L)$ to count random walks $(N \times N)$

$$\langle j | W(L) | i \rangle = \#\text{ of random walks of } L \text{ steps from } i \text{ to } j$$

$$\frac{\ln Z'}{N} = \sum_L \frac{t^L}{L} \langle 0 | W(L) | 0 \rangle$$

Correlation Functions
$$\langle \sigma_0 \sigma_r \rangle = \frac{\sum_{\{\sigma_i\}} \sigma_0 \sigma_r \prod_{\langle ij \rangle} (1 + t\sigma_i\sigma_j)}{\sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} (1 + t\sigma_i\sigma_j)}$$



$\langle \sigma_0 \sigma_r \rangle \overset{\text{ignore}}{\underset{\text{intersections}}{\approx}}$

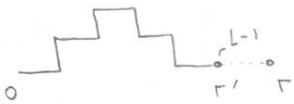
(because numerator can be factorised)

$$= \sum_L t^L \#\text{ of random walks from 0 to } r \text{ in } L \text{ steps}$$

$$= \sum_L t^L \langle 0 | W(L) | r \rangle$$

Count Random Walks using the Markovian (recursive) nature. i.e.

L step walk $= (L-1)$ walk $+ 1$ step



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# of ways from 0 to r in l steps $= \sum_{r'}$ # of ways from 0 to r' in (L-1) steps × # of ways from r' to r in 1 step

$$\langle r|W(L)|0\rangle = \sum_{r'} \langle r|W(1)|r'\rangle\langle r'|W(L-1)|0\rangle$$

$$= \sum_{r'} \langle r|W(1)\,W(L-1)|0\rangle$$

$W(L) = T\,W(L-1)$, where $T \equiv W(1)$, is the Transfer or Generating Matrix

$$= T\,T\,W(L-2)$$

$$= T^{L}$$

$$\langle j|T|i\rangle = \begin{cases} 1 & \text{if } i,j \text{ are nearest neighbours (or perhaps some more complicated connectivity)} \\ 0 & \text{otherwise} \end{cases}$$

e.g. for d = 2 square lattice

$$\langle xy|T|x'y'\rangle = \delta_{xx'}\left(\delta_{y,y'+1} + \delta_{y,y'-1}\right) + \delta_{y,y'}\left(\delta_{x,x'+1} + \delta_{x',x+1}\right)$$

To find $\langle r|W(l)|0\rangle$, multiply $\underset{\text{1 at 0 and 0 everywhere else}}{T^{L} \text{ on } |0\rangle}$

$$T \times \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}, \times T = \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix} \quad \text{etc.}$$

* Since $\langle r|T|r'\rangle = f(r-r')$ has translation symmetry, it can be diagonalised in the Fourier basis.

$$\langle r|q\rangle = \frac{1}{\sqrt{N}}\, e^{iq\cdot r}$$

$$T|q\rangle = T(q)\,|q\rangle$$

e.g. in d = 2

$$\langle x,y|T|q\rangle = \sum_{x'y'} \langle x,y|T|x',y'\rangle\langle x',y'|q\rangle$$

$$= \sum_{x'y'} \frac{e^{i(q_x x' + iq_y y')}}{\sqrt{N}} \times \left[\delta_{xx'}\left(\delta_{y,y'+1} + \delta_{y,y'-1}\right) + \delta_{yy'}\left(\delta_{x,x'+1} + \delta_{x,x'-1}\right)\right]$$

$$= \frac{e^{i(q_x x + q_y y)}}{\sqrt{N}}\left(e^{iq_y} + e^{-iq_y} + e^{iq_x} + e^{-iq_x}\right) = \langle x,y|q\rangle \times \left(2\cos q_x + 2\cos q_y\right)$$

for d-dim.

$$T(q) = 2\sum_{\alpha=1}^{d} \cos q_\alpha$$

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$$\langle \sigma_0 \sigma_r \rangle' = \sum_L t^L \langle r | T^L | 0 \rangle$$

$$= \langle r | \sum_L (tT)^L | 0 \rangle = \langle r | \frac{1}{1-tT} | 0 \rangle$$

$$= \sum_q \frac{\langle r|q\rangle \langle q|0\rangle}{1 - tT(q)} = \cancel{N} \int \frac{d^dq}{(2\pi)^d} \frac{e^{iq.r}}{\cancel{N}} \frac{1}{1 - 2t\sum_\alpha \cos q_\alpha}$$

(For very high T, $t \to 0$ expect contribution of shortest path to be dominant

$$\langle \sigma_0 \sigma_r \rangle' \simeq t^{|r|} = e^{-|r|/\xi}$$

At lower T, other trajectories appear with high weight but large entropy – like fluctuations of line with tension)

First singularity occurs for denominator $= 0$, at $(q=0)$ $1 - 2td = 0$

$$\Rightarrow t_c = 1/2d$$

represents entropy

For large r behaviour, look at small q

denominator $= 1 - 2td + t\underline{q}^2 + O(q^4) \simeq t(q^2 + \xi^{-2})$

$$\langle \sigma_0 \sigma_r \rangle' = \frac{1}{t} \int \frac{d^dq}{(2\pi)^d} \frac{e^{iq.r}}{(q^2 + \xi^{-2})}$$

$$= \begin{cases} \dfrac{1}{r^{d-2}} & |r| \ll \xi \\[2ex] \dfrac{e^{-|r|/\xi}}{r^{(d-1)/2}} & |r| \gg \xi \end{cases} \qquad \eta = 0$$

* Correlation length

$$\xi = \left(\frac{t}{1 - 2dt} \right)^{1/2} \sim (t_c - t)^{-1/2}, \qquad \nu = 1/2$$

* Find behaviour identical to Gaussian model

$$\langle m(r) m(0) \rangle \simeq \int \mathcal{D}m(x)\, m(r)\, m(0)\, e^{-\beta H[m]}$$

Have Field / Particle duality (Parisi ch 13 + 16)

$$\frac{\ln Z}{N} = \sum_L \frac{t^L}{L} \langle 0 | T^L | 0 \rangle = \langle 0 | \sum_L \frac{(tT)^L}{L} | 0 \rangle$$

$$= - \langle 0 | \ln(1 - tT) | 0 \rangle$$

$$= -\sum_q |\langle 0|q\rangle|^2 \ln(1 - tT(q)) = -N \int \frac{d^dq}{(2\pi)^d} \frac{1}{N} \ln\left(1 - 2t\sum_\alpha \cos q_\alpha\right)$$

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$$C \sim -\frac{\partial^2 f}{\partial T^2} \sim -\frac{\partial^2 f}{\partial t^2} \sim \int \frac{d^dq}{(2\pi)^d} \frac{1}{\left(1 - 2t \sum_\alpha \cos q_\alpha\right)^2}$$

$$\sim \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + \xi^{-2})^2} \overset{2<d<4}{\sim} \xi^{4-d} \sim (t_c - t)^{d/2 - 2}$$

$$\alpha = 2 - d/2$$

as in Gaussian approximation

Does the lattice approximation become reliable in high dimensions ?

(in GM, when $t<0$ get $m \to \infty$ nonsense analagously get  multiple intersections)

* Random Walk of L steps covers a size $R \sim L^{1/2}$

Define fractal dimension from Mass $M \sim R^{d_f}$. For random walk, $M \sim L \sim R^2$

Fractal dimension $d_f = 2$ for random walks

* Two objects of dimensions d_1 and d_2 geometrically intersect only in dimensions $d \leq d_1 + d_2$

e.g.

two lines must intersect in $d=2$, but not in $d=3$

So two random walks generally intersect in $d \leq 2+2 = 4$

Intersections corresponds to interactions (like m^4)

analogy is clear.

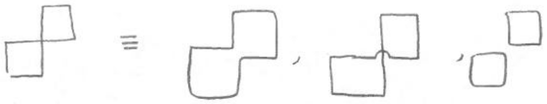
VI F EXACT PARTITION FUNCTION FOR ISING MODEL ON A SQUARE LATTICE

$$Z = \sum_{\{\sigma_i\}} e^{K \sum_{\langle ij \rangle} \sigma_i \sigma_j} = 2^N \cosh^{2N} K \underbrace{\sum \text{all graphs with even \# bond per site}}_{S} \times \underbrace{t^{\# \text{bonds}}}_{t = \tanh K}$$

Previously we found that $S \neq S' = \sum$ all multi-loop random walks on the lattice $\times\, t^{\# \text{bonds}}$

$S' > S$ overcounts graphs due to intersections. Two problems

a) Intersection's on a site → over-counting



3 possible random walks

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(b) Unacceptable graphs due to intersections on a bond

e.g.

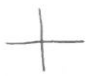
Assertion

S = $\sum$ all closed multi-loop RW's, with no backward steps, i.e  ← no U'turns

$\times\ (-1)^{\#\text{ of intersections},\ n_i}$

defined properly later

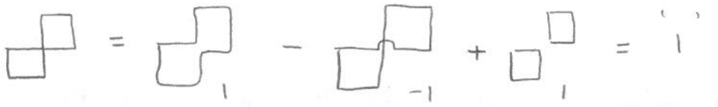
Proof

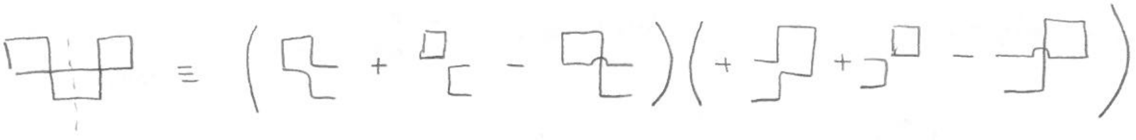
(a) Site intersection problem removed since for each  , correspond

to the 3 configurations  adding to give

Intersection +1 +1 −1 = +1

So





b) An unacceptable double occupation of the bond corresponds to two RW

Only  back steps are ambiguous but excluded by definition

If RW* = RW;  , $(-1)^{n_c}$

$$S = 1 + \sum \text{all 1 loop RW}^* + \sum \text{all 2 loops RW}^* + \cdots$$

$$= \exp\left(\sum \text{1 loop RW}^*\right)$$

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$$\exp\left(\sum \square\right) = 1 + \sum \square + \frac{1}{2}\left(\sum \square\right)^2 + \cdots$$

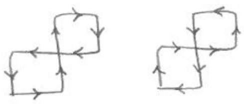
Note two loops intersect an even no. of times
and the signs are preserved

$$\ln Z = N \ln(\text{Cosh}^2 K) + \sum_{L,r} \frac{t^L}{L} \langle r | W^*(L) | r \rangle$$

where $\langle r | W^*(L) | r \rangle$ = sum over an L step RW from r to r, $\oslash$, $(-1)^{n_c}$

- n_c is a non-local (or non Markovian) quantity that makes the computation of $W^*(L)$ difficult.

* Give walks direction by putting an arrow on bonds

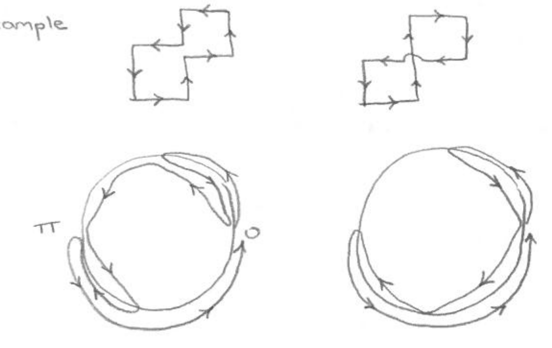


Then $\langle r | W^*(L) | r \rangle = \frac{1}{2} \sum$ all directed RW's from r to r, $\oslash$, $(-1)^{n_c}$

* Whitney Theorem

Consider a planar loop with n_c intersections. Let Θ indicate the total angle that the tangent vector rotates in going around the loop. Then $(n_c)_{\text{mod } 2} = \left(1 + \frac{\Theta}{2\pi}\right)_{\text{mod } 2}$

for example



$\Theta = 2\pi$ $\qquad\qquad$ $\Theta = 0$

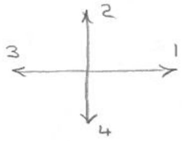
$n_c = \left(1 + \frac{2\pi}{2\pi}\right)_{\text{mod} 2} = 0$ $\qquad$ $n_c = 1_{\text{mod } 2} = 1$

$$(-1)^{n_c} = e^{i\pi n_c} = e^{i\pi(1 + \Theta/2\pi)} = -e^{i\Theta/2} = -e^{+\frac{i}{2}\sum_{j=1}^{L} \Delta\Theta_j}$$

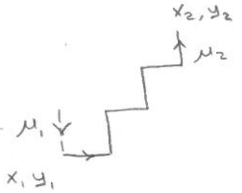
Angle of rotation after step j
and is a local quantity.

$$\langle r | W^*(L) | r \rangle = -\frac{1}{2} \sum \text{all directed RW's from } r \text{ to } r', \oslash, \; e^{\frac{i}{2}\text{Angle related by tangent}}$$

To calculate rotation of tangent keep track of the orientation of the bonds, μ



Define $\langle x_2, y_2; \mu_2 | W^*(L) | x_1, y_1; \mu_1 \rangle = \sum$ all directed RW's, $\oslash$ departing x_1, y_1 with respect to μ_1 and arriving at x_2, y_2 along direction μ_2 $\times\, e^{i/2 \text{ Angle of rotation of tangent}}$



μ_1 is the direction before starting walk

$$\langle x_2, y_2; \mu_2 | W^*(L) | x_1, y_1; \mu_1 \rangle$$

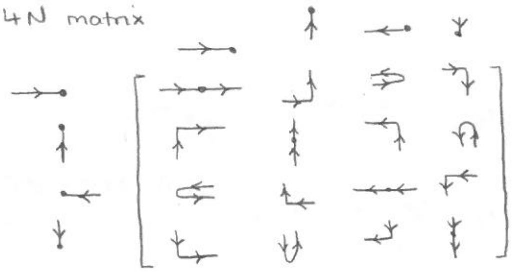
$$= \sum_{x' y' \mu'} \langle x_2, y_2; \mu_2 | W^*(1) | x', y', \mu' \rangle \langle x', y', \mu' | W^*(L-1) | x_1, y_1, \mu_1 \rangle$$

$$W^*(L) = W^*(1) W^*(L-1)$$

$$= W^*(1)^L = T^L, \quad T = W^*(1)$$

T is a 4N × 4N matrix

$T =$



$\langle xy | T | x', y' \rangle$

$$= \begin{bmatrix} \langle xy | x'+1, y' \rangle & \langle xy | x', y'+1 \rangle e^{i\pi/4} & 0 & \langle xy, x', y'-1 \rangle e^{-i\pi/4} \\ \langle xy | x'+1, y' \rangle e^{-i\pi/4} & \langle xy | x', y'+1 \rangle & \langle xy, x'-1, y' \rangle e^{i\pi/4} & 0 \\ 0 & \langle xy | x', y'+1 \rangle e^{-i\pi/4} & \langle xy, x'-1, y' \rangle & \langle xy, x', y'-1 \rangle e^{i\pi/4} \\ \langle xy | x'+1, y' \rangle e^{i\pi/4} & 0 & \langle xy, x'-1, y' \rangle e^{-i\pi/4} & \langle xy, x', y'-1 \rangle \end{bmatrix}$$

* Make Bloch diagonal by going to Fourier basis $\langle xy | q_x q_y \rangle = \frac{1}{\sqrt{N}} e^{iq_x x + iq_y y}$

$$T[q] = \begin{bmatrix} e^{-iq_x} & e^{-iq_y + i\pi/4} & 0 & e^{iq_y - i\pi/4} \\ e^{-iq_x - i\pi/4} & e^{-iq_y} & e^{iq_x + i\pi/4} & 0 \\ 0 & e^{-iq_y - i\pi/4} & e^{iq_x} & e^{iq_y + i\pi/4} \\ e^{-iq_x + i\pi/4} & 0 & e^{iq_x - i\pi/4} & e^{iq_y} \end{bmatrix}$$

$$\ln \mathcal{Z} = N \ln [2 \cosh^2 K] - \underset{\text{rotations}}{\frac{1}{2}} \underset{\text{directions}}{\sum_L} \frac{t^L}{L} \underbrace{\sum_{r\mu} \langle r\mu | T^L | r\mu \rangle}_{\mathrm{Tr}\, T^L}$$

directions μ must be same

$$= N \ln [2 \cosh^2 K] - \frac{1}{2} \mathrm{Tr} \left(\sum_L \frac{(tT)^L}{L} \right)$$

$$= N \ln [2 \cosh^2 K] + \frac{1}{2} \mathrm{Tr} \ln (1 - tT)$$

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$$\mathrm{Tr}\ln M = \sum_\alpha \ln M_\alpha = \ln \prod_\alpha M_\alpha = \ln\det M$$

$$\ln Z = N\ln 2\cosh^2 K + \frac{1}{2}\underbrace{\ln\det(1-tT)}_{\sum_q \ln\det(1-tT(q))}$$

$$\frac{\ln Z}{N} = \ln 2\cosh^2 K + \frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\ln\det\begin{vmatrix} 1-te^{-iq_x} & -te^{-iq_y+i\pi/4} & 0 & -te^{iq_y-i\pi/4} \\ -te^{-iq_x-i\pi/4} & 1-te^{-iq_y} & -te^{iq_x+i\pi/4} & 0 \\ 0 & -te^{-iq_y-i\pi/4} & 1-te^{iq_x} & -te^{iq_y+i\pi/4} \\ -te^{-iq_x+i\pi/4} & 0 & -te^{iq_x-i\pi/4} & -te^{iq_y} \end{vmatrix}$$

Recall what we've done so far

$$Z = \sum e^{K\sum\sigma\sigma} = 2^N\cosh^{2N}K \sum \text{even} \times t^{\#\text{bonds}}$$

$$= 2^N\cosh^{2N}K \sum \text{all closed, multi-loop RW } ⊘,\ (-1)^{\#\text{intersections}} \times t^{\#\text{bonds}} \qquad \text{in 2d}$$

$$\frac{\ln Z}{N} = \frac{1}{N}\ln 2^N\cosh^{2N}K \pm\frac{1}{2N}\sum \text{all one-loop closed RW, } ⊘\text{, directed, } e^{\frac{i}{2}\text{angle of tangent}}\ t^{\#\text{bonds}}$$

$$= \ln 2\cosh^2 K - \frac{1}{2N}\sum\frac{t^l}{L}\sum\langle r,\mu|T^{*L}|r,\mu\rangle$$

$$= \ln 2\cosh^2 K - \frac{1}{2N}\mathrm{Tr}\left(\sum\frac{(tT^*)^l}{L}\right)$$

$$= \ln 2\cosh^2 K + \frac{1}{2N}\mathrm{Tr}\left[\ln(1-tT^*)\right]$$

$$= \ln 2\cosh^2 K + \frac{1}{2N}\sum_q \mathrm{Tr}\ln\left[1-tT^*(q)\right]$$

$$= \ln 2\cosh^2 K + \frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\ln\det\left[\text{shown above}\right]$$

$$= \ln 2\cosh^2 K + \frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\ln\left[(1+t^2)^2 - 2t(1-t^2)(\cos q_x + \cos q_y)\right]$$

$$\frac{\ln Z}{N} = \ln 2\cosh^2 K + \frac{1}{2}\int_0^{2\pi}\frac{dq_x\,dq_y}{(2\pi)^2}\ln\left[\cosh^2(2K) - \sinh^2(2K)(\cos q_x + \cos q_y)\right]$$

We need to understand the singular behaviour of the partition function

* Compare singularity of $\ln Z$ with unrestricted RW's

$$\frac{\ln Z'}{N} = \ln 2\cosh^2 K - \int\frac{d^2q}{(2\pi)^2}\ln\left[\underbrace{1-2t(\cos q_x+\cos q_y)}_{A(t,q)}\right]$$

* Singularity when argument of ln vanishes

first vanishing occurs at $q=0$; $A(t_c,0) = 0 = 1-4t_c \Rightarrow t_c = 1/4$

$$A(t\sim t_c, q\sim 0) = 1-4t + tq^2 + O(q^4)$$

$$= t_c\left[q^2 + 4\frac{\delta t}{t_c}\right] \qquad \delta t = t_c - t$$

$$\frac{\ln Z'}{N}\Big|_{\rm Sing.} = -\int_0^{\Lambda}\frac{2\pi q\,dq}{4\pi^2}\ln\left[q^2+4\delta t/t_c\right]$$

$$= -\frac{1}{4\pi}\left\{\left(q^2+4\delta t/t_c\right)\ln\left(\frac{q^2+4\delta t/t_c}{e}\right)\right\}\Big|_0^{\Lambda}$$

$$\frac{\ln Z'}{N}\Big|_{\rm Sing} = \frac{1}{\pi}\frac{\delta t}{t_c}\ln\left(\frac{\delta t}{t_c}\cdot\frac{4}{e}\right)$$

$$C_{\rm sing} \sim \frac{\partial^2 \ln Z}{\partial t^2} \sim (\delta t)^{-1}, \qquad \alpha = 1, \quad (\alpha = 2 - d/2 \text{ from Gaussian model})$$

$$A^*(t,q) = (1+t^2)^2 - 2t(1-t^2)(\cos q_x + \cos q_y)$$

$$A^*(t_c,0) \underset{,}{=} 0 = (1-t_c^2)^2 + 4t_c^2 - 4t_c(1-t_c^2)$$

$$= (1-t^2-2t)^2$$ – So the argument of the log never becomes negative

$$t_c^2 + 2t_c - 1 = 0 \Rightarrow t_c = -1+\sqrt{2}$$ Other solution doesn't apply

(c.f. Duality $e^{-2K_c} = \tanh K_c = \sqrt{2}-1 \equiv t_c$

$$K_c = \tfrac{1}{2}\ln(\sqrt{2}+1) = 0.44$$)

$$A^*(t\sim t_c, q\sim 0) = \Big[\underbrace{(-2t_c-2)}_{-2\sqrt{2}}\delta t\Big]^2 + t_c\underbrace{(1-t_c)^2}_{2t_c}q^2 + O(q^4)$$

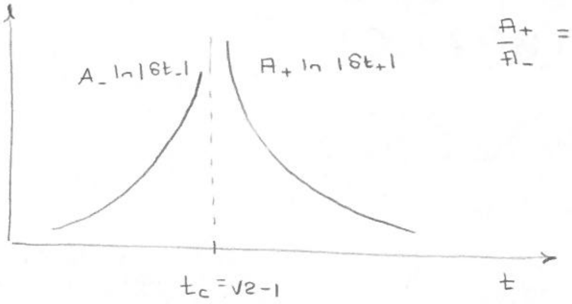
$$= 2t_c^2\left[q^2 + \frac{8(\delta t)^2}{2t_c^2}\right]$$

$$\frac{\ln Z}{N}\Big|_{\rm Sing} = \frac{1}{2}\int_0^{\Lambda}\frac{2\pi q\,dq}{4\pi^2}\ln\left[q^2+\frac{8(\delta t)^2}{2t_c^2}\right]$$

$$= \frac{1}{8\pi}\left\{\left(q^2+4(\delta t/t_c)^2\right)\ln\left(q^2+4(\delta t/t_c)^2\right)\right\}_0^{\Lambda}$$

$$\frac{\ln Z}{N}\Big|_{\rm Sing} = -\frac{1}{\pi}\left(\frac{\delta t}{t_c}\right)^2\ln\left|\frac{\delta t}{t_c}\right|$$

$$C_{\rm sing} \sim \frac{\partial^2\ln Z}{\partial t^2} \sim \ln|\delta t|$$



Since δt enters as square

$\frac{A_+}{A_-} = 1$ i.e. symmetric peak

$\alpha = 0$, "ln"

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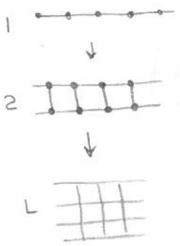
Ising model solved by Lars Onsager (1944) Phys. Rev.

Solved by Transfer matrix

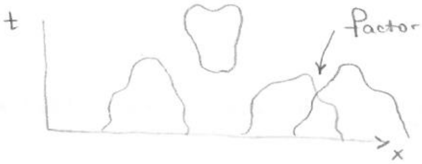


— diagonalise $2^L \times 2^L$

B Kaufman (1949) Chapter 15 of Huang

Kac-Wood, Feynman (50's) Graphical method in Parisi

Mattis, Lieb and Schulz : World lines of fermions



Factor of (-1) at intersection

cancels with)(and corresponds to Pauli exclusion

* Correlations

$$\langle \sigma_0 \sigma_r \rangle \sim \frac{e^{-r/\xi}}{r^2} \quad , \quad \xi \propto (\delta t)^{-\nu} \quad , \quad \nu = 1 \qquad \left(\text{c.f.} \quad q^2 + (\delta t)^2 \right)$$

Hyperscaling $\alpha = 2 - d\nu$ satisfied

At T_c $\eta = 1/4$ – harder to determine

Susceptibility $\chi \sim \int d^2r \langle \sigma_0 \sigma_r \rangle \sim (\delta t)^{-\gamma}$ $\quad \gamma = 7/4$

Magnetisation (Onsager) C.N. Yang

$$m \sim (\delta t)^{\beta} \qquad \beta = 1/8$$

But $f(t,h)$ has not been solved and 3d

VII Continuous Spin Systems near Two-Dimensions

VII A Non-Linear σ Model

O(n) model on lattice $\underline{S}_i$, $|\underline{S}_i| = 1$

$$\beta H = K \sum_{\langle ij \rangle} \underline{S}_i \cdot \underline{S}_j = -\frac{K}{2} \sum_{\langle ij \rangle} \left[(\underline{S}_i - \underline{S}_j)^2 - 2 \right]$$

$$\xrightarrow[\text{limit}]{\text{continuum}} \; -\frac{K}{2} \int d^dx \, (\nabla \underline{S})^2 \qquad \text{where} \quad |\underline{S}(x)| = 1$$

$$* \quad \sum_{\alpha} S_\alpha(x)^2 = 1$$

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$T=0, \quad \underline{S}(x) = (0,0,\dots,1)$

At T small $\quad \underline{S}(x) = (\pi_1(x), \pi_2(x), \dots \underbrace{(1-\pi^2)^{1/2}}_{\sigma(x)})$

$$\sigma(x) = \left(1 - \sum_{\alpha}^{n-1} \pi_\alpha^2\right)^{1/2}$$

$$-\beta H = \frac{K}{2}\int d^dx \left[(\nabla\pi)^2 + \left(\nabla\sqrt{(1-\pi^2)}\right)^2 \right]$$

$$= \frac{K}{2}\int d^dx \left[(\nabla\pi)^2 + \left(\nabla \frac{\pi^2}{2}\right)^2 + \dots \right]$$

* At lowest order

$$-\beta H = \frac{K}{2}\int d^dx\, (\nabla\pi)^2 \xrightarrow{F.T.} \frac{K}{2}\sum_q q^2 |\pi_q|^2$$

$$\langle |\pi_q|^2 \rangle = \frac{n-1}{Kq^2} \quad \propto (k_B T)$$

$$\langle |\pi(x)|^2 \rangle = \int \frac{d^dq}{(2\pi)^d} \langle |\pi_q|^2\rangle = \int_{1/L}^{1/a} \frac{d^dq}{(2\pi)^d} \frac{n-1}{Kq^2}$$

$$= \frac{n-1}{K} \frac{S_d}{(2\pi)^d} \frac{a^{2-d} - L^{2-d}}{d-2}$$

$$\underset{L\to\infty}{=} \begin{cases} \dfrac{(n-1)K_d}{K} a^{2-d} \propto T & d > 2 \text{ (and always be chosen} \ll 1) \\ \dfrac{(n-1)K_d}{K} L^{2-d} \to \infty & d \lesssim 2 \end{cases}$$

($\log L/a$ in $d=2$ and a low T expansion runs into trouble. Fluctuations are large, no LRO possible)

* Polyakov (75)

$$T_c = O(d-2)$$

* Combine low T expansion with a d-2 expansion to get critical behaviour.

* $$Z = \int \mathcal{D}S(x) \exp\left[-\frac{K}{2}\int d^dx\, (\nabla S)^2\right] \delta(S^2-1)$$

$$= \int \mathcal{D}\pi\, \mathcal{D}\sigma\, \delta(\pi^2+\sigma^2-1) \exp\left[-\tfrac{K}{2}\int d^dx \left[(\nabla\pi)^2 + (\nabla\sqrt{(1-\pi^2)})^2\right]\right]$$

$$\prod_i \int d\sigma_i\, d\pi_i\, \delta(\pi_i^2+\sigma_i^2-1) = \prod_i \int d\sigma_i\, d\pi_i\, \delta\left[(\sigma_i - \sqrt{1-\pi_i^2})(\sigma_i + \sqrt{1-\pi_i^2})\right]$$

$\delta(xa) = \dfrac{\delta(x)}{|a|}$

$$= \prod_i \int \frac{d\pi_i}{2\sqrt{(1-\pi_i^2)}}$$

$$= \prod_i \int \frac{d\pi_i}{2} e^{-1/2 \ln(1-\pi_i^2)}$$

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$$\int \mathcal{D}\underline{S}(x) \sim \;=\; \int \mathcal{D}\Pi(x) \exp\left[-\frac{1}{2}\int d^dx \underbrace{\ln(1-\Pi^2)}_{\text{Corresponds to new measure}} . \underbrace{\rho}_{\text{density of sites.}}\right]$$

$$Z = \int \mathcal{D}\Pi(x) \; \exp\left[-\frac{K}{2}\int d^dx \left[(\nabla\Pi)^2 + (\nabla\sqrt{1-\Pi^2})^2\right] - \frac{\rho}{2}\int d^dx \ln(1-\Pi^2)\right]$$

Order terms in Hamiltonian in powers of $1/K$

$$K\langle\Pi^2\rangle \sim 1/K \times K \sim 1$$

$$\therefore \quad K(\nabla\Pi^2)^2 \sim 1/K$$

$$\rho\Pi^2 \sim 1/K$$

Treat $+\beta H_0 = \frac{K}{2}\int d^dx \; (\nabla\pi)^2$ as unpeturbed Hamiltonian

$O(1/K)$ correction $\quad \frac{K}{2}\int d^dx \; (\underbrace{\Pi_\alpha\nabla\Pi_\alpha}_{\text{To keep track of contracted indicies}})^2 - \frac{\rho}{2}\int \pi^2 d^dx \equiv \int d^dx \; U(x)$

Terms of $O(1/K^2)$ neglected

In terms of normal modes

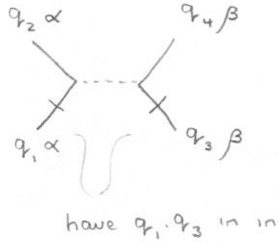
$$-\beta H_0 = -\frac{K}{2}\int \frac{d^dq}{(2\pi)^d} \quad q^2 |\Pi(q)|^2$$

$$U = -\frac{K}{2}\int \frac{d^dq_1 \, d^dq_2 \, d^dq_3}{(2\pi)^{3d}} \; \Pi_\alpha(q_1)\Pi_\alpha(q_2)\Pi_\beta(q_3)\Pi_\beta(-q_1-q_2-q_3) \; q_1\cdot q_3$$

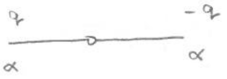
$$-\frac{\rho}{2}\int \frac{d^dq}{(2\pi)^d} \quad |\Pi(q)|^2$$

$$\langle\Pi_\alpha(q)\Pi_\beta(q_1)\rangle_0 = \frac{\delta_{\alpha\beta}\;(2\pi)^d\,\delta^d(q+q_1)}{Kq^2}$$

$U \rightarrow$



have $q_1\cdot q_3$ in interaction



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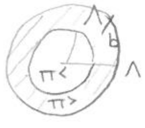
× Combine perturbation theory with Renormalisation Group

1/ Coarse Grain



$$\pi = \pi^> + \pi^< = \begin{cases} \pi^> & \Lambda/b < q < \Lambda \\ \pi^< & 0 < q < \Lambda/b \end{cases}$$

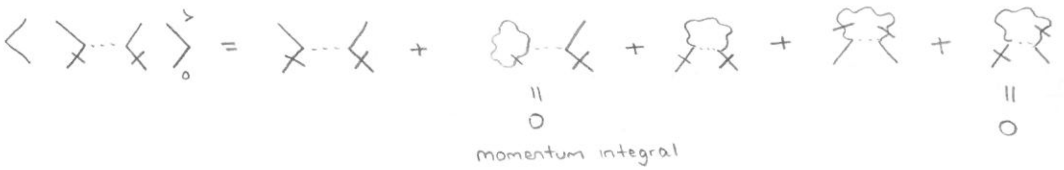
$$Z \sim \int \mathcal{D}\pi^< \mathcal{D}\pi^> \exp\left[-\beta H_0[\pi^>] - \beta H_0[\pi^<] - U[\pi^>, \pi^<]\right]$$

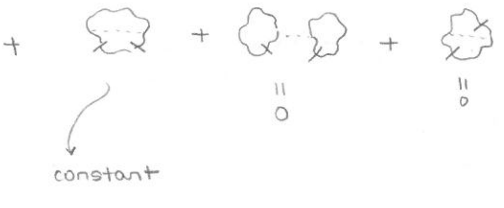
$$\sim \int \mathcal{D}\pi^< \, e^{-\underbrace{\delta f_b^0}_{\text{from bare integral}} - \beta H_0[\pi^<]} \left\langle e^{-U[\pi^>, \pi^<]} \right\rangle_0^>$$

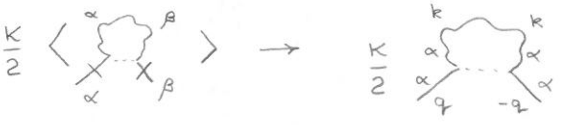
$$-\ln\langle e^{-U}\rangle = \langle U\rangle - \frac{1}{2}\langle U^2\rangle_0^c + O(U^3)$$ Cumulant expansion

$-\widetilde{\beta H} = +\delta f_b^0 - \beta H_0[\pi^<] +$  $+ \; O(1/K^2)$

$\pi = \pi^< + \pi^>$: 







$$= \frac{K}{2}\int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^d} \, |\pi^<(q)|^2 (-q^2) \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{Kk^2}$$

$$= -\frac{K}{2}\int_0^{\Lambda/b} \frac{d^dq}{(2\pi)^d} \, q^2 |\pi^<(q)|^2 \; \frac{I_d(b)}{K}$$

where $I_d(b) = \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} = K_d \int_{\Lambda/b}^{\Lambda} dk \; k^{d-3}$

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$\frac{K}{2}\Big\langle$  $\Big\rangle = \frac{K}{2}\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\,|\Pi^<(q)|^2\int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{(-k^2)}{Kk^2}$

$$= -\frac{1}{2}\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\,|\Pi^<(q)|^2\rho\,(1-b^{-d})$$

$$\rho = \frac{N}{V} = \int_0^{\Lambda}\frac{d^dq}{(2\pi)^d} = b^d\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}$$ ⇒ Subtracting and find

$$-\beta\tilde{H} = \delta f_b^0 + \delta f_b^1 - \frac{K}{2}\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\,q^2|\Pi^<(q)|^2\left(1+\frac{I_d(b)}{K}\right)$$

$$+\frac{\rho}{2}\int_0^{\Lambda/b}\frac{d^dq}{(2\pi)^d}\,|\Pi^<(q)|^2\left(1-1+b^{-d}\right)$$

$+\ \frac{K}{2}$  $+\ O(1/K^2)$

$$-\beta\tilde{H} = \delta f_b^0 + \delta f_b^1 - \frac{\tilde{K}}{2}\int_b d^dx\,(\nabla\Pi^<)^2 + \frac{\rho}{2}b^{-d}\int_b d^dx\,|\Pi^<(x)|^2 - \frac{K}{2}\int_b d^dx\,(\Pi_\alpha^<\nabla\Pi_\alpha^<)^2 + O(1/K^2)$$

$$\tilde{K} = K\left(1+\frac{I_d(b)}{K}\right)$$

2) Rescale

$x' = x/b$ and

3) Renormalise

$\underline{S}' = \tilde{\underline{S}}/\xi\ ,\qquad \Pi^< = \xi\,\Pi'$

$$-\beta H' = \text{const.} - \frac{\tilde{K}}{2}b^{d-2}\xi^2\int d^dx'\,(\nabla\Pi')^2 + \frac{\rho}{2}\xi^2\int d^dx'\,\Pi'^2 - \frac{K}{2}b^{d-2}\xi^4\int d^dx'\,(\Pi'\nabla\Pi')^2$$

$$+\ O(1/K^2)$$

Spherical symmetry ensures that there is only one K', but need high orders — i.e. get $\tilde{K}$ at all stages.

orrection $O(K^0)$

$$K' = b^{d-2}\xi^2K\left(1+\frac{I_d(b)}{K}\right)$$

Finding ξ

$$\underline{S} = \left(\Pi_1^> + \Pi_1^<, \dots, \sqrt{(1-\Pi^{>2}-\Pi^{<2}}\ )\right)$$

$$\tilde{\underline{S}} = \langle\underline{S}\rangle_0^> = \left(\Pi_1^<, \dots, 1-\frac{\Pi^{<2}}{2}-\frac{\langle\Pi^{>2}\rangle}{2}+\dots\right)$$

$$= \underbrace{\left(1-\frac{\langle\Pi^{>2}\rangle}{2}\right)}_{\xi}\underbrace{\left(\Pi_1^<, \dots, 1-\frac{\Pi^{<2}}{2}\right)}_{\underline{S}'}$$

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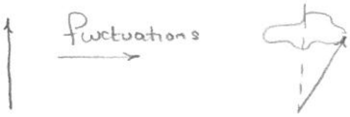


average along z diminishes

0 in other directions

$$\xi = 1 - \frac{1}{2}(n-1)\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\,\frac{1}{Kk^2} = 1 - \frac{n-1}{2K}\, I_d(b)$$

$$K' = Kb^{d-2}\left[1 + \frac{I_d(b)}{K}\right]\left[1 - \frac{n-1}{2K}\, I_d(b)\right]^2$$

$$= K\,b^{d-2}\left[1 - \frac{n-2}{K}\, I_d(b)\right]$$

* Differential recursion relations

$$b = e^{\delta l} = 1 + \delta l$$

$$K + \delta l\,\frac{dK}{dl} = K\left(1 + (d-2)\delta l\right)\left(1 - \frac{n-2}{K}\,.\,K_d\,\Lambda^{d-2}\delta l\right)$$

$$\frac{dK}{dl} = (d-2)K - (n-2)K_d\,\Lambda^{d-2}$$

* Set $T = K^{-1}$

$$\frac{dT}{dl} = -\frac{1}{K^2}\frac{dK}{dl} = -(d-2)T + (n-2)K_d\,\Lambda^{d-2}\,T^2 + O(T^3)$$

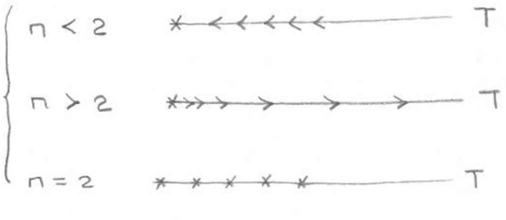
Defines effective T at different length scales

* RG flows

$d<2$:

System looks hotter at larger length scales — Disordered

$d=2$: marginal



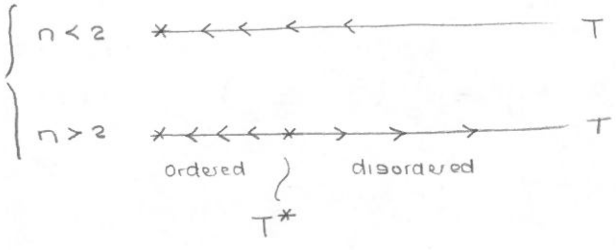
Ordered (1d Ising ordered)

Disordered

$\frac{dT}{dL} = 0$ to all orders. Why is $n=2$ special? of perturbation theory.

(Non-perturbative ?)

$d>2$



ordered

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Fixed Point $\frac{dT}{dl} = 0 \Rightarrow T^* = \frac{d-2}{(n-2) K_d \Lambda^{d-2}}$

$$= \frac{2\pi \epsilon}{(n-2)} + O(\epsilon^2) \qquad d = 2+\epsilon$$

Transition temperature at $O(\epsilon)$

* Linearise RG

$$\frac{d\,\delta T}{dl} = -(d-2)\delta T + 2(n-2) K_d \Lambda^{d-2} T^* \delta T$$

$$= \delta T \left(-\epsilon + \frac{2(n-2)}{2\pi} \frac{2\pi\epsilon}{n-2} \right) = \epsilon\, \delta T$$ relevant perturbation

$$\delta T(l) = e^{\epsilon l} \delta T(0)$$

$$= b^{\epsilon} \delta T$$

$(\delta T') = b^{y_t} (\delta T)$, $y_t = \epsilon + O(\epsilon^2)$ Thermal exponents are independent of n at this order

$\nu = \frac{1}{y_t} = \frac{1}{\epsilon}$ $d=3, n=3$ $\nu \simeq 0.7$ Best estimate

$$\begin{cases} \nu(2+\epsilon) = 1 \\ \nu(4-\epsilon) = 0.67 \end{cases}$$

As a general rule $2+\epsilon$ expansion does not do as well as $4-\epsilon$ expansion.

* Magnetic exponent

Add $-\underline{h} \int d^d x\, \underline{S} \xrightarrow{RG} -\underline{h} \cdot b^d \xi \int d^d x' \underline{S}'$

$$h' = b^d \underset{\sim}{\xi} h = b^{y_h} h$$

$$b^{y_h} = b^d \left[1 - \frac{n-1}{2K^*} I_d(b) \right]$$

$$\frac{dh}{dl} = h\left(d - \frac{n-1}{2(n-2)} \epsilon \right)$$

$$y_t = \epsilon$$

$$y_h = 2 + \epsilon - \frac{n-1}{2(n-2)} \epsilon = 2 + \frac{n-3}{2(n-2)} \epsilon$$

$$\gamma = \nu(2y_h - d) = \frac{1}{\epsilon} \left(4 + \frac{n-3}{n-2}\epsilon - 2 - \epsilon \right) = \frac{1}{\epsilon} \left(2 - \frac{\epsilon}{n-2} \right)$$

$d = 3, n = 3$

$\gamma_{est} = 1.38$, $\gamma(4-\epsilon) = 1.35$

$\gamma(2+\epsilon) = 1$

So magnetic exponends depend on n.

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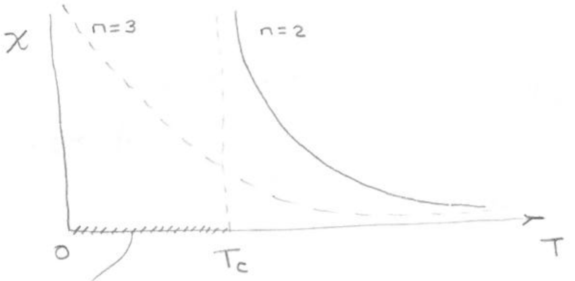
η: from χ of $\int$ of two-point correlation

$$\gamma = \nu(2-\eta) \quad , \quad \eta = \frac{\epsilon}{n-2} \qquad \left(\begin{array}{c} \eta(d=3,\ n=3)_{est} = 0.05 \\ \eta_{2+\epsilon} = 1 \end{array} \right)$$

V. 3 2d XY Model + Topological Defects

* Stanley + Kaplan (1966) did high T series that indicated a phase transition at a finite T_c



$\langle m \rangle = 0$ for n=2 and 3

by Mermin - Wagner theorem

A phase transition without symmetry breaking ?

c.f. Wegner : Z_2 lattice gauge theory - dual to 3d Ising model

Asymptotic behaviour of correlation function distinguishes phases calculated in high and low T series. In both cases it must decay because no symmetry breaking

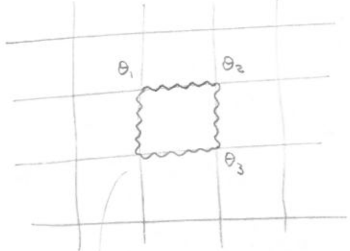
High T series

$$Z = \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{K \sum \cos(\theta_i - \theta_j)}$$

$$\underset{K \to 0}{\overset{\text{high } T}{\simeq}} \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} \left[1 + K \cos(\theta_i - \theta_j) + \underbrace{O(K^2)} \right]$$

represent graphically

alternatwely choose $\ln(1 + K\cos(\theta_i - \theta_j))$ and assume in same universality class



$2\left(\frac{K}{2}\right)^4$ or generally

$2\left(\frac{K}{2}\right)^L$

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2) = 0$$

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2)\cos(\theta_2 - \theta_3) = \frac{1}{2\pi}\,\frac{1}{2}\cos(\theta_1 - \theta_3)$$

$$\int_0^{2\pi} \frac{d\theta_1\, d\theta_4}{(2\pi)^2} \cos(\theta_1 - \theta_4)^2 = \frac{1}{2}$$

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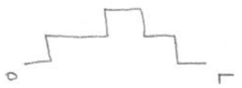
We require $\langle \underline{S}_0 . \underline{S}_r \rangle = \langle \cos(\theta_r - \theta_0) \rangle$

$$= \frac{1}{Z}\int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \cos(\theta_r - \theta_0) \prod_{\langle ij \rangle} \left(1 + K\cos(\theta_i - \theta_j) + ..\right)$$

$$\simeq \left(\frac{K}{2}\right)^r$$



$$\simeq \exp\left[-r\ln\left(\frac{2}{K}\right)\right] = e^{-r/\xi}$$

$$\xi = 1/\ln(2/K)$$

exponential decay in disordered phase

(but not ξ is actually a function of angle because of the lattice)

Low T Series

$$Z = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi}\, e^{K\sum \cos(\theta_i - \theta_j)}$$

$$\simeq e^{2NK} \int \mathcal{D}\theta(x)\, e^{-\frac{K}{2}\int d^2x\, (\nabla\theta)^2}$$

ground state

- all aligned.

$$\langle \cos(\theta_r - \theta_0) \rangle = \mathrm{Re} \langle e^{i(\theta_r - \theta_0)} \rangle$$

$$= \frac{1}{Z}\int \mathcal{D}\theta(x)\, e^{-\frac{K}{2}\int d^2x\left((\nabla\theta)^2 + i(\theta(r) - \theta(0))\right)}$$

$$= \mathrm{Re}\, e^{-1/2\langle[\theta(r) - \theta(0)]^2\rangle}$$ because Gaussian

$$\frac{1}{2}\langle[\theta(r)-\theta(0)]^2\rangle = \frac{1}{2}\int \frac{d^2q\, d^2q'}{(2\pi)^4}(e^{iq.r}-1)(e^{iq'.r}-1)\langle\theta(q)\theta(q')\rangle$$

$$= \frac{1}{2}\int \frac{d^2q}{(2\pi)^2}\frac{(2-2\cos q.r)}{Kq^2} = \frac{C(r)}{K}$$

$$\nabla^2 C(r) = \int \frac{d^2q}{(2\pi)^2}\cos q.r = \delta^2(r)$$

$2\pi r \frac{dC}{dr} = 1$ by Gauss' theorem

$C(r) = \frac{1}{2\pi}\ln r$ + constant

$$\frac{1}{2}\langle[\theta(r)-\theta(0)]^2\rangle = +\frac{1}{2\pi K}\ln\left(\frac{r}{a}\right)$$

fluctuations at lattice spacing assumed zero

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$$\langle \mathrm{Cos}[\theta(r)-\theta(0)] \rangle = e^{-\frac{1}{2\pi K}\ln(r/a)}$$

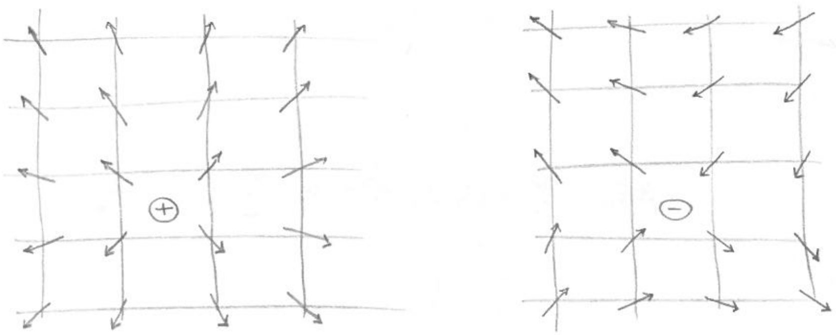
$$= \left(\frac{a}{r}\right)^{1/2\pi K}$$ decays as a power law

Exponential is much faster than power law decay where the correlation length is infinite

However this argument works for different n. Why is n = 2 special ?

* Kosterlitz + Thouless

Periodicity of θ leads to possibility of topological defects (vorticies) which are responsible for the transition.



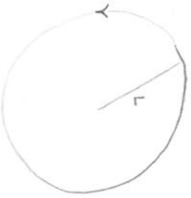
(Velocity fields or gradients are vorticies)

e.g. superfluid

$V = \nabla\theta$

* Distortion small for $r \gg a$

far away $\mathrm{Cos}(\theta_i - \theta_j) \simeq 1 - \frac{(\nabla\theta)^2}{2}$



Change of angle in circling vortex

$$\Delta\theta = \oint \underline{dr} \cdot \nabla\theta = 2\pi m$$

Above are shown $m = \pm 1$ vorticies

Then $2\pi r \frac{d\theta}{dr} = 2\pi n \Rightarrow \frac{d\theta}{dr} = \frac{n}{r}$

$$\nabla\theta = \frac{n}{r}\,\underline{\hat{r}} \wedge \underline{\hat{z}} = n\,\nabla_\wedge(\hat{z}\ln r)$$

Energy for lowest energy distortions

$$\beta\mathcal{E}_m = \frac{K}{2}\int d^2r\,(\nabla\theta)^2 = \frac{K}{2}\int 2\pi r\,dr\left(\frac{n}{r}\right)^2 = \pi K n^2 \int_a^L \frac{dr}{r}$$

$$= \pi n^2 K \ln\left(\frac{L}{a}\right)$$ c.f. dislocation

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Objects also have entropy due to $\left(\frac{L}{a}\right)^2$ possible locations

$$Z = \left(\frac{L}{a}\right)^2 e^{-\beta \mathcal{E}_m} = \left(\frac{L}{a}\right)^2 e^{-\pi^2 K n^2 \ln(L/a)} 2\pi$$

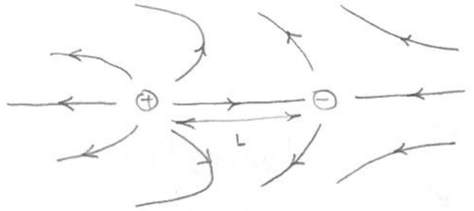
$$= 2\pi \left(\frac{L}{a}\right)^{2-\pi K n^2}$$

Spontaneous appearance of vorticies for

$$K < K_c = \frac{2}{\pi} \quad (n = \pm 1)$$

Note that for $n \geqslant 3$ energy of objects become finite because of 3rd degree ~~or~~ of freedom and proliferate at all T

* Pairs of vorticies have finite energy



$$\nabla\Theta \simeq \nabla\Theta_+ + \nabla\Theta_-$$

$$\simeq L\, \partial_x \left(\frac{1}{r}\right)$$

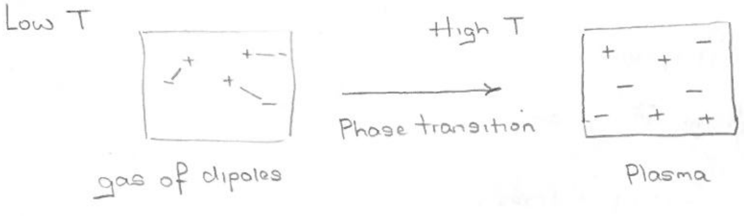
$$\sim L/r^2$$

$$\beta \mathcal{E}_d = \frac{K}{2} \int (\nabla\Theta)^2 d^2r \;\propto\; \frac{K}{2} \int d^2r \, \frac{L^2}{r^4}$$

$$\sim \pi K \ln\left(\frac{L}{a}\right)$$

Probability to create dipole $\sim e^{-\beta \mathcal{E}_d}$

· New picture of phase transition



Can determine which phase by introducing test charges and ask about polarizability

Recall $$Z = \int \prod_i d\Theta_i \; e^{K \sum \cos(\Theta_i - \Theta_j)}$$

$$\beta H = \frac{K}{2} \int d^2x \, (\nabla\Theta)^2$$

Superfluid $\quad V = \nabla\Theta = V_0 + V_1$

$V_0 = \nabla\phi \qquad \nabla\wedge V_0 = 0 \qquad \nabla . V_0 = \nabla^2\phi \neq 0$ No vorticy, Irrotational

$V_1 = \nabla_\wedge(\hat{z}\Psi(r)) \quad \nabla . V_1 = 0 \quad \nabla\wedge V_1 = -\nabla^2\Psi \neq 0$ Vortex part

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We have seen

$$\psi(r) = \sum_i n_i \ln|r_i - r| \qquad \nabla^2\psi = 2\pi \sum_i n_i \delta^2(r-r_i) \qquad \text{c.f. electrostatics}$$

Assuming sufficiently dilute to apply superposition

$$\left(\nabla^2 C = \delta^2(x) \ ; \quad C(x) = \frac{\ln|x|}{2\pi} \right)$$

$$\{\theta_i\} \longrightarrow \{\phi(x), (n_i, r_i)\}$$

$$\int \prod_i d\theta_i \longrightarrow \int \mathcal{D}\phi(x) \sum_{n_i r_i}$$

$$\beta H = \frac{K}{2} \int d^2x \left(\nabla\phi + \nabla_\wedge(\hat{z}\psi) \right)^2$$

$$= \frac{K}{2} \int d^2x \left[(\nabla\phi)^2 + 2\nabla\phi . \nabla_\wedge \hat{z}\psi + (\nabla_\wedge \hat{z}\psi)^2 \right]$$

2nd term $\quad K\int d^2x \ \nabla\phi . \nabla_\wedge \hat{z}\psi = -K\int d^2x \ \phi . \nabla . \nabla_\wedge \hat{z}\psi = 0$

Systems are therefore decoupled

3rd term $\quad \frac{K}{2}\int d^2x \ (\nabla_\wedge(\hat{z}\psi))^2 = \frac{K}{2}\int d^2x \ (\nabla\psi)^2$

$$\nabla\psi = (\partial_x\psi, \partial_y\psi, 0)$$
$$\nabla_\wedge \hat{z}\psi = (+\partial_y\psi, -\partial_x\psi, 0)$$

$$= -\frac{K}{2}\int d^2x \ \psi \nabla^2\psi$$

$$= -\frac{K}{2}\int d^2x \Big(\underbrace{\sum_i n_i \ln|r-r_i|}_{2\pi C(r-r_i)\text{, coulomb}} \Big)\Big(2\pi \sum_j n_j \delta(r-r_j) \Big)$$

$$= -2\pi^2 K \sum_{\langle ij \rangle} n_i n_j C(r_i - r_j)$$

$$= -4\pi^2 K \sum_{i<j} n_i n_j C(r_i - r_j) - \underbrace{2\pi^2 K \sum_i n_i^2 C(0)}$$

divergence reflects breakdown of approximation at core − $\beta\mathcal{E}(n)$ will denote the true finite core energy of the vortex

$$\mathcal{Z} = \int \mathcal{D}\phi(x) \sum_{\{n_i, r_i\}} e^{-K/2 \int d^2x (\nabla\phi)^2}$$
$$\times \ e^{-\sum_i \beta\mathcal{E}(n_i) + 4\pi^2 K \sum_{i<j} n_i n_j C(r_i - r_j)}$$

$$= \underbrace{\mathcal{Z}_{sw}}_{\text{Spin Waves}} \ \underbrace{\mathcal{Z}_{Q}}_{\text{Coulomb Gas}}$$

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At low T, it suffices to look at $n_i = \pm 1$ (q_i)

$$\text{Let} \quad y_0 = e^{-\beta \mathcal{E}(1)} = e^{-\beta \mathcal{E}(-1)} \quad \text{fugacity of } |n| = 1 \text{ vortices}$$

Then
$$\mathcal{Z}_Q = \sum_{N=0}^{\infty} y_0^N \int \prod_{i=1}^{N} d^2 r_i \exp\left[4\pi^2 K \sum_{i<j} q_i q_j C(r_i - r_j) \right]$$

If we had been careful with boundary terms in earlier integrations we would have found that only $\sum q_i = 0$, neutral configurations have finite energy in infinite system - so we will assume this to be true.

$$\mathcal{Z}_Q = 1 + y_0^2 \int d^2x\, d^2x' \;\underset{x}{\oplus}\cdots\underset{x'}{\ominus} \;+\; y_0^4 \int \begin{matrix} \oplus & \ominus \\ \ominus & \oplus \end{matrix} + \cdots$$

$4\pi^2 K\, C(x-x')$ gets modified by other charges to $4\pi^2 K_{eff}\, C(x-x')$

* Compute the effective interaction perturbatively in y_0

$$e^{-\beta V_{eff}(x-x')} = \overset{x}{\oplus}\!\sim\!\sim\!\sim\!\overset{x'}{\ominus} = \frac{\overset{x}{\oplus}\cdots\overset{x'}{\ominus} + y_0^2\, \overset{x}{\oplus}\cdots \overset{\overset{y}{\oplus}\cdot\overset{y'}{\ominus}}{} \cdots\overset{x'}{\ominus} + O(y_0^4)}{1 + y_0^2\, \overset{y}{\oplus}\cdot\overset{y'}{\ominus} + O(y_0^4)}$$

So
$$e^{-\beta V_{eff}(x-x')} = e^{-4\pi^2 K C(x-x')} \frac{\left[1 + y_0^2 \int d^2y\, d^2y'\, e^{-4\pi^2 K C(y-y') + 4\pi^2 K [C(x-y) - C(x-y') - C(x'-y) + C(x'-y')]} + O(y_0^4)\right]}{1 + y_0^2 \int d^2y\, d^2y'\, e^{-4\pi^2 K C(y-y')} + O(y_0^4)}$$

$$= e^{-4\pi^2 K C(x-x')} \left\{ 1 + y_0^2 \int d^2y\, d^2y'\, e^{-4\pi^2 K C(y-y')} \left[e^{4\pi^2 K D(x,x',y,y')} - 1 \right] + O(y_0^4) \right\}$$

Look at small $r = y - y'$; Set $R = \frac{y+y'}{2}$ or $\begin{cases} y = R + r/2 \\ y' = R - r/2 \end{cases}$ $\quad \int d^2y\, d^2y' \rightarrow \int d^2R\, d^2r$

$$D = C(x - R - \tfrac{r}{2}) - C(x - R + \tfrac{r}{2}) - C(x' - R \mp \tfrac{r}{2}) + C(x' - R + r/2) \qquad \text{expansion in } r$$

$$= C(x-R) - \tfrac{r}{2} \nabla_x C(x-R) - C(x-R) - \tfrac{r}{2} \nabla_x C(x-R)$$
$$- C(x'-R) + \tfrac{r}{2} \nabla_{x'} C(x'-R) + C(x'-R) + \tfrac{r}{2} \nabla_{x'} C(x'-R) + \cdots$$

$$= -r \nabla_x C(x-R) + r \nabla_{x'} C(x'-R)$$

$$= r \cdot \nabla_R C(x-R) - r \nabla_R C(x'-R) + O(r^3)$$

$$= \underline{r} \cdot \nabla_R\, \delta C + O(r^3)$$

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$$e^{4\pi^2 K D} - 1 = 4\pi^2 K D + 8\pi^4 K^2 D^2 + O(r3)$$

$$y_0^2 \int d^2r\, d^2R \left\{ 4\pi^2 K \underbrace{r . \nabla_R \delta C}_{0} + 8\pi^4 K^2 \underbrace{(r . \nabla_R \delta C)^2}_{\frac{1}{2} r^2 (\nabla_R \delta C)^2} \right\} e^{-4\pi^2 K C(r)}$$

r Angular integration

R integration $\int d^2R\, (\nabla_R \delta C)^2$

$$= - \int d^2R\, [C(x-R) - C(x'-R)][\delta(x-R) - \delta(x'-R)]$$

$$= - C(0) + C(x-x') - C(0) + C(x-x')$$

$$= -2\cancel{C}(0) + 2C(x-x')$$

0 by proper choice of cut-off — put in logarithm

$$e^{-\beta V_{eff}(x-x')} = e^{-4\pi^2 K C(x-x')} \left[1 + 8\pi^4 K^2 y_0^2 \int 2\pi r\, dr\, r^2 e^{-2\pi K \ln r} C(x-x') + O(y_0^4) \right]$$

$$+ O(r^4)$$

$$= e^{-4\pi^2 K_{eff} C(x-x')}$$

$$K_{eff} = K - 4\pi^3 K^2 y_0^2 \int_{\underset{\text{Short distance cut-off}}{a}}^{\infty} dr\, r^{2-2\pi K+1} + O(y_0^4)$$

Perturbation is valid if integral converges and correction at $O(y_0^2)$ is finite

$$K > K_c = \frac{2}{\pi}$$

(c.f. calculation of susceptibility — resort to RG)

Treat the divergence of the Perturbation series via RG

Interactions (K, y_0) + cut-off 'a'

Coarse grain to find theory with cut-off $ba = e^{\delta L} a$

$$K_{eff} = K - 4\pi^3 K^2 y_0^2 (a^{4-2\pi K})\delta L - 4\pi^3 K^2 y_0^2 \int_{ae^{\delta L}}^{\infty} dr\, r^{3-2\pi K} + O(y_0^4)$$

$$K_{eff} = \tilde{K} - 4\pi^3 y_0^2 \tilde{K}^2 \int_{ae^{\delta L}}^{\infty} dr\, r^{3-2\pi \tilde{K}} + O(y_0^4)$$

Maps to model $(\tilde{K}, y_0)$ cut-off $ae^{\delta L}$

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Rescale

$$r' = r/e^{\delta L}$$

$$K_{eff} = \tilde{K} - 4\pi^3 \tilde{K}^2 \underbrace{y_0^2 \, e^{(4-2\pi K)\delta L}}_{\tilde{y}_0^2} \int_a^\infty dr' \, r'^{\,3-2\pi\tilde{K}}_{1} + O(\tilde{y}_0^4)$$

$(\tilde{K}, \tilde{y}_0)$ + cut-off a

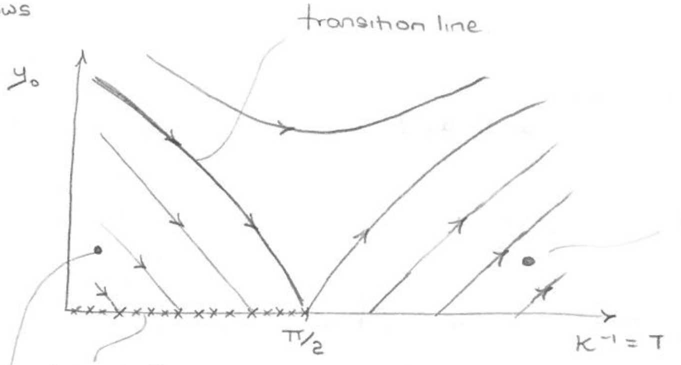
$$K' = K(1+\delta L) = K + \delta L \frac{\partial K}{\partial L} = K - 4\pi^3 K^2 y_0^2 \, a^{4-2\pi K} \, \delta L + O(y_0^4)$$

$$y_0' = y_0(1+\delta L) = y_0 + \delta L \frac{dy_0}{dL} = y_0 \, e^{\delta L(2-\pi K)} + O(y_0^3)$$

$$\begin{cases} \dfrac{dy_0}{dL} = y_0(2-\pi K) + O(y_0^3) \\[2ex] \dfrac{dK^{-1}}{dL} = 4\pi^3 y_0^2 \, a^{4-2\pi K} + O(y_0^4) \end{cases}$$

Kosterlitz 1975

* RG flows



line of fixed points, stable for $K > 2/\pi$

$K_{in} \longrightarrow K_{eff} = 0$ like plasma or metal

debye-Huckel like behaviour

$e^{-r/\xi}$ screening

1 vortex

$K_{in} \xrightarrow{\quad} \overset{\text{Screened}}{K_{eff}}$ Coulomb interaction + gas of dipoles preserved

Corresponds to insulator $\epsilon = K_{in}/K_{eff}$

Low T expansion valid

Recall $Z_{xy} = Z_{sw} Z_Q$

$$Z_Q = \sum_{N=0}^{\infty} y_0^N \int_{\text{"a"}} \prod_i^N d^2 r_i \, e^{4\pi^2 K \sum_{i<j} q_i q_j C(r_i - r_j)}$$

* Expand recursions around $K^{-1} = \pi/2$ fixed point to learn about transition

$x = K^{-1} - \frac{\pi}{2}$ $\qquad \frac{1}{K} = \frac{\pi}{2}\left(1 + \frac{2x}{\pi}\right)$ $\qquad K = \frac{2}{\pi}\left(1 - \frac{2x}{\pi} + O(x^2)\right)$

$y = y_0$

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$$\begin{cases} \dfrac{dx}{dl} = 4\pi^3 y^2 + O(xy^2, y^4) \\[2ex] \dfrac{dy}{dl} = \dfrac{4x}{\pi} y + O(x^2 y, y^3) \end{cases}$$

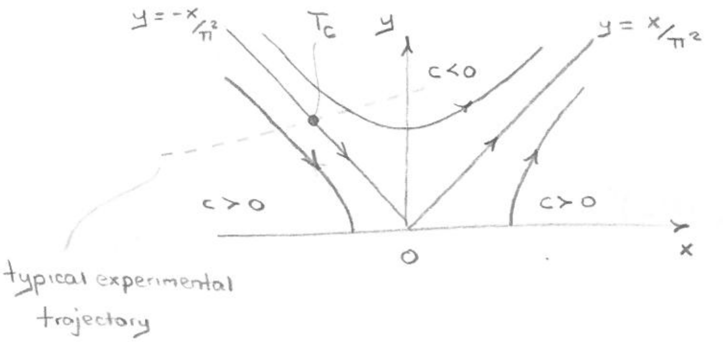
These equations are non-linear (we can't linearise around this point to find exponents)

$$\frac{d}{dl} x^2 = 8\pi^3 x y^2 = \pi^4 \frac{d}{dl} y^2$$

$$\Rightarrow \frac{d}{dl}(x^2 - \pi^4 y^2) = 0$$

$$x^2 - \pi^4 y^2 = c$$

As x and y evolve along l the must obey this formula (for RG trajectories of hyperbolae)



$$\text{Estimate } T_c, \quad x_c = -\pi^2 y_0 = K_c^{-1} - \pi/_2$$

$$T_c \simeq K_c^{-1} = \frac{\pi}{2} - \pi^2 y_0$$

lowest order reduction in T_c due to finite fugacity y_0

* For $T < T_c$, spin-spin correlation function

$$\langle \cos(\theta_r - \theta_0) \rangle \sim \frac{1}{|r|^\eta} \qquad \eta = \frac{1}{2\pi K_R} \qquad K_R = \lim_{l\to\infty} K(l) \equiv \frac{K}{\varepsilon} > \frac{2}{\pi}$$

$$\text{So} \quad \eta \lesssim \frac{1}{4} \quad , \text{ meaningful for } K_R^{-1} \lesssim \pi/_2$$

How does $K \to 2/_\pi$ as $T \to T_c$?

Remember that y_0 is activated, $y_0 = e^{-\varepsilon_{core}/T}$ and so we can include its T dependence explictly.

$$\text{Parameter } c = x_0^2(T) - \pi^4 y_0^2(T) = b^2(T_c - T) \quad \text{for some +ve } b^2$$

assuming analytical behaviour for c, and knowing that it vanishes at T_c

reasonal because K^{-1} and y_0 assumed analytic in T

$$\text{and} \quad c = x_\infty^2$$

$$x_\infty = -b\sqrt{(T_c - T)} \quad \text{taking obvious sign}$$

$$= K_R^{-1} - \frac{\pi}{2}$$

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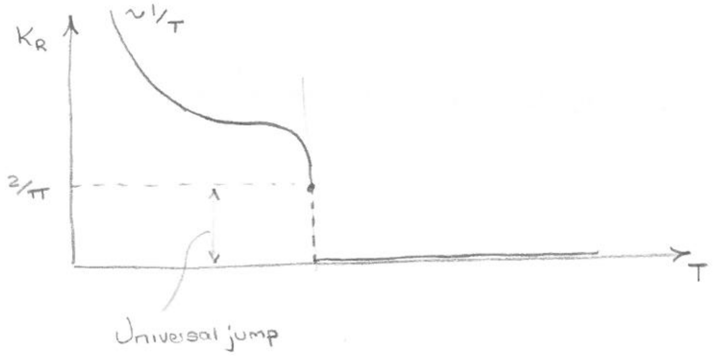
$$K_R = \frac{2}{\pi} - \frac{4}{\pi^2} x_\infty = \frac{2}{\pi} + \frac{4b}{\pi^2}\sqrt{(T_c - T)}$$



Universal Jump is confirmed by experiments on Superfluid films (Bishop and Reppy 1975)

$$H = \int d^2x \left[\underbrace{\psi^* \left(-\frac{\hbar^2}{2m_{He}} \nabla^2 \right) \psi}_{\text{Kinetic energy}} + \underbrace{U(\psi^* \psi)}_{\text{Potential energy}} \right]$$

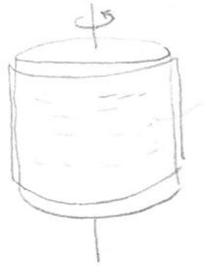
$$\overset{\text{Low } T}{\underset{\text{Condensate}}{=}} \quad \underset{\substack{\text{minimised} \\ \text{by } \bar{\psi}}}{(H)_0} + \frac{\hbar^2 \bar{\psi}^2}{2m_{He}} \int d^2x \, (\nabla\theta)^2$$

$$\psi = \bar{\psi} e^{i\theta(x)}$$

So $K \equiv \dfrac{\hbar^2 \bar{\psi}^2}{kT m_{He}}$

$\bar{\psi}^2$ is related to the superfluid density

* Superfluid density measured by torsional oscillators



Below T_c the superfluid fraction, $m_{He}\bar{\psi}^2 \equiv \rho_s$ does not contribute inertia. frequency of oscillation $\rightarrow \rho_s$ allowing plot of K_R

The jump is independent of film thickness, or He_3 impurities which reduces T_c

i.e. The jump in K_R at transition is Universal

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* For $T > T_c$

$$\langle \cos(\theta_r - \theta_0) \rangle \sim e^{-r/\xi}$$

How does ξ depend on T_c ?

$$c = x_0(T)^2 - \pi^4 y_0(T)^2 = -b^2(T - T_c) = x_\infty^2 - \pi^4 y_\infty^2$$

$$\frac{dx}{dl} = 4\pi^3 y^2 = \frac{4}{\pi}\left(x^2 + b^2(T - T_c)\right)$$

$$\int \frac{dx}{x^2 + b^2(T - T_c)} = \frac{4}{\pi}\int dl$$

$$\tan^{-1}\left(\frac{x(l)}{b\sqrt{(T - T_c)}}\right)\frac{1}{b\sqrt{(T - T_c)}} = \frac{4}{\pi}\, l$$

But equations apply where x and y ~ O(1) at scale of ξ. y, the fugacity can not exceed 1, and when close to 1 particles are independent

$$\Rightarrow \frac{4}{\pi}\underbrace{\ln\left(\frac{\xi}{a}\right)}_{l} = \frac{1}{b\sqrt{(T - T_c)}}\,\frac{\pi}{2}$$

$$\xi = a \exp\left[\frac{\pi^2}{8b\sqrt{(T - T_c)}}\right]$$

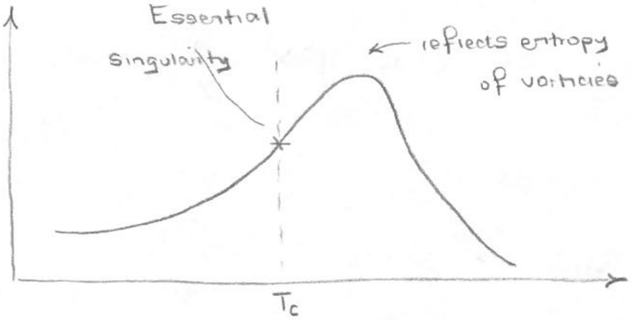
Divergence is not like a power law as we would have if equations were linear.

Actually faster than power law ($\nu \to \infty$)

$$* \quad f_{sing} \simeq \xi^{-2} \sim e^{-\pi^2/4b\sqrt{(T-T_c)}} \sim n_F$$

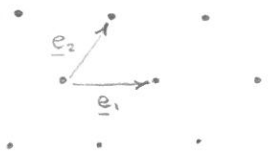
density of vortices

$$C \sim \frac{\partial^2 f}{\partial T^2}$$



VII C 2d Melting (Kosterlitz + Thouless, Halperin + Nelson, Young (KT+HNY))

a) Start at $T = 0$; forming a perfect lattice, say triangular



Atomic positions

$$\underline{r}_0(m,n) = m\underline{e}_1 + n\underline{e}_2$$

* At finite T

$$\underline{r}(m,n) = \underline{r}_0(m,n) + \underline{u}(m,n) \qquad |u| \ll a$$

* Coarse grain

to define an average distortion or deformation field $\underline{u}(x_1, x_2)$

b) Elastic Theory for distortions

$H[\underline{u}]$, must be invariant under translations or rotations

Depends only on the strain field

$$u_{ij}(\underline{x}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

$$\beta H[\underline{u}] = \frac{1}{2}\int d^2x \left[2\mu\, u_{ij} u_{ij} + \lambda u_{ii} u_{jj}\right] + O(u^3)$$

$\mu, \lambda \equiv$ Lamé coefficients

For isotropic solids – of which triangular is a member

$$= \int \frac{d^2q}{(2\pi)^2}\left[\frac{\mu}{2} q^2 |u(q)|^2 + \frac{\mu+\lambda}{2}(\underline{q}\,.\,\underline{u}(q))^2\right] + O(q^4, u^3)$$

c) Normal modes of Vibration (phonons)

Transverse $\underline{q}\,.\,\underline{u}$

$$\beta H = \int \frac{d^2q}{(2\pi)^2}\frac{\mu q^2}{2}|u_T(q)|^2 \Rightarrow \langle |u_T(q)|^2\rangle = \frac{1}{\mu q^2}$$

Longitudinal $\underline{q} \parallel \underline{u}$

$$\beta H = \int \frac{d^2q}{(2\pi)^2}\frac{2\mu+\lambda}{2} q^2 |u_L(q)|^2 \Rightarrow \langle |u_L(q)|^2\rangle = \frac{1}{(2\mu+\lambda)q^2}$$

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$$\langle u_i(q)\,u_j(q')\rangle = \frac{(2\pi)^2\,\delta^2(q+q')}{\mu\,q^2}\left(\delta_{ij} - \frac{\mu+\lambda}{2\mu+\lambda}\,\frac{q_i q_j}{q^2}\right)$$

In real space

$$\langle [u(x)-u(0)]^2\rangle = \int \frac{d^2q\,d^2q'}{(2\pi)^4}\,(e^{iqx}-1)(e^{iq'x'}-1)\,\langle u_i(q)\,u_i(q)\rangle$$

$$= \int \frac{d^2q}{(2\pi)^2}\,\frac{(2-2\cos q.x)}{q^2}\,\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}$$

$$\frac{(2\pi)^2\,\delta^2(q+q')}{\mu q^2}\left(\delta_{ii} - \frac{\mu+\lambda}{2\mu+\lambda}\,\frac{q_i q_i}{q^2}\right)$$

$$= \frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\cdot 2\,\frac{\ln|x|}{2\pi}$$

Alternatively $\langle |u|^2\rangle = \dfrac{3\mu+\lambda}{2\mu(2\mu+\lambda)}\ln L$

So lattice assumption was incorrect - fluctuations always destroy lattice

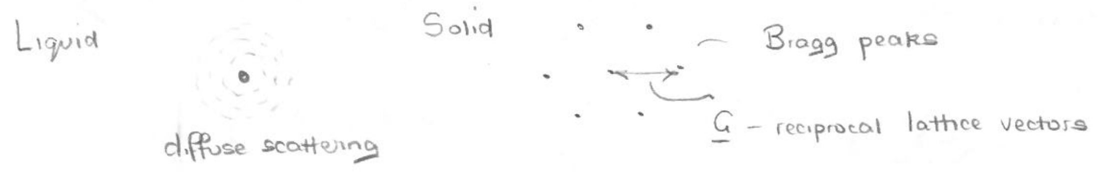
Lindeman Criterion

Empirical formula for crystal melting where

$$\sqrt{\langle u^2\rangle} \sim a/_{10}$$

predicts that 2d crystal always melts at finite T (shouldn't be surprise because continuous symmetry => no LRO in 2d)

d) Translation Order probed by diffraction



Bragg peak intensity can be used as an order parameter

$S(q)$

Order parameter $\langle e^{i\underline{G}\cdot\underline{r}(x)}\rangle = \rho_G(x)$ unity at finite T

reciprocal lattice vector

$\underline{G}.\underline{r}_0 = 2\pi n$ integer

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At T=0 $\langle |\rho_G(r)| \rangle = 1$

Translational correlations

$$\langle \rho_G(x)\rho_G^*(0) \rangle = \langle e^{i\underline{G}.(\underline{u}(x)-\underline{u}(0))} \rangle$$

$$= \exp\left[-\frac{1}{2} G_\alpha G_\beta \langle (u_\alpha(x)-u_\alpha(0))(u_\beta(x)-u_\beta(0)) \rangle \right]$$

$$= \exp\left[-\frac{1}{2} G_\alpha G_\beta \int \frac{d^2q}{(2\pi)^2} \frac{(2-2\cos q.x)}{\mu q^2} \left(\delta_{\alpha\beta} - \frac{\mu+\lambda}{2\mu+\lambda} \frac{q_\alpha q_\beta}{q^2} \right) \right]$$

$\to \frac{1}{2} G^2$ on angular integration

$$= \exp\left[-\frac{G^2}{2\mu} \frac{3\mu+\lambda}{2\mu+\lambda} \frac{1}{2\pi} \ln\left(\frac{|x|}{a}\right) \right] \times$$ diffeent functions of x, G with faster decay

$$= \left(\frac{a}{|x|}\right)^{\eta_G} \times \ldots$$

$$\eta_G = \frac{G^2}{4\pi} \frac{3\mu+\lambda}{\mu(2\mu+\lambda)}$$ — decay as power law

Quasi-Long-Range order

In 3d get $\frac{1}{x} - \frac{1}{a}$ rather than $\ln|x|$

$$\lim_{x\to\infty} \langle \rho_G(x)\rho_G^*(0) \rangle \sim \exp\left(-\underbrace{\frac{G^2}{2\mu} \frac{3\mu+\lambda}{2\mu+\lambda}}_{\propto T} \overset{\text{cut-off}}{\Lambda} \right)$$ — constant at T=0 and diminishes at finite T

Debye-Waller factor — can use as criterion for melting when ~ 1

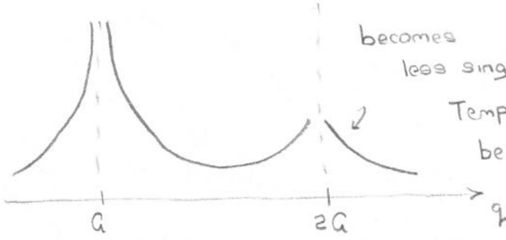
(e) Translational Order probed by Diffraction

$$S(q) \propto \langle |A(q)|^2 \rangle = \langle | \sum_x e^{i\underline{q}.\underline{r}(x)} |^2 \rangle = N \langle \sum_x e^{iq.(r(x)-r(0))} \rangle$$

$$= N \sum_x \underbrace{e^{iq.(r_0(x)-r_0(0))}}_{\text{Oscillates rapidly except for } q \simeq G} \underbrace{\langle e^{iq.(u(x)-u(0))} \rangle}_{\left(\frac{a}{|x|}\right)^{\eta_q}}$$

$$S(\underline{q} \simeq \underline{G}) = N \int d^2x \, e^{i(q-G).x} \left(\frac{a}{|x|}\right)^{\eta_G}$$

$$\propto \frac{1}{|q-G|^{2-\eta_G}}$$



becomes less singular with G [illegible]d Temperature — eventually becomes non-singular

In 3d, get δ functions with amplitude of Debye-Waller factor

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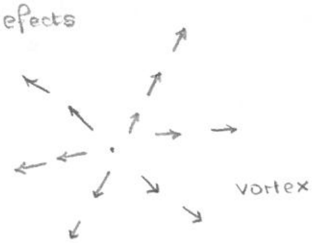
(f) Topological Defects

may destroy quasi-long-range order as in X-Y model

XY model

- Defects



- Topological charge

$$\oint \nabla\theta . d\underline{s} = 2\pi m$$

$$\nabla\theta = \nabla_\wedge \hat{\underline{z}} \sum_i n_i \ln|r_i - r|$$

$$\nabla\theta = \underbrace{\nabla\tilde{\theta}}_{\text{Vorticies}} + \underbrace{\nabla\phi}_{\text{Spin Waves}}$$

$$\beta H = \frac{K}{2}\int d^2x\,(\nabla\theta)^2$$

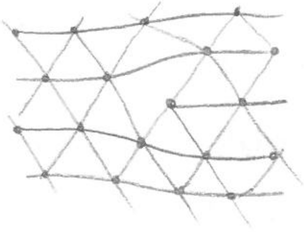
$$= \frac{K}{2}\int d^2x\,(\nabla\phi)^2 - \overline{K}\sum_{i<j} n_i n_j \ln\left(\frac{|r_i - r_j|}{a}\right)$$

$$\underbrace{-\sum_i \ln y(n_i)}_{\text{Core energy}}$$

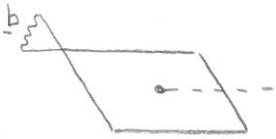
$$\overline{K} = 2\pi K$$

2d Solid

- Dislocation



- Burger's Circuit



Closure failure by a lattice vector $\underline{b}$

$$\underbrace{\oint \nabla u_\alpha . d\underline{s}}_{\text{distortion field}} = b_\alpha$$

$$\nabla u_\alpha = \nabla_\wedge \hat{z} \sum_i b_i^\alpha \ln|r - r_i|$$

$$u_{ij} = \underbrace{\tilde{u}_{ij}}_{\text{dislocations}} + \underbrace{\phi_{ij}}_{\text{phonons}}$$

$$\beta H = \int d^2x\left[\mu u_{ij} u_{ij} + \frac{\lambda}{2} u_{ii} u_{jj}\right]$$

$$= \int d^2x\left[\mu \phi_{ij}\phi_{ij} + \frac{\lambda}{2}\phi_{ii}\phi_{jj}\right]$$

$$- \overline{K}\sum_{i<j}\left[\underline{b}_i . \underline{b}_j \ln\left(\frac{|r_i - r_j|}{a}\right) - \underbrace{\frac{b_i .(r_i - r_j)\; b_j .(r_i - r_j)}{|r_i - r_j|^2}}_{\text{vector Coulomb interaction}}\right]$$

$$\underbrace{-\sum_i \ln y(b_i)}_{\text{Core energy}}$$

$$\overline{K} = \frac{1}{\pi}\frac{\mu(\mu+\lambda)}{2\mu+\lambda}$$

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$$Z_{XY} = Z_{SW} Z_Q$$

RG.

$$\left\{\begin{aligned} \frac{dK^{-1}}{dl} &= 4\pi^3 y^2 + O(y^4) \\ \frac{dy}{dl} &= (2-\pi K) y + O(y^3) \end{aligned}\right\} \text{from quadrupole}$$

entropy $\sim l^2$

energy

from hyperbolic flow

$T < T_c$

$$K_R = \frac{2}{\pi} + c(T_c - T)^{1/2}$$

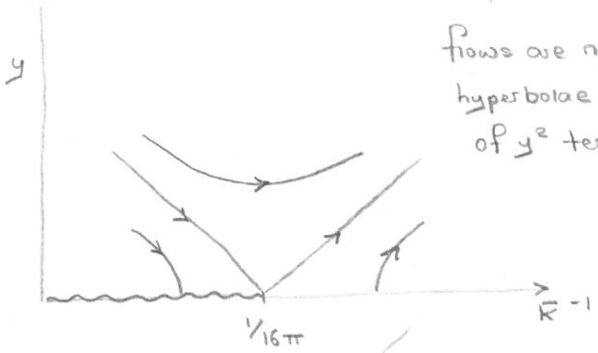
$T > T_c$

$$\xi \simeq a \exp\left[\frac{c'}{\sqrt{(T_c - T)}}\right]$$

$$Z = Z_{Ph} Z_{VG}$$

$$\left\{\begin{aligned} \frac{d\bar{K}^{-1}}{dl} &= A y^2 + O(y^3) \\ \frac{dy}{dl} &= \left(2 - \frac{\bar{K}}{8\pi}\right) y + O(y^2) \end{aligned}\right.$$

~ from



flows are not hyperbolae because of y^2 term

$\bar{K} \to 0 \implies \bar{\mu} \to 0$

transverse modes become soft – cannot support shear stress.

$T < T_c$

$$\mu_R = \mu_1 + c\,|T_c - T|^{\bar{\nu}}$$

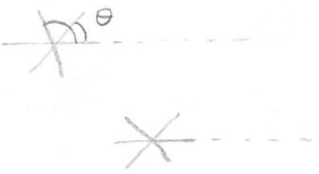
$T > T_c$

$$\xi \simeq a \exp\left[\frac{c'}{|T_c - T|^{\bar{\nu}}}\right]$$

$\bar{\nu} = 0.36963\ldots$ characteristic of the vector Coulomb gas

(9) A solid also has orientational order which can be measured by the order parameter

$$\Psi(x) = e^{6i\theta(x)}$$



At zero T $\quad \Psi(x) = e^{6i\theta_0}$

In terms of distortion

$$\theta(x) = -\frac{1}{2}\,\hat{z}\cdot\nabla\wedge\underline{u}$$

from rotation

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Orientational Correlations

$$\langle \Psi(x)\Psi^*(0)\rangle = \langle e^{6i(\theta(x)-\theta(0))}\rangle$$

$$= \exp\left[-\frac{36}{2}\left(\frac{1}{2}\right)^2\langle|\nabla_\wedge \underline{u}(x) - \nabla_\wedge \underline{u}(0)|^2\rangle\right]$$

$$= \exp\left[-\frac{9}{2}\int\frac{d^2q}{(2\pi)^2}\,(2-2\cos q.x)\,\underbrace{\langle(iq_\wedge \underline{u}(q))(-i\underline{q}_\wedge u(-\underline{q}))\rangle}_{\text{only transverse}}\right]$$

$$q^2\langle|u_T(q)|^2\rangle = q^2\frac{1}{\mu q^2}$$

$$= \exp\left[-\frac{9}{\mu}\int\frac{d^2q}{(2\pi)^2}(1-\underbrace{\cos q.x}_{\text{averages to zero}})\right]$$

$$= \exp\left[-\underbrace{\frac{9}{\mu}}_{\propto T}\underbrace{n}_{\text{density}}\right]$$

True Long-Range Order - more robust than translational order

This is the effect of phonons on orientational order

* For $T > T_c$ what is the effect of dislocations

$$\Rightarrow \quad \tilde{\theta}(r) = \sum_i \frac{b_i.(r-r_i)}{|r-r_i|^2}$$

$$= -\int d^2x\,\frac{\underline{b}(x).(\underline{x}-\underline{r})}{|x-r|^2} \qquad \text{many dislocations}$$

$$\tilde{\theta}(q) = \frac{i\underline{b}(q).\underline{q}}{q^2} \qquad \text{Since its just a convolution}$$

$$\langle|\tilde{\theta}(q)|^2\rangle = \frac{q_i q_j}{q^4}\langle b_i(q)\,b_j^*(q)\rangle$$

$$\langle b_i(q)b_j^*(q)\rangle = \frac{1}{A}\int d^2x\,d^2x'\,e^{iq.(x-x')}\underbrace{\langle b_i(x)\,b_j(x')\rangle}_{\delta_{ij}\,b_0^2\,e^{-|x-x'|/\xi}\ \text{by assumption}}$$

$$= \frac{b_0^2}{q^2+\xi^{-2}}$$

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$$\langle |\Theta(q)|^2 \rangle \underset{q \to 0}{=} \frac{b_0^2}{\xi^{-2} q^2}$$

So $\langle \Psi(x) \Psi^*(0) \rangle$ decays like a power law

So phase is not a liquid but a liquid crystal

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Mean–Field Theory

1. *The Binary Alloy:* A binary alloy (as in β brass) consists of N_A atoms of type A, and N_B atoms of type B. The atoms form a simple cubic lattice, each interacting only with its six nearest neighbors. Assume an attractive energy of $-J$ $(J > 0)$ between like neighbors $A - A$ and $B - B$, but a repulsive energy $+J$ for an $A - B$ pair.

(a) What is the minimum energy configuration, or the state of the system at zero temperature?

(b) Estimate the total interaction energy assuming that the atoms are randomly distributed among the N sites; i.e. each site is occupied independently with probabilities $p_A = N_A/N$ and $p_B = N_B/N$.

(c) Estimate the mixing entropy of the alloy with the same approximation. Assume $N_A, N_B \gg 1$.

(d) Using the above, obtain a free energy function $F(x)$, where $x = (N_A - N_B)/N$. Expand $F(x)$ to the fourth order in x, and show that the requirement of convexity of F breaks down below a critical temperature T_c. For the remainder of this problem use the expansion obtained in (d) in place of the full function $F(x)$.

(e) Sketch $F(x)$ for $T > T_c$, $T = T_c$, and $T < T_c$. For $T < T_c$ there is a range of compositions $x < |x_{sp}(T)|$ where $F(x)$ is not convex and hence the composition is locally unstable. Find $x_{sp}(T)$.

(f) The alloy globally minimizes its free energy by separating into A rich and B rich phases of compositions $\pm x_{eq}(T)$, where $x_{eq}(T)$ minimizes the function $F(x)$. Find $x_{eq}(T)$.

(e) In the (T, x) plane sketch the phase separation boundary $\pm x_{eq}(T)$; and the so called spinodal line $\pm x_{sp}(T)$. (The spinodal line indicates onset of metastability and hysterisis effects.)

****

2. *The Ising Model of Magnetism:* The local environment of an electron in a crystal sometimes forces its spin to stay parallel or antiparallel to a given lattice direction. As a model of magnetism in such materials we denote the direction of the spin by a single variable $\sigma_i = \pm 1$ (an Ising spin). The energy of a configuration $\{\sigma_i\}$ of spins is then given by

$$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i \quad ;$$

where h is an external magnetic field, and J_{ij} is the interaction energy between spins at sites i and j.

(a) For N spins we make the drastic *approximation* that the interaction between all spins is the same, and $J_{ij} = -J/N$. Show that the energy can now be written as $E(M, h) = -N[Jm^2/2 + hm]$, with a magnetization $m = \sum_{i=1}^{N} \sigma_i/N = M/N$.

(b) Show that the partition function $Z(h,T) = \sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H})$ can be re-written as $Z = \sum_M \exp[-\beta F(m,h)]$; with $F(m,h)$ easily calculated by analogy to problem (1). For the remainder of the problem work only with $F(m,h)$ expanded to 4th order in m.

(c) By saddle point integration show that the actual free energy $F(h,T) = -kT\ln Z(h,T)$ is given by $F(h,T) = \min[F(m,h)]_m$. When is the saddle point method valid? Note that $F(m,h)$ is an analytic function but not convex for $T < T_c$, while the true free energy $F(h,T)$ is convex but becomes non-analytic due to the minimization.

(d) For $h = 0$ find the critical temperature T_c below which spontaneous magnetization appears; and calculate the magnetization $\overline{m}(T)$ in the low temperature phase.

(e) Calculate the singular (non-analytic) behavior of the response functions

$$C = \frac{\partial E}{\partial T}|_{h=0}, \quad \text{and} \quad \chi = \frac{\partial \overline{m}}{\partial h}|_{h=0}.$$

****

3. *The Lattice-Gas Model:* Consider a gas of particles subject to a Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} v(r_i - r_j), \quad \text{in a volume } V.$$

(a) Show that the grand partition function Ξ can be written as

$$\Xi = \sum_{N=0}^{\infty} \left(\frac{e^{\beta\mu N}}{\lambda^{3N} N!}\right) \int \prod_{i=1}^{N} d^3r_i \exp[-\frac{\beta}{2}\sum_{i,j} v(r_i - r_j)].$$

(b) The volume V is now subdivided into $\mathcal{N} = V/a^3$ cells of volume a^3, with the spacing a chosen small enough so that each cell α is either empty or occupied by one particle; i.e. the cell occupation number n_α is restricted to 0 or 1 ($\alpha = 1, 2, ..., \mathcal{N}$). After approximating the integrals $\int d^3r$ by sums $a^3 \sum_{\alpha=1}^{\mathcal{N}}$, show that

$$\Xi \approx \sum_{n_\alpha = 0,1} \left(\frac{e^{\beta\mu} a^3}{\lambda^3}\right)^{\sum_\alpha n_\alpha} \exp[-\frac{\beta}{2}\sum_{\alpha,\beta=1}^{\mathcal{N}} n_\alpha n_\beta v(r_\alpha - r_\beta)].$$

(c) By setting $n_\alpha = (1+\sigma_\alpha)/2$ and approximating the potential by $v(r_\alpha - r_\beta) = -J/N$ show that this model is identical to the one studied in problem (2). What does this imply about the behavior of this imperfect gas?

****

Comments:

(a) The manifest equivalence between these three systems is a straightforward consequence of their mapping onto the same (Ising) Hamiltonian. However, there is a more subtle equivalence relating the critical behavior of systems that cannot be so easily mapped onto each other due to the Universality Principle.

(b) Chapter (3) of Parisi provides yet another perspective on the mean-field approximation. For details of saddle point integration, see Chapter (9) of Huang.

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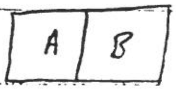
8.334 PS 1 Solutions

1a) at $T=0$, the phases are separated



1b) a random distribution corresponds to $T \to \infty$.

$p_A = N_A/N$; $p_B = N_B/N$; $q=6$; $x = p_A - p_B$; $N = N_A + N_B$

$$E = -Jq/2 \sum_{sites} \left(p_A^2 + p_B^2 - 2p_A p_B\right) = -3JNx^2$$

1c) $S = k_B \ln \Omega = k_B \ln\left(\frac{N!}{N_A!\,N_B!}\right) \simeq -k_B N \left(p_A \ln p_A + p_B \ln p_B\right)$, using Stirlings approximation

1d) $\mathcal{F}(x,T) = E(x) - TS(x) = N\left(-3Jx^2 + k_BT\left(\frac{1+x}{2}\ln\frac{1+x}{2} + \frac{1-x}{2}\ln\frac{1-x}{2}\right)\right) - NkT\ln 2$

Note that $\mathcal{F}(x) = \mathcal{F}(-x)$, so only even powers of x occur in the expansion around $x=0$

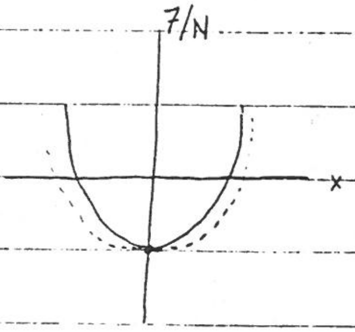
$$\mathcal{F}(x,T) = \mathcal{F}(0,T) + \frac{1}{2!}\frac{\partial^2 \mathcal{F}}{\partial x^2}\Big|_{x=0} x^2 + \frac{1}{4!}\frac{\partial^4 \mathcal{F}}{\partial x^4}\Big|_{x=0} x^4 + O(x^6)$$

$$\mathcal{F}(x,T)/N \overset{(x\ll 1)}{=} k_BT\left(-\ln 2 + \frac{1}{2}\left(1 - \frac{6J}{k_BT}\right)x^2 + x^4/12\right)$$

Convexity requires that $\forall x$ $\partial^2\mathcal{F}/\partial x^2 > 0$. $\frac{1}{N}\frac{\partial^2 \mathcal{F}}{\partial x^2} = k_BT\left(1 - \frac{6J}{k_BT} + x^2\right)$

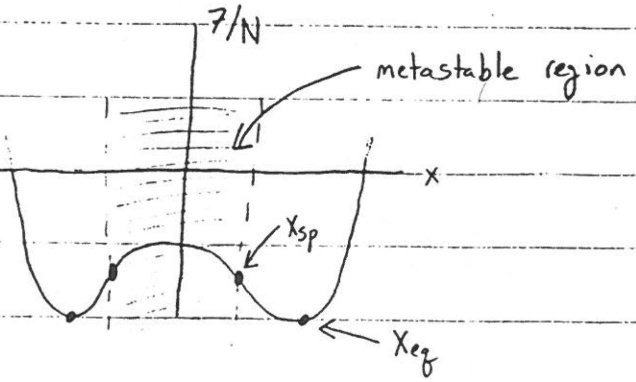
For $T < T_c = 6J/k_B$ & $x^2 < 6J/k_BT - 1$, $\mathcal{F}$ is not convex.

1e)



— $T > T_c$

---- $T = T_c$



$T < T_c$

$\partial^2\mathcal{F}/\partial x^2\Big|_{x_{sp}} = 0 \Longrightarrow x_{sp} = \pm\left(6J/k_BT - 1\right)^{1/2}$ for $T < T_c$, and $|x_{sp}| \ll 1$

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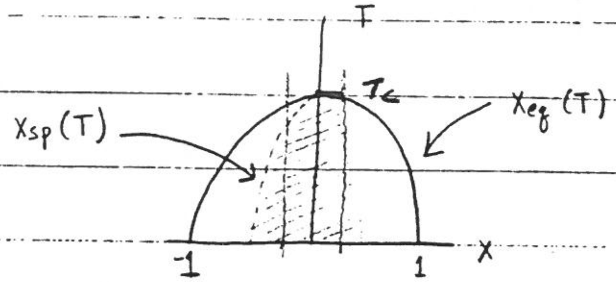
1f) $\partial\mathcal{F}/\partial x\big|_{x_{eq}} = 0 \quad \Rightarrow \quad x_{eq} = \pm\sqrt{3}\,|x_{sp}|$ for $T<T_c$, and $|x_{eq}| \ll 1$

$x_{eq} = 0$ for $T \geq T_c$

1g) The above results are for $T \lesssim T_c$ & $\boxed{x \ll 1.}$ Also note that $|x_{eq}|$ & $|x_{sp}|$ are bounded by 1 & that $|x_{eq}|_{T\to 0} \to 1$



2a) $$H\{\sigma_i\} = \frac{1}{2}\sum_{i,j}^{N} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i$$

For $J_{ij} = -J/N$, $\quad E(m)/N = -\frac{1}{2}Jm^2 - hm \quad$ where $m = \frac{\sum\sigma_i}{N}$

2b) Note that $$\sum_{\{\sigma_i\}} = \sum_{Nm=-N}^{N}\left[\sum_{\{\sigma_i\}} \delta\Big(\sum_i\sigma_i - Nm, 0\Big)\right] = \sum_{Nm} \frac{N!}{\left(\frac{N}{2}(1+m)\right)!\left(\frac{N}{2}(1-m)\right)!}$$

This has the same form as Ω in problem 1, by analogy $\left(x \to m,\ 3J \to J/2\right)$:

$$\mathcal{F}(m,T)/N = k_BT\left(-\ln 2 - \beta hm + \frac{1}{2}(1-\beta J)m^2 + m^4/12\right) + \mathcal{O}(m^6)$$

2c) To evaluate $Z(h,T)$ by the saddle point method, consider $m=\bar{m}$ which minimizes $\mathcal{F}(m,T)$, then

$$q\,e^{-\beta\mathcal{F}(\bar{m},T)} \leq \left(Z(h,T) \equiv e^{-\beta F(h,T)} \equiv \sum_{mN} e^{-\beta\mathcal{F}(m,T)}\right) \leq 2Ne^{-\beta\mathcal{F}(\bar{m},T)}$$

where $1 \leq q \leq 2N$. The lower limit counts only the q states which minimizes $\mathcal{F}$, and the upper limit gives this maximum weight to all $2N$ states. By construction, $\mathcal{F}$ is extensive so $\lim_{N\to\infty} \mathcal{F}(\bar{m},T)/N \sim \mathcal{O}(1)$. This leads to

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$$-(\ln g)/\beta N + \mathcal{F}(\bar{m},T)/N \geq F(h,T)/N \geq (-\ln 2N)/\beta N + \mathcal{F}(\bar{m},T)/N$$

In the limit $N \to \infty$ $\mathcal{F}(\bar{m},T) = F(h,T)$ & the saddle point method is valid.

N.B. one can also evaluate $Z(h,T)$ by integration. $\sum_{mN} \Rightarrow N\int_{-1}^{1} dm \underset{SP}{\Longrightarrow} N\int_{-\infty}^{\infty}$

$$e^{-\beta F(h,T)} = N e^{-\beta \mathcal{F}(\bar{m},T)} \int_{-\infty}^{\infty} dm\, e^{-\beta/2\, \partial^2\mathcal{F}/\partial m^2|_{\bar{m}} (m-\bar{m})^2 + \text{h.o.t.}}$$

since $\partial^2\mathcal{F}/\partial m^2 \sim N$, $e^{-\beta F(h,T)} \sim N e^{-\beta\mathcal{F}(\bar{m},T)}\left(1/\sqrt{N} + \text{h.o.t}\right)$

& $F(h,T)/N = \mathcal{F}(\bar{m},T)/N - \frac{(\ln N)}{2\beta N} + \text{h.o.t.} \underset{N\to\infty}{\Longrightarrow} F(h,T) = \mathcal{F}(\bar{m},T)$

2d) at equilibrium $\partial^2\mathcal{F}/\partial m^2 \Big|_{\substack{m=0\\ T=T_c\\ h=0}} = 0 \Rightarrow T_c = J/k_B$

$$\partial\mathcal{F}/\partial m\Big|_{\bar{m}} = 0 \Rightarrow \bar{m} = \begin{cases} 0 & T > T_c \\ \pm\sqrt{3}\left(\frac{T_c - T}{T}\right)^{1/2} & T \leq T_c \end{cases}$$

2e) for singular behavior, $|t|, |m|, |h| \ll 1$ where $t \equiv \frac{T-T_c}{T_c}$

Note that $\frac{T-T_c}{T} = \frac{T-T_c}{T_c(1+t)} = t + O(t^2)$

$\frac{\partial \mathcal{F}}{\partial m}\Big|_{\substack{\bar{m}\\ h\neq 0}} \overset{!}{=} 0 \Rightarrow \chi^{-1}_{\text{SING}} = \partial h/\partial\bar{m}\big|_{h=0} = k_B T_c(t + \bar{m}^2)$

$\chi = 1/k_BT_c\, t^{-1}$ for $t > 0$; $\gamma_+ = \gamma_- = 1$; $A_+/A_- = 2$

$\chi = 1/2k_BT_c\, |t|^{-1}$ for $t < 0$

From (2a) $\bar{E} \underset{h=0}{=} -1/2\, NJ\bar{m}^2$ $C = \frac{\partial\bar{E}}{\partial T}\Big|_{h=0} = \frac{1}{T_c}\frac{\partial\bar{E}}{\partial t}\Big|_{h=0}$

For $T > T_c$, $\bar{m} = 0$, $\bar{E} = 0 \Rightarrow C = 0$

For $T < T_c$, $\bar{m}^2 = -3t$, $\bar{E} = 3/2\, NJt \Rightarrow C = 3/2\, Nk_B$

$C \sim |t|^{-\alpha}$ $\alpha = 0$

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See Kubo, "Statistical Mechanics" &/or Landau & Lifshitz, "Stat. Mech. V1"

3a) $\Xi = \sum_{N=0} e^{\beta\mu N} Z(N, \beta, V_{ij})$

Divide phase space into squares of area h. N indistinguishable particles gives 1/N! term.

$$\Xi = \sum_{N=0}^{\infty} \frac{1}{h^{3N} N!} \left[\left(\int \prod_{i=1}^{N} d^3r_i \, d^3p_i \right) e^{-\beta \sum_i p_i^2/2m \, - \, \beta/2 \sum_{ij} V(\vec{r}_i - \vec{r}_j)} \right] e^{-\beta\mu N}$$

momentum integrals give $\left(2m\pi/\beta\right)^{3N/2}$, using $\lambda = \left(\frac{\hbar^2 \beta}{2\pi m}\right)^{1/2}$

$$\Xi = \sum_{N=0}^{\infty} e^{\beta\mu N} \frac{1}{\lambda^{3N} N!} \int_V \left(\prod_{i=1}^{N} d^3r_i \right) e^{-\beta/2 \sum_{ij} V(\vec{r}_i - \vec{r}_j)}$$

3b) Divide V into η cells, $V = \eta a^3$, $\int d^3r_i \to a^3 \sum_{\alpha=1}^{\eta}$

$n_\alpha = \{0, 1\}$ for each cell. Since "α" labels are indistinguishable, but "i" are not $\sum_N 1/N! \Rightarrow \sum_{\{n_\alpha\}}$ where $\sum_\alpha n_\alpha = N$

Since the potential acts only between filled sites $\sum_{ij} V(\vec{r}_i - \vec{r}_j) \Rightarrow \sum_{\alpha,\beta} n_\alpha n_\beta V(\vec{r}_\alpha - \vec{r}_\beta)$

$$\Xi = \sum_{\{n_\alpha\}} e^{(\beta\mu + 3\ln(a/\lambda)) \sum_{\alpha=1}^{\eta} n_\alpha \, - \, \beta/2 \sum_{\alpha\beta}^{\eta} n_\alpha n_\beta V(\vec{r}_\alpha - \vec{r}_\beta)}$$

3c) $\Xi \sim \sum_{\{\sigma_\alpha = \pm 1\}} e^{1/2 (\beta\mu + 3\ln(a/\lambda) + \beta J/2) \sum_\alpha^{\eta} \sigma_\alpha \, + \, \beta J/8\eta \sum_{\alpha,\beta} \sigma_\alpha \sigma_\beta}$

Where ~~[illegible]~~ $V(\vec{r}_\alpha - \vec{r}_\beta) = -J/\eta$ and $\sigma_\alpha = 2n_\alpha - 1 = \begin{cases} 1 & (n_\alpha = 1) \\ -1 & (n_\alpha = 0) \end{cases}$

For $\left(\beta h \to \frac{1}{2}(\beta\mu + 3\ln(a/\lambda) + \beta J/2) \, ; \, \beta J \to \beta J/4 \, ; \, N \to \eta\right)$

$Z_{ISING} \to \Xi$. This implies that at low temperatures the lattice gas phase separates. There is a line of first order phase transitions for $\beta\tilde{h} = \frac{1}{2}(\beta\mu + 3\ln(a/\lambda) + \beta J/2) = 0$ terminating at $T_c = J/4k_B$ (which is a critical point).

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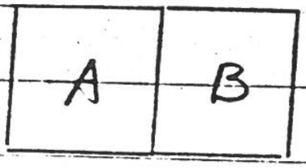
8.334 Problem Set #1 Solution

2/11/93

1.(a) Type A atoms and type B atoms are completely separated.



(b) $E = -\frac{J}{z}\, b \sum_{\substack{\text{sites}\\ \text{of Alloy}}} p_A^2 + p_B^2 - 2p_Ap_B = -3JNX^2$

b: # of nearest neighbors

p_A^2, p_B^2: Prob. that the site and its neighbor are both the same type atoms

p_Ap_B: the site and its neighbor are different type atoms

, where $X = P_A - P_B$

(c) $S = k_B \ln \Omega = k_B \ln \frac{N!}{N_A!\, N_B!} \cong k_B (N \ln N - N_A \ln N_A - N_B \ln N_B)$

$= -k_B N \left[\frac{(1+X)}{2} \ln\left(\frac{1+X}{2}\right) + \frac{(1-X)}{2} \ln\left(\frac{1-X}{2}\right) \right]$

$(\ln N! \approx N \ln N - N \quad \text{for } N \gg 1)$

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(d) $F = E - TS = -3JNx^2 + k_BT\left[\left(\frac{1+x}{2}\right)\ln\left(\frac{1+x}{2}\right) + \left(\frac{1-x}{2}\right)\ln\left(\frac{1-x}{2}\right)\right]$

$$\cong -(\ln 2)Nk_BT + N\left(\frac{k_BT}{2} - 3J\right)x^2 + \frac{Nk_BT}{12}x^4$$

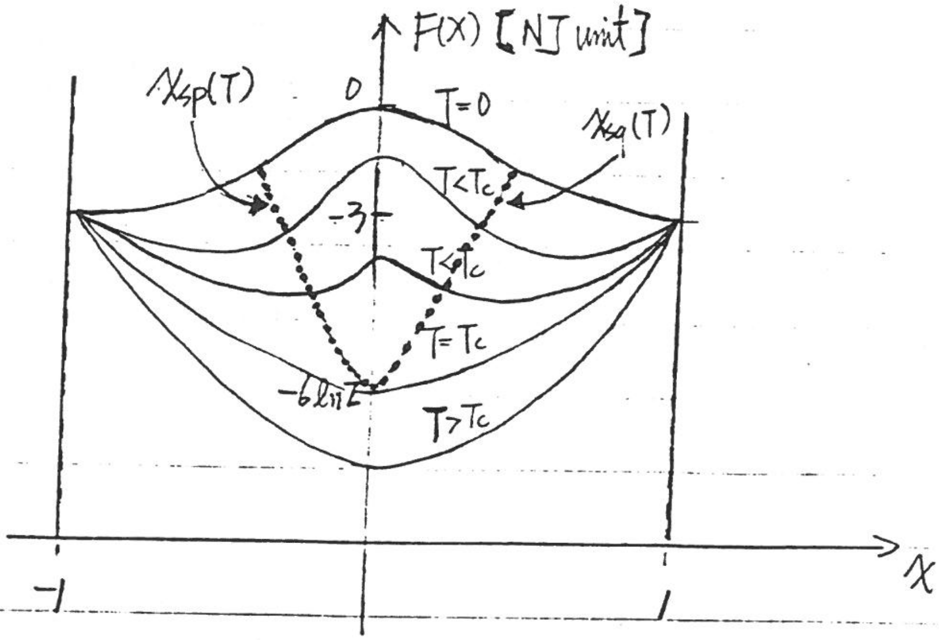
for $|x| \ll 1$

$$\frac{\partial^2 F}{\partial x^2} = N(k_BT - 6J) + Nk_BT\,x^2$$

For $k_BT - 6J < 0$, $\frac{\partial^2 F}{\partial x^2}$ could be less than zero, i.e. the convexity of F breaks down.

Therefore, $T_c = \frac{6J}{k_B}$.

(e)



For $\frac{\partial^2 F}{\partial x^2} < 0 \Rightarrow N(k_BT - 6J) + Nk_BT\,x^2 < 0 \Rightarrow x^2 < \left(\frac{6J}{k_BT} - 1\right)$

$$\Rightarrow |x_{sg}(T)| = \sqrt{6J/k_BT \;- 1}$$

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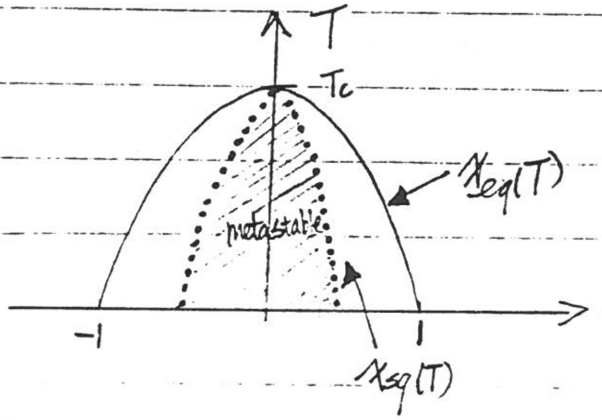
(f) $\left.\frac{\delta F}{\delta x}\right|_{x=x_{eq}} = 0 \Rightarrow x_{eq}\left[x_{eq}^{2} - 3\left(\frac{bJ}{k_BT} - 1\right)\right] = 0$

For $T > T_c$, $x_{eq} = 0$

$T < T_c$, $x_{eq} = \pm\sqrt{3}\sqrt{\frac{bJ}{k_BT} - 1} = \pm\sqrt{3}\,|x_{sq}|$

(g)



2.(a) $E = -\frac{NJ}{2}\sum_{ij=1}^{N}\frac{\sigma_i\sigma_j}{N} - h\sum_{i=1}^{N}\sigma_i$

$= -\frac{NJ}{2}\left(\sum_{i=1}^{N}\frac{\sigma_i}{N}\right)\left(\sum_{j=1}^{N}\frac{\sigma_j}{N}\right) - hN\left(\sum_{i=1}^{N}\frac{\sigma_i}{N}\right)$

$= -N\left(\frac{J}{2}m^2 + hm\right)$

(b) $Z = \sum_{\{\sigma_i\}} e^{-\beta H} = \sum_{M=-N}^{N}\frac{N!}{N_\uparrow!\,N_\downarrow!}\,e^{-\beta H}$

$= \sum_{M=-N}^{N} e^{\frac{S}{k_B} - \beta H} = \sum_{M=-N}^{N} e^{-\beta F}$

$\left(S = k_B \ln\Omega = k_B \ln\frac{N!}{N_\uparrow!\,N_\downarrow!}\right)$

Where $N_\uparrow$ and $N_\downarrow$ are # of spin-up and spin-down electrons respectively.

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$$N_\uparrow = \left(\frac{1+m}{2}\right)N \quad , \quad N_\downarrow = \left(\frac{1-m}{2}\right)N$$

$$\Rightarrow F(m,h) = E - TS \cong N\left[-(\ln 2)\,k_BT - hm + \frac{1}{2}(k_BT - J)\,m^2 + \frac{k_BT}{12}m^4\right]$$

(c) By saddle point approximation, there is $m = \bar{m}$ that minimizes $F(m,h)$

Let $\min[F(m,h)] \equiv F(\bar{m}, h)$

$$\underset{\substack{\uparrow \\ \#\text{ of } \bar{m}\text{'s}}}{g}\, e^{-\beta F(\bar{m},h)} \leq Z = e^{-\beta F(h,T)} \leq \underset{\substack{\uparrow \\ \#\text{ of all possible } m\text{'s}}}{2N}\, e^{-\beta F(\bar{m},h)}$$

$$\Rightarrow \frac{-\ln g}{\beta N} + \frac{F(\bar{m},h)}{N} \geq \frac{F(h,T)}{N} \geq \frac{F(\bar{m},h)}{N} - \frac{\ln(2N)}{N}$$

In (b), we notice F has the same order as N.

For $\lim\limits_{N\to\infty} \frac{\ln N}{N} \to 0 \Rightarrow \underset{\substack{\uparrow \\ \text{microscopic}}}{F(\bar{m}, h)} = \underset{\substack{\uparrow \\ \text{macroscopic}}}{F(h,T)}$

Therefore, saddle point method is valid as long as N is sufficient large.

(d) For $h=0$,

$$\frac{\partial F}{\partial m^2}\Big|_{T=T_c}=0 \quad \Rightarrow \quad T_c = J/k_B$$

$$\frac{\partial F}{\partial m}\Big|_{m=\bar{m}}=0 \quad \Rightarrow \quad \bar{m} = \begin{cases} 0 & \text{for } T>T_c \\ \pm\sqrt{3}\left(\frac{T_c-T}{T}\right)^{1/2} & \text{for } T<T_c \end{cases}$$

(e) Let $t \equiv \frac{T_c-T}{T_c}$

$$C = \frac{\partial E}{\partial T}\Big|_{h=0,\, m=\bar{m}} = \begin{cases} 0 & (\text{for } T>T_c) \\ \frac{3NJT_c}{2T^2} & (\text{for } T<T_c) \longrightarrow \frac{3NJ}{2T_c}\, t^0,\ T\to T_c^- \end{cases}$$

Therefore, $\alpha_\pm = 0$ and $A_+/A_- = 0$

For $h\neq 0$, $\frac{\partial F}{\partial m}\Big|_{m=\bar{m}}=0 \Rightarrow h=(k_BT-J)+k_BT\,\bar{m}^2$

$$\chi = \frac{\partial \bar{m}}{\partial h}\Big|_{h=0} = \begin{cases} \frac{1}{k_B(T-T_c)} = \frac{T_c}{k_B}\, t^{-1} & \text{for } T>T_c \\ \frac{1}{2k_B(T-T_c)} = \frac{T_c}{2k_B}\, t^{-1} & \text{for } T<T_c \end{cases}$$

Therefore, $\gamma_\pm = -1$ and $A_+/A_- = 2$

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3(a) $$\Xi = \sum_{N=0}^{\infty} \frac{e^{-N\beta\mu}}{N!} Z_N$$

$$= \sum_{N=0}^{\infty} \frac{e^{-N\beta\mu}}{N!} \int \frac{\prod_{i=1}^{N} d^3\vec{p}_i \, d^3\vec{q}_i}{h^{3N}} e^{-\beta H}$$

$$= \sum_{N=0}^{\infty} \frac{e^{-N\beta\mu}}{N! \, h^{3N}} \left\{ \prod_{i=1}^{N} \left(\int d^3\vec{p}_i \, e^{-\beta \frac{\vec{p}_i^{\,2}}{2m}} \right) \right\} \left\{ \int \prod_{i=1}^{N} d^3\vec{r}_i \, e^{-\frac{\beta}{2} \sum_{ij} \mathcal{V}} \right\}$$

$$\int d^3\vec{p}_i \, e^{-\frac{\beta \vec{p}_i^{\,2}}{2m}} = \left(\frac{2\pi m}{\beta}\right)^{3/2} \Rightarrow \Xi = \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{e^{\beta\mu}}{\lambda^3}\right)^N \int \prod_{i=1}^{N} d^3\vec{r}_i \, e^{-\frac{\beta}{2} \sum_{ij} \mathcal{V}}$$

where $\lambda = h / \sqrt{2\pi m k_B T}$

(b) $$N = \sum_{\alpha=1}^{\mathcal{N}} n_\alpha$$

$$H = \sum_{\alpha=1}^{\mathcal{N}} n_\alpha \frac{p_\alpha^2}{2m} + \frac{1}{2} \sum_{\alpha,\beta=1}^{\mathcal{N}} n_\alpha n_\beta \, \mathcal{V}(\vec{r}_\alpha - \vec{r}_\beta)$$

$$\int \prod_{i=1}^{N} d^3\vec{r}_i \rightarrow \left(a^3\right)^{\sum_{\alpha=1}^{\mathcal{N}} n_\alpha}$$

Each cell is distinguishable $\Rightarrow \sum_N \frac{1}{N!} \rightarrow \sum_{\{n_\alpha = 0,1\}}$

Therefore, $\Xi \approx \sum_{\{n_\alpha=0,1\}} \left(\frac{e^{\beta\mu}a^3}{\lambda^3}\right)^{\sum_\alpha n_\alpha} \exp\left[-\frac{\beta}{2}\sum_{\alpha,\beta=1}^{N} n_\alpha n_\beta V(\vec{r}_\alpha-\vec{r}_\beta)\right]$

(c) $$\Xi = \sum_{\{\sigma_\alpha=\pm1\}} \exp\left[\left(\beta\mu+3\ln\frac{a}{\lambda}\right)\sum_\alpha^N\left(\frac{1+\sigma_\alpha}{2}\right)+\frac{\beta J}{2N}\sum_{\alpha\beta}^N\left(\frac{1+\sigma_\alpha}{2}\right)\left(\frac{1+\sigma_\beta}{2}\right)\right]$$

$$= \sum_{\{\sigma_\alpha=\pm1\}} \exp\left[\frac{N}{2}\left(\beta\mu+3\ln\frac{a}{\lambda}-\frac{\beta J}{4}\right)\right]\cdot\exp\left[\frac{N}{2}\left(\beta\mu+3\ln\frac{a}{\lambda}-\frac{\beta J}{2N}\right)\left(\sum_{\alpha=1}^N\frac{\sigma_\alpha}{N}\right)-\frac{N\beta J}{8}\left(\sum_{\alpha\beta}^N\frac{\sigma_\alpha\sigma_\beta}{N^2}\right)\right]$$

Let $\sum_\alpha \frac{\sigma_\alpha}{N} = m \qquad \frac{1}{2}\left(\mu+\frac{3}{\beta}\ln\frac{a}{\lambda}-\frac{J}{2N}\right)=h' \qquad \frac{J}{4}=J'$

$$\Rightarrow \quad \Xi = (\text{constant})\times\sum_{\{\sigma_\alpha\}}\exp\left[N\beta\left(h'm+\frac{J'}{2}m^2\right)\right]$$

the same functional form as the partition function in problem 2.

Identify $m = \sum_\alpha \frac{\sigma_\alpha}{N} = \sum_\alpha \frac{(2n_\alpha-1)}{N} = 2a^3\rho-1$

where $\rho = \frac{N}{V}$ is the # density of the gas.

We can expect a critical temperature, $T_c = 4J/k_B$, of the imperfect gas. Below T_c, there is a phase separation of ρ when $h'=0$:

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Discontinuous Transitions

When the order parameter m goes to zero discontinuously, the phase transition is said to be first order. The most commonly encountered first order transitions in Landau theory are described in the following problems.

1. *Cubic Invariants:* In some systems (e.g. liquids) symmetry considerations do not exclude a cubic term in the Landau free energy. Consider

$$\mathcal{H} = \int d^d\mathbf{x}[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + cm^3 + um^4] \qquad (c, u > 0).$$

(a) By plotting $F(m)$ for uniform m at various values of t show that as t is reduced there is a discontinuous jump to $\overline{m} \neq 0$ for a positive $\overline{t}$.

(b) By writing down the two conditions that $\overline{m}$ and $\overline{t}$ must satisfy at the transition, solve for $\overline{m}$ and $\overline{t}$.

(c) Recall that the correlation length ξ is related to the curvature of $F(m)$ at its minimum by $K\xi^{-2} = \partial^2 f/\partial m^2|_{eq.}$. Plot ξ as a function of t.

****

2. *Tricritical Point:* In class we examined the Landau Hamiltonian

$$\mathcal{H} = \int d^d\mathbf{x}[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 + vm^6 - hm]$$

with $u > 0$ and $v = 0$. If $u < 0$, then a positive v is necessary for stability.

(a) By sketching $F(m)$ for various t, show that there is a first-order transition for $u < 0$ and $h = 0$.

(b) Calculate $\overline{t}$ and the discontinuity $\overline{m}$ at this transition.

(c) For $h = 0$ and $v > 0$ plot the phase boundary in the (u, t) plane, identifying the phases, and order of the phase transitions.

(d) The special point $u = t = 0$, separating first- and second-order phase boundaries, is called a tricritical point. For $u = 0$ calculate the exponents α, β, γ, and δ.

(Recall: $C \sim t^{-\alpha}$; $\overline{m} \sim t^{\beta}$; $\chi \sim t^{-\gamma}$; and $\overline{m} \sim h^{1/\delta}$.)

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Suggested Reading: Huang, Chapter 17

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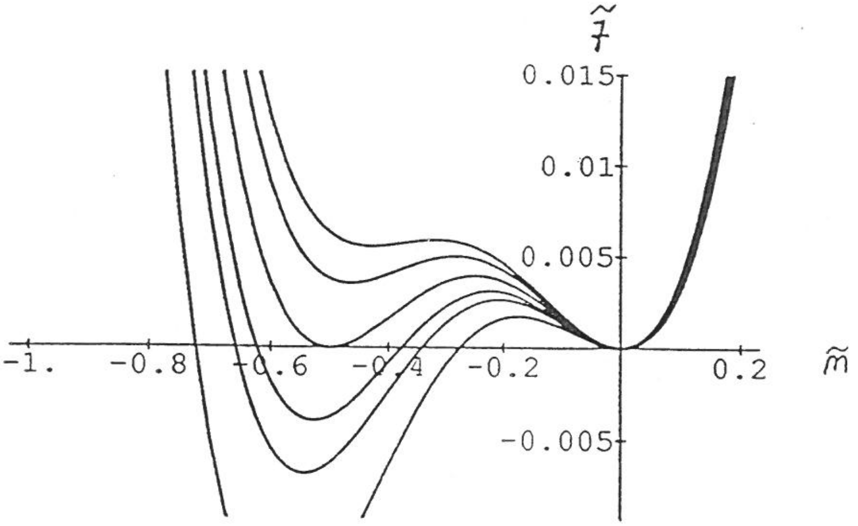
8.334 PS2 SOLUTIONS

1a) $f(m) = \frac{t}{2} m^2 + cm^3 + um^4$: Rescale $m = \tilde{m} * \left(\frac{c}{u}\right)$, $t = \tilde{t}\left(c^2/u\right)$, $f = \tilde{f}\left(c^4/u\right.$

$$\tilde{f} = \frac{\tilde{t}}{2}\tilde{m}^2 + \tilde{m}^3 + \tilde{m}^4$$

$\tilde{f}$ is plotted below for $\tilde{t} = 0.55$, 0.53, 0.50, 0.47, 0.45, 0.40



The discontinous transition occurs when the local minimu for $\tilde{m} < 0$ becomes the absolute minimum. There i a corresponding jump from $\tilde{m} = 0$ to $\tilde{m} < 0$ as calcula below:

1b) For the 1st order transition $\tilde{f}(0) = \tilde{f}(\bar{m}) = 0$ and $\left.\frac{\partial \tilde{f}}{\partial \tilde{m}}\right|_{\bar{\tilde{m}}} = 0$

$$\Rightarrow \quad \bar{t}/2 + \bar{m} + \bar{m}^2 = 0 \quad \text{and} \quad \bar{t} + 3\bar{m} + 4\bar{m}^2 = 0$$

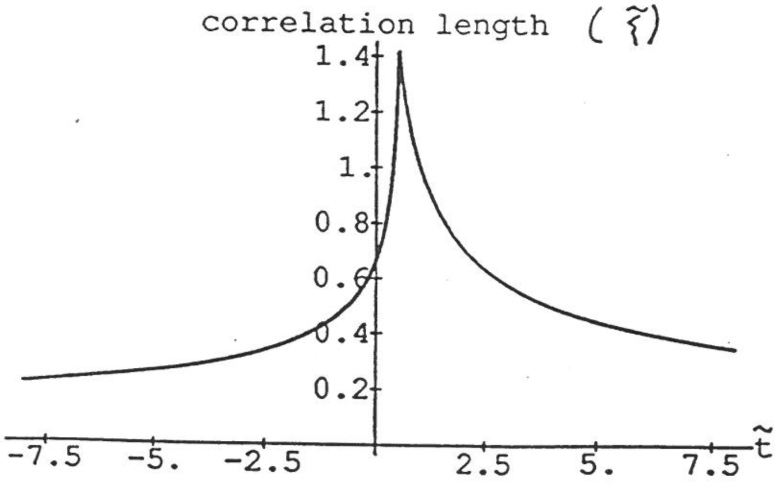
$$\bar{\tilde{t}} = 1/2, \quad \bar{\tilde{m}} = -1/2 \quad \text{or} \quad \bar{t} = 1/2\left(c^2/u\right), \quad \bar{m} = -1/2\left(c/u\right)$$

1c) For $\tilde{t} > 1/2$ $\tilde{m}_{eq} = 0$, for $\tilde{t} < 1/2$ $\tilde{m}_{eq} = \dfrac{-3 - \sqrt{9 - 16\tilde{t}}}{8}$

$$K\xi^{-2} = \left.\frac{\partial^2 f}{\partial m^2}\right|_{m_{eq}} = \left(\frac{c^2}{u}\right)\left.\frac{\partial^2 \tilde{f}}{\partial \tilde{m}^2}\right|_{\tilde{m}_{eq}} \qquad \xi = \tilde{\xi}\sqrt{\frac{uK}{c^2}}$$

$$\frac{1}{\tilde{\xi}^2} = \tilde{t} + 6\tilde{m}_{eq} + 12\tilde{m}_{eq}^2$$

$$\frac{1}{\tilde{\xi}^2} = \begin{cases} -2\tilde{t} - 3\tilde{m}_{eq}, & \tilde{t} < 1/2 \\ \tilde{t}, & \tilde{t} > 1/2 \end{cases}$$



Note- The correlation length ξ is finite for the discontinous transition.

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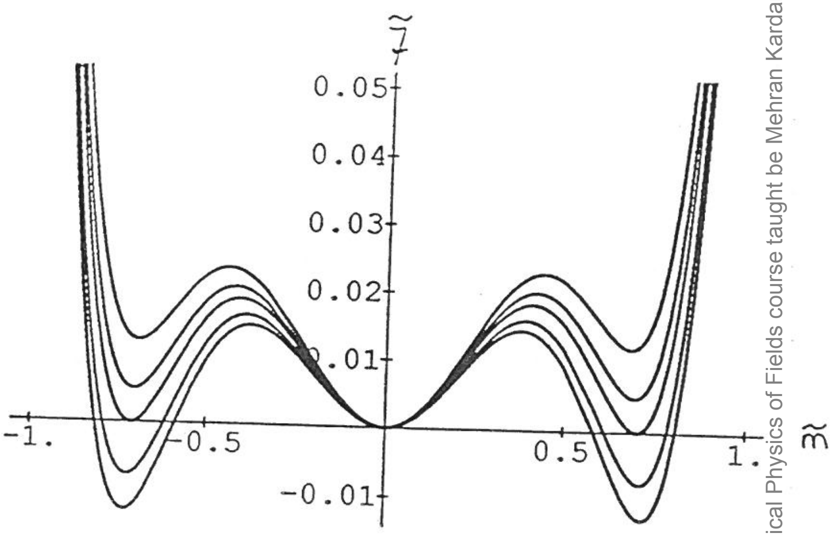
2a) for $h=0$ $u<0$ $f(m)= t/2\, m^2 - |u| m^4 + v m^6$

Again rescale $m^2 = \frac{|u|}{v}\tilde m^2$, $t = \frac{|u|^2}{v}\tilde t$, and $f = \frac{|u|^3}{v^2}\tilde f$

$\tilde f = \tilde t/2\, \tilde m^2 - \tilde m^4 + \tilde m^6$

$\tilde f$ is plotted for $\tilde t = \{0.55, 0.52, 0.50, 0.47, 0.45\}$

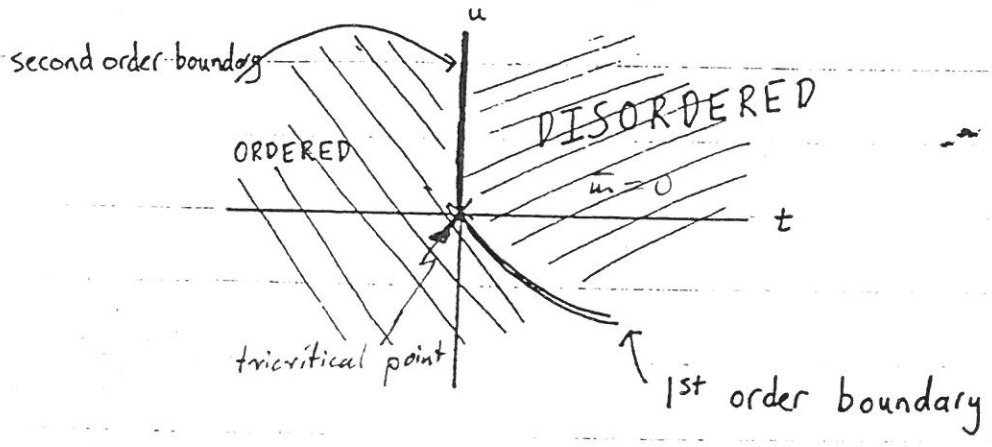
there is a first order transition at $\tilde t = 1/2$, $\tilde m^2 = 1/2$



2b) $\tilde f(\bar{\tilde m}) = \tilde f(0) = 0$ and $\partial\tilde f/\partial\tilde m|_{\bar{\tilde m}} = 0$

$\tilde t/2 - \tilde m^2 + \tilde m^4 = 0$ and $\tilde t - 4\tilde m^2 + 6\tilde m^4 = 0 \Rightarrow \bar{\tilde t} = 1/2$, $\bar{\tilde m} = \pm\sqrt{1/2}$

2c) or $\bar t = \frac{1}{2}\, u^2/v$ $\bar m = \pm\sqrt{1/2}\sqrt{\frac{|u|}{v}}$ for $u<0$



2d) at $u=0$ $\tilde f = \tilde t/2\, \tilde m^2 + \tilde m^6 - \tilde h\tilde m$ where $h = u^2/v\sqrt{u/v}\,\tilde h$

For α and β set $\tilde h = 0$ $\partial\tilde f/\partial\tilde m|_{\tilde m_{eq}} = 0 = \tilde m(\tilde t + 6\tilde m^4)$

$\tilde m_{eq} = 0$ $t>0$, $\tilde m_{eq} = (-t/6)^{1/4}$ for $t<0$ $\Rightarrow$ $\boxed{\beta = 1/4}$

$\tilde f_{eq} = 0$ for $t>0$: $\tilde f_{eq} \sim |\tilde t|^{3/2}$ for $t<0$ $\Rightarrow$ $2-\alpha = 3/2$, $\boxed{\alpha = 1/2}$

For δ set $\tilde t = 0$: $\partial\tilde f/\partial\tilde m|_{\tilde m_{eq}} = 0 = 6\tilde m_{eq}^5 - h$ $\Rightarrow$ $\boxed{\delta = 5}$

For $\tilde h$ finite, $\tilde t$ finite $\tilde h = \tilde t\tilde m_{eq} + 6\tilde m_{eq}^5$, $\chi^{-1} \sim t^{\gamma} \sim \partial\tilde h/\partial\tilde m_{eq}$

$\partial\tilde h/\partial\tilde m_{eq} = \tilde t + 30\tilde m_{eq}^4 = \begin{cases}\tilde t, & \tilde t>0\\ -4\tilde t, & \tilde t<0\end{cases}$ $\boxed{\gamma = 1}$

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8.334 PS.2 Solutions

1a) $F(m) = \frac{t}{2} m^2 + cm^3 + um^4$ - free energy

It is possible to eliminate c and u by the rescaling:

$$m_r = \frac{u}{c} m$$

Then

$$F_r(m_r) = \frac{1}{2} t_r m_r^2 + m_r^3 + m_r^4$$

where

$$F_r = \left(\frac{c^4}{u^3}\right) F \quad \text{and} \quad t_r = \left(\frac{u}{c^2}\right) t$$

To the minima and maxima of $F_r(m_r)$ - function consider the derivative

$$\frac{dF_r}{dm_r} = m_r(4m_r + 3m_r + t_r)$$

This derivative vanishes at $m_r = 0$. This corresponds to the minimum of $F_r(m_r)$ at $m_r = 0$ provided $t_r > 0$

In addition, if $t_r < 9/16$ the derivative vanishes also at

$$m_r = \frac{-3 \pm \sqrt{9-16t_r}}{8}$$

It means that $F_r(m_r)$ has a maximum at

$$m_r = m_{max} = \frac{\sqrt{9-16t_r}}{8} - \frac{3}{8}$$

and a minimum at

$$m_r = m_{min} = -\frac{3}{8} - \frac{\sqrt{9-16t_r}}{8}$$

(the maximum should be located between the minima)

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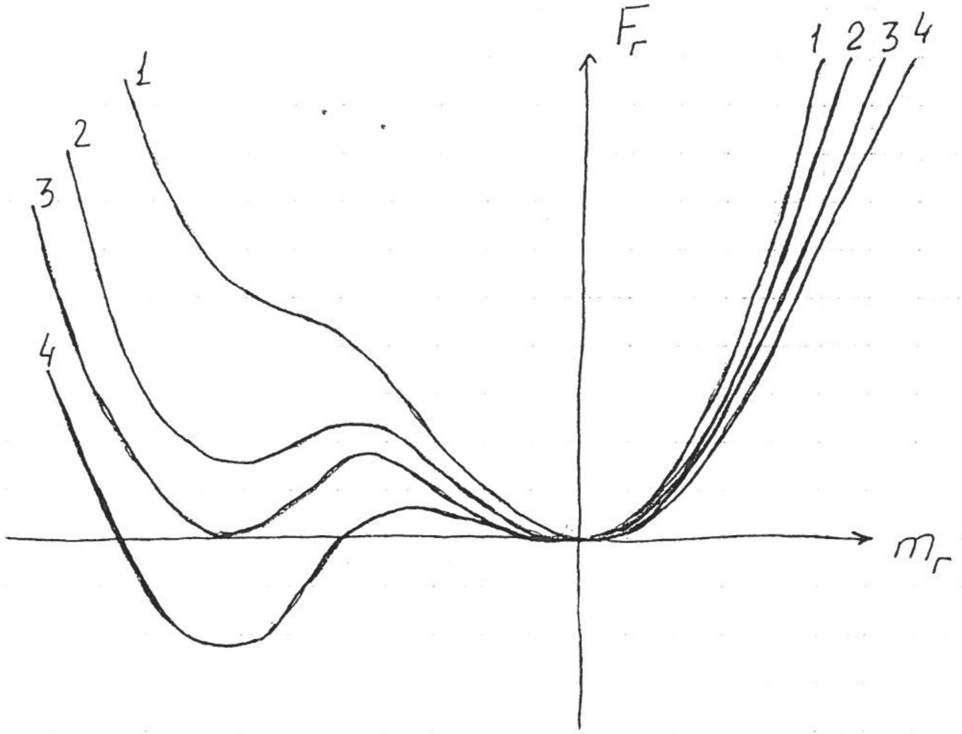
It is also clear that $F_r(0)=0$. Therefore, $F_r(m_r)$ dependance at different t_r can be sketched as



1 - $t_r > \frac{9}{16}$ only one minimum at $m_r=0$

2 - $\bar{t}_r < t_r < \frac{9}{16}$ two minima but $F(m_{min}) > F(0)=0$

3. - $t_r = \bar{t}_r$ $F(m_{min}) = F(0) = 0$

4 - $t_r < \bar{t}_r$ $F(m_{min}) < F(0)$

The discontinous transition occurs when the local minimum at $m_{min} < 0$ becomes the absolute minimum There is a corresponding jump of m_r from $m_r=0$ to $m_r = \bar{m}$ where $\bar{m}_r = m_{min} \, (t_r = \bar{t}_r)$

1b) To determine $\bar{m}_r$ and $\bar{t}_r$ we have therefore to solve the two equations

$$\frac{dF(m_r)}{dm_r} = 0 \quad \text{and} \quad F(m_r) = F(0) = 0$$

simaltaneously

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Excluding the trivial solution $m_r = 0$ we get

$$\left.\begin{array}{l} m_r^2 + m_r + \frac{t_r}{2} = 0 \\ 4m_r^2 + 3m_r + t_r = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} m_r = -\frac{1}{2} \\ t_r = \frac{1}{2} \end{array}\right.$$

In the original units

$$\bar{m} = -\frac{c}{2u} \qquad \bar{t} = \frac{c^2}{2u}$$

1c) The equilibrium value of $m = m_{eq}$ equals ~~to~~

$$m_{eq} = \begin{cases} 0 & \text{for } t > \bar{t} = \frac{c^2}{2u} \\ -\frac{\sqrt{9-16ut/c^2} + 3}{8} & \text{for } t < \bar{t} \end{cases}$$

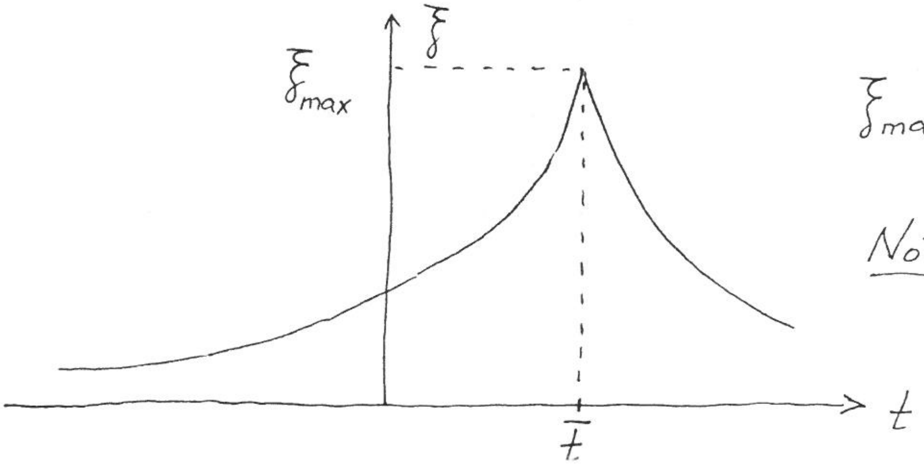
For the correlation length ξ we have

$$\frac{K}{\xi^2} = \left.\frac{\partial^2 F}{\partial m^2}\right|_{m=m_{eq}} = \cancel{2}'t + 6cm_{eq} + 12um_{eq}^2$$

Therefore

$$\frac{1}{\xi^2} = \begin{array}{ll} t/K & \text{for } t > \bar{t} \\ -\frac{1}{K}(t+3cm_{eq}) & \text{for } t < \bar{t} \end{array}$$

(We have used the equation $dF(m)/dm \big|_{m=m_{eq}} = 0$



$$\xi_{max} = \xi(\bar{t}) = \sqrt{\frac{K}{\bar{t}}} = \frac{\sqrt{2uK}}{c}$$

<u>Note</u> The correlation length ξ is <u>finite</u> for the discontinuous phase transition

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2a) For $h=0$ $\qquad F(m) = \frac{t}{2} m^2 + u m^4 + v m^6$

For $t>0$ $F(m)$ has a minimum at $m=0$.

In addition, at some u and v the free energy can have minima at nonzero m.

$$\frac{dF}{dm^2} = 3vm^4 + 2um^2 + \frac{t}{2} = 0$$

$$m^2_{1,2} = -\frac{u}{3v} \pm \frac{\sqrt{4u^2 - 6vt}}{6v}$$

We have real and positive solutions of the equation

$$dF/dm^2 = 0$$

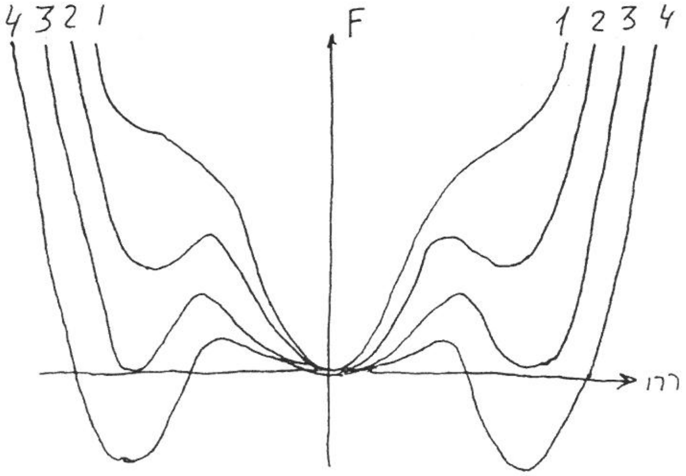
provided a) $u<0$ and b) $t < \frac{2u^2}{3v}$.

Under these conditions there are two maxima at

$$m_r^2 = m^2_{max} = \frac{|u|}{3v} - \frac{\sqrt{4u^2-6vt}}{6v}$$

and two minima of $F(m)$ at

$$m_r^2 = m^2_{min} = \frac{|u|}{3v} + \frac{\sqrt{4u^2-6vt}}{6v}$$



1 $\quad t > 2u^2/3v$

2 $\quad \bar{t} < t < 2u^2/3v$

3 $\quad t = \bar{t}$

4 $\quad t < \bar{t}$

There is a discontinuous phase transition at $u<0$ and $t = \bar{t}$

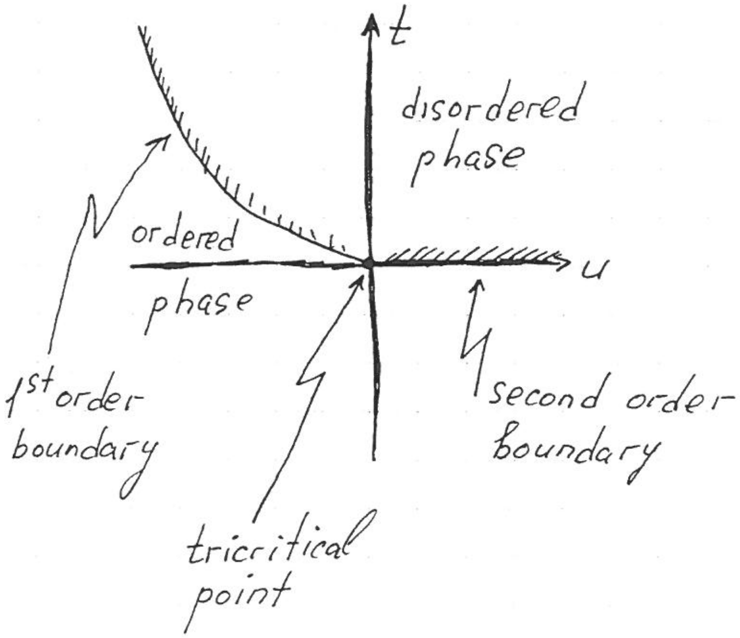
2b) To determine $\bar{m}$ and $\bar{u}$ we again have to solve the two equations

$$\frac{dF}{dm^2}\bigg|_{\bar{m}^2} = 0 \qquad \text{and} \qquad F(\bar{m}^2) = F(0) = 0$$

simultaneously.

$$\left.\begin{aligned} 3vm^4 + 2um^2 + \frac{t}{2} &= 0 \\ vm^4 + um^2 + \frac{t}{2} &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \bar{m}^2 &= -\frac{u}{2v} = \frac{|u|}{2v} \\ \bar{t} &= -u\bar{m}^2 = \frac{u^2}{2v} \end{aligned}\right.$$

2c) On the (u,t) plane the line $t = \frac{u^2}{2v}$ is a 1st order phase transition boundary. If $t=0, u>0$ there is the second order phase transition.



2d) At $u=0$ $\quad F = \frac{t}{2}m^2 + vm^6 - hm$

To calculate α and β set $h=0$

$$\frac{\partial F}{\partial m}\bigg|_{m=m_{eq}} = m_{eq}\left(t + 6vm_{eq}^4\right) = 0 \Rightarrow m_{eq} = \begin{cases} 0 & \text{for } t > \bar{t} = 0 \\ \left(-\frac{t}{6}\right)^{1/4} & \text{for } t < 0 \end{cases}$$

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$$m_{eq} \sim (-t)^{1/4} \Rightarrow \boxed{\beta = \frac{1}{4}}$$

$$F(m_{eq}) \sim m_{eq}^6 \sim (-t)^{3/2}$$

If $t \sim T - T_c$ then $c \sim \frac{\partial^2 F}{\partial T^2} \sim \frac{\partial^2 F}{\partial t^2} \sim t^{-\frac{1}{2}}$

Therefore $\boxed{\alpha = \frac{1}{2}}$

To calculate δ set $t = 0$ $h \neq 0$

$$F(m_{eq}) = v m_{eq}^6 - h\, m_{eq}$$

$$\left.\frac{\partial F}{\partial m}\right|_{m_{eq}} = 0 \Rightarrow 6 m_{eq}^5 = h \Rightarrow m_{eq} \sim h^{1/5} \Rightarrow \boxed{\delta = 5}$$

Finally for finite h and t

$$\frac{dF}{dm} = 0 \Rightarrow h = t m_{eq} + 6 m_{eq}^5 v$$

$$\chi = \frac{\partial m_{eq}}{\partial h} = \left(\frac{\partial h}{\partial m_{eq}}\right)^{-1} = \left(t + 30 v m_{eq}^4\right)^{-1}$$

Therefore

$\chi \sim \frac{1}{|t|}$ for both $t > 0$ and $t < 0$

As a result $\boxed{\gamma = 1}$

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Goldstone Modes

1. *Spin Waves:* In the XY model of $n = 2$ magnetism, a unit vector $\vec{s} = (s_x, s_y)$ (with $s_x^2 + s_y^2 = 1$) is placed on each site of a d–dimensional lattice. There is an interaction that tends to keep nearest–neighbors parallel, i.e. a Hamiltonian $-\beta\mathcal{H} = K\sum_{<ij>} \vec{s}_i.\vec{s}_j$. The notation $< ij >$ is conventionally used to indicate summing over all *nearest–neighbor* pairs (i, j).

(a) Rewrite the partition function $Z = \int \prod_i d\vec{s}_i \exp(-\beta\mathcal{H})$ as an integral over the set of angles $\{\theta_i\}$ between the spins $\{\vec{s}_i\}$ and some arbitrary axis.

(b) At low temperatures ($K \gg 1$), the angles $\{\theta_i\}$ vary slowly from site to site. In this case expand $-\beta\mathcal{H}$ to get a quadratic form in $\{\theta_i\}$.

(c) For $d = 1$ consider L sites with periodic boundary conditions (i.e. forming a chain). Find the normal modes θ_q that diagonalize the quadratic form (by Fourier transformation), and the corresponding eigenvalues $K(q)$. Pay careful attention to whether the modes are real or complex, and to the allowed values of q.

(d) Generalize the results from (c) to a d–dimensional simple cubic lattice with periodic boundary conditions.

(e) Calculate the contribution of these modes to the free energy and specific heat. (Evaluate the classical partition function, i.e. do not quantize the modes.)

(f) Find an expression for $\langle \vec{s}_0.\vec{s}_{\mathbf{x}}\rangle = \mathrm{Re}\langle \exp[i\theta_{\mathbf{x}} - i\theta_0]\rangle$ using standard formulas of Gaussian integrals. Convince yourself that for $|\mathbf{x}| \to \infty$, only $q \to 0$ modes contribute appreciably to this expression, and hence calculate the asymptotic limit.

(g) Calculate the transverse susceptibility from $\chi_t \propto \int d^d\mathbf{x} \langle \vec{s}_0.\vec{s}_{\mathbf{x}}\rangle_c$. How does it depend on the system size L?

(h) In $d = 2$ show that χ_t only diverges for K larger than a critical value $K_c = 1/4\pi$.

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2. *Capillary Waves:* A reasonably flat surface in d–dimensions can be described by its height h, as a function of the remaining $(d-1)$ coordinates $\mathbf{x} = (x_1, ...x_{d-1})$. Convince yourself that the "area" is in general given by $\mathcal{A} = \int d^{d-1}\mathbf{x}\sqrt{1 + (\nabla h)^2}$. With a surface tension σ, the Hamiltonian is simply $\mathcal{H} = \sigma\mathcal{A}$.

(a) At sufficiently low temperatures there will be only slow variations in h. Expand the energy to quadratic order, and write down the partition function as a functional integral.

(b) Use Fourier transformation to diagonalize the quadratic Hamiltonian into normal modes $h_{\mathbf{q}}$ (capillary waves).

(c) What symmetry breaking is responsible for these Goldstone modes?

(d) Calculate the height–height correlations $\langle (h(\mathbf{x}) - h(\mathbf{x}'))^2 \rangle$.

(e) Comment on the form of the result (d) in dimensions $d = 4$, 3, 2, and 1.

(f) By estimating typical values of ∇h, comment on when it is justified to ignore higher order terms in the expansion for $\mathcal{A}$.

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Suggested reading: Huang, Chapter 16, and Negele and Orland, Chapter 4.

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1) a,b) $$Z = \int \prod_i d\theta_i \exp K \sum_{\langle ij\rangle} \cos(\theta_i-\theta_j)$$

$$\approx \int \prod_i d\theta_i \exp\left\{-\frac{1}{2} K \sum_{\langle ij\rangle}(\theta_i-\theta_j)^2\right\}$$

c,d) Label lattice sites by vectors $\vec{r}$ with each component $0, 1, 2, \ldots L-1$

Let $\theta(\vec{r}) = \frac{1}{L^{d/2}}\sum_{\vec{q}}\theta(\vec{q})e^{i\vec{q}\cdot\vec{r}}$, $\theta(\vec{q}) = \frac{1}{L^{d/2}}\sum_{\vec{r}}\theta(\vec{r})e^{-i\vec{q}\cdot\vec{r}}$

with each component of $\vec{q}$ taking values $0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \ldots, \pi$

[$\theta(-\vec{q}) = \theta^*(\vec{q})$]. Let $\vec{e}_1, \ldots \vec{e}_d$ be unit vectors along axes. Then $\sum_{\langle ij\rangle}(\theta_i-\theta_j)^2 = \sum_{\vec{r}}\sum_{\vec{e}}\left\{\theta(\vec{r})-\theta(\vec{r}+\vec{e})\right\}^2$

$$\theta(\vec{r}+\vec{e})-\theta(\vec{r}) = \frac{1}{L^{d/2}}\sum\theta(\vec{q})\left\{e^{i\vec{q}\cdot\vec{e}}-1\right\}e^{i\vec{q}\cdot\vec{r}}$$

so $\sum_{\langle ij\rangle}(\theta_i-\theta_j)^2 = \sum_{\vec{q},\vec{e}}\theta(\vec{q})\theta(-\vec{q})\left|e^{i\vec{q}\cdot\vec{e}}-1\right|^2$

and $+\frac{1}{2}K\sum_{\langle ij\rangle}(\theta_i-\theta_j)^2 = \frac{1}{2}\sum_{\vec{q}}K(\vec{q})\,|\theta(\vec{q})|^2$

$$K(\vec{q}) = 2K\sum_{\vec{e}}(1-\cos\vec{q}\cdot\vec{e}) \approx K|\vec{q}|^2 \text{ for } |\vec{q}| \ll 1$$

[ie. wavelength >> lattice spacing]

e) Contribution of a normal mode with weight $e^{-\frac{1}{2}K\theta^2}$ to Z is $\int d\theta\, e^{-\frac{1}{2}K\theta^2} = \sqrt{\frac{2\pi}{K}}$ so $\ln Z = \text{const} - \frac{1}{2}\sum_{\vec{q}}\ln K(\vec{q})$

[2 complex modes $\vec{q}, -\vec{q}$ are equivalent to 2 real modes $\cos\vec{q}\cdot\vec{r}$, $\sin\vec{q}\cdot\vec{r}$ so counting is correct] and free energy $F = \frac{1}{2\beta}\frac{L^d}{(2\pi)^d}\int d^dq\,\ln K(\vec{q})$

[density of $\vec{q}$ values is $\left(\frac{L}{2\pi}\right)^d$] Since $K \propto \beta$, temperature dependence of F is $\frac{T}{2}(-\ln T + \text{const})$ per site, and $C = -T\frac{\partial^2 F}{\partial T^2} = \frac{1}{2}$ per site ($k_B = 1$)

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F) $\theta_{\vec{x}} - \theta_{\vec{0}} = \frac{1}{L^{d/2}} \sum \theta(\vec{q}) \{ e^{i\vec{q}\cdot\vec{x}} - 1 \}$

so $$\left\langle e^{i(\theta_{\vec{x}} - \theta_{\vec{0}})} \right\rangle = \frac{1}{\mathcal{N}} \int \prod_{\vec{q}} d\theta(\vec{q}) \exp\left\{ -\frac{1}{2} K(q)\theta(\vec{q})\theta(-\vec{q}) + \frac{i}{2L^{d/2}} \theta(\vec{q})(e^{i\vec{q}\cdot\vec{x}} - 1) + \frac{i}{2L^{d/2}} \theta(-\vec{q})(e^{-iq\cdot x} - 1) \right\}$$

(where $\mathcal{N}$ is some integral without linear terms)

$$= \frac{1}{\mathcal{N}} \int \prod_{q} \left[d\theta(\vec{q}) \exp -\frac{1}{2} K(\vec{q}) \left\{ \theta(\vec{q}) - \frac{i}{L^{d/2}} \frac{(e^{-iq\cdot x} - 1)}{K(q)} \right\} \left\{ \theta(-q) - \frac{i}{L^{d/2}} \frac{(e^{i\vec{q}\cdot\vec{x}} - 1)}{K(q)} \right\} \times \exp -\frac{1}{2} \frac{1}{L^d} \frac{|e^{iq\cdot x} - 1|^2}{K(q)} \right]$$

Integrals cancel and $\left\langle e^{i(\theta_x - \theta_0)} \right\rangle = \exp -\frac{1}{2} \frac{1}{(2\pi)^d} \int d^dq \frac{2(1 - \cos\vec{q}\cdot\vec{x})}{K(\vec{q})}$

We are interested in case $K \gg 1$ so that $\langle(\theta_i - \theta_j)^2\rangle$ is small for neighbors. In fact Away from $\vec{q} = 0$, integral is bounded and $\propto \frac{1}{K}$, $\cos\vec{q}\cdot\vec{x}$ oscillates rapidly and integral is independent of $\vec{x}$, for $|\vec{x}|$ large. Contribution from values close to $\vec{q} = \vec{0}$ $\approx \frac{1}{(2\pi)^d} \int_{|\vec{q}|<A} d^dq \frac{1 - \cos\vec{q}\cdot\vec{x}}{K|\vec{q}|^2}$

[A is some small number, independent of $\vec{x}$]

i) For $d \geq 3$, integral is convergent at $\vec{q} = 0$ even without $\cos\vec{q}\cdot\vec{x}$ term so we get a constant $\times \frac{1}{K}$ for $|\vec{x}|$ large, i.e. fluctuations of $\vec{S}_x \cdot \vec{S}_0$ remain small however large $|\vec{x}|$ is, long range order.

ii) $d = 1$, $\left\langle e^{i(\theta_x - \theta_0)} \right\rangle = \exp -\frac{1}{4\pi K} \int_{-\pi}^{\pi} \frac{1 - \cos q|x|}{1 - \cos q} = e^{-\frac{|x|}{2K}}$

[x is an integer], which is of course consistent with the fact that $\theta_x - \theta_0$ is just the sum of x Gaussian, independent random variables, $\theta_1 - \theta_0, \theta_2 - \theta_1, \cdots, \theta_x - \theta_{x-1}$

iii) $d = 2$. Integral $= \int_0^A \frac{2\pi q\, dq\, 2(1 - J_0(q|x|))}{Kq^2}$

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$$= \frac{4\pi}{K}\int_0^{A|x|}\frac{dq}{q}\left\{1-J_0(q)\right\} = -\frac{4\pi}{K}\int_A^{A|x|}\frac{dq}{q}\left\{1-J_0(q)\right\} + \text{constant indep of } |x|$$

$$= \frac{4\pi}{K}\ln|x| + \text{constant} \quad \left(J_0(q) \sim \frac{\sin(q+\delta)}{\sqrt{q}}\right)$$

so $\langle \vec{S}_0 \cdot \vec{S}_x \rangle \propto |x|^{-\frac{1}{2\pi K}}$, no long range order.

g, h) i) $d>2$. There is long range order and we are interested in states where each $\vec{S}_x$ fluctuates round a fixed direction

We have $\langle e^{i(\theta_x-\theta_0)}\rangle = 1 - \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dq \, \frac{2(1-\cos\vec{q}\cdot\vec{x})}{K(q)}$

A similar calculation gives $\langle e^{i\theta_x}\rangle = 1 - \frac{1}{2}\frac{1}{(2\pi)^d}\int\frac{d^dq}{K(q)}$

(N.B. this converges for $d>2$)

so $\langle e^{i\theta_x} e^{-i\theta_0}\rangle_c = \langle e^{i(\theta_x-\theta_0)}\rangle - \langle e^{i\theta_x}\rangle\langle e^{-i\theta_0}\rangle$

$$= \frac{1}{(2\pi)^d}\int\frac{d^dq}{K(q)}\cos\vec{q}\cdot\vec{x} \propto \int\frac{d^dq}{q^2}\cos\vec{q}\cdot\vec{x} \propto \frac{1}{|x|^{d-2}} \text{ for large } |x|$$

so $\chi_t = \int\langle\sigma_x\cdot\sigma_0\rangle_c \, d^dx \propto L^2$ for $d>2$.

ii) For $d=2$, no long range order, $\langle\vec{\sigma}_x\rangle = 0$ and

$\langle\vec{\sigma}_x\cdot\vec{\sigma}_0\rangle_c \propto |x|^{-\frac{1}{2\pi K}}$ so $\chi_t \sim \int d^2x\,|x|^{-\frac{1}{2\pi K}}$

converges (ie $\propto L^0$) if $\frac{1}{2\pi K} > 2$ ie $K < \frac{1}{4\pi} = K_c$

If $K>K_c$, $\chi_t \propto L^{2-2\frac{K_c}{K}}$

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2a) If α is angle between normal to surface and x_d axis,

$$dA = \frac{dx_1 \cdots dx_{d-1}}{\cos\alpha}.$$

Equation of surface is $x_d - h(x_1, \cdots x_{d-1}) = 0$

so unit normal is $\left(-\frac{\partial h}{\partial x_1}, \cdots, -\frac{\partial h}{\partial x_{d-1}}, 1\right) \Big/ \sqrt{1+\Sigma\left(\frac{\partial h}{\partial x_i}\right)^2}$

so $\cos\alpha = 1\Big/\sqrt{1+(\nabla h)^2}$, $\quad A = \int\sqrt{1+(\nabla h)^2}\, d^{d-1}x$

and at low temperature, $Z \propto \int \mathcal{D}h(x)\, e^{-\beta\sigma\frac{1}{2}\int(\nabla h)^2 d^{d-1}x}$

b) Let $h(\vec{x}) = \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int h(\vec{q})\, e^{i\vec{q}\cdot\vec{x}}\, d^{d-1}\vec{q}$

$$Z \propto \int \mathcal{D}h \,\exp\left[-\frac{1}{2}\beta\sigma\int d^{d-1}q\, |q|^2\, |h(\vec{q})|^2\right]$$

[We have got an integral over continuous $\vec{q}$ values because we have not made the total space $(x_1 \cdots x_{d-1})$ finite in extent by e.g. imposing periodic boundary conditions. Also we have high values of $|q|$ in the integral because we have not imposed a minimum wavelength on the capillary waves]

c) $h(x) \to h(x) + h_1$ does not change energy of configuration

so $h(x) \to h(x)$ + slowly varying $h_1(x)$ makes small energy increase

ie as $q \to 0$, energy of mode $\to 0$. Expanding round $h(x) = 0$

is breaking of symmetry, (translation in x_d direction)

d,e) $h(x) - h(x') = \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int h(\vec{q})\left(e^{iq\cdot x} - e^{iq\cdot x'}\right) dq$

so $(h(x) - h(x'))^2 = \int |h(\vec{q})|^2\, 2\{1 - \cos q\cdot x - x'\}\, d^{d-1}q$

<u>Or</u> as in problem (1),

$$\langle e^{it(h-h')}\rangle$$

$$= \exp\left[-\frac{t^2}{2}\underbrace{\frac{1}{\beta\sigma}\frac{1}{(2\pi)^{d-1}}\int d^{d-1}q\; \frac{2\{1-\cos q\cdot x - x'\}}{|q|^2}}_{\langle(h-h')^2\rangle}\right]$$

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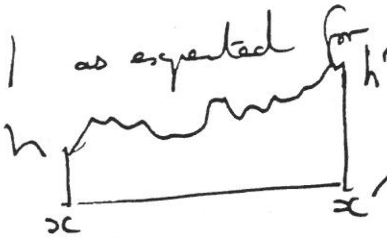
Dimension d in this problem corresponds to $d-1$ in problem 1, so for $d=4$, $\langle (h-h')^2 \rangle \rightarrow$ finite limit as $|x|$ gets big

and $\langle h \cdot h' \rangle_c \propto \dfrac{1}{|x|}$

For $d=3$, $\langle (h-h')^2 \rangle \approx \dfrac{1}{\pi \beta \sigma} \ln|x-x'|$ so heights at distant points are very different.

For $d=2$, $\langle (h-h')^2 \rangle \propto |x-x'|$ as expected for a random walk in one dimension



We can think of d as taking non-integral values.

For $d<3$, $\displaystyle\int d^{d-1}q \, \frac{1-\cos q \cdot x}{q^2} \propto |x|^{3-d}$ [let $\vec{q} = \dfrac{\hat{q}'}{|x|}$, integral converges for large q'] while for $d>3$, $\propto$ constant (which depends of short wavelength cut-off)

To estimate $(\nabla h)^2$, use $\left\langle \dfrac{(h(x)-h(x'))^2}{|x-x'|^2} \right\rangle$ which is $\propto |x|^{1-d}$ $(d<3)$ or $|x|^{-2}$ $(d>3)$ for large enough $|x-x'|$

So for $d>1$ we can make it small by coarse-graining on a large enough scale, which is what we want to do — we are looking at long wavelength effects.

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8.33 PS.3 Goldstone Modes

Solutions

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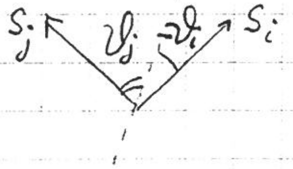
1a) $$Z = \int \prod_i d\vec{S}_i \exp\left(+K \sum_{\langle ij \rangle} \vec{S}_i \vec{S}_j\right) \delta(|S_i| - 1)$$



$$\vec{S}_i \vec{S}_j = \cos(\vartheta_i - \vartheta_j)$$

since $s_i^2 = s_j^2 = 1$

$$d\vec{S}_i = d\vartheta_i |\vec{S}_i| \, d|S_i|$$

The integral over $d|S_i|$ disappears because of δ-function

$$Z = \int \prod_i d\vartheta_i \exp\left(K \sum_{\langle ij \rangle} \cos(\vartheta_i - \vartheta_j)\right)$$

1b) For $K \gg 1$

$$Z = e^{KL^d} \int \prod_i d\vartheta_i \exp\left\{-\frac{K}{2}(\vartheta_i - \vartheta_j)^2\right\}$$

Higher order terms in the expansion of $\cos(\vartheta_i - \vartheta_j)$ can be neglected since the integral is determined by $|\vartheta_i - \vartheta_j| \sim \sqrt{2/K}$. L^d is a total number of sites

1c) For a 1d chain of L sites

$$\vartheta(q) = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} \vartheta_j e^{-iqj} \qquad (1)$$

$$\vartheta_j = \frac{1}{\sqrt{L}} \sum_q \vartheta(q) e^{iqj} \qquad (2)$$

Periodic boundary conditions:

$$\vartheta_{L+1} = \vartheta_1 \qquad (3)$$

require

$$qL = 2\pi n \quad \text{or} \quad q = \frac{2\pi}{L} n\, ; \qquad n = 0, \pm 1, \pm 2, \ldots, \pm \frac{L}{2}$$

ϑ_j should be real. Therefore

$$\vartheta(-q) = \vartheta^*(q) \qquad (4)$$

From (2) we have

$$\vartheta_j - \vartheta_k = \frac{1}{\sqrt{L}} \sum_q \vartheta(q) e^{iqj} \left[1 - e^{-iq(j-k)}\right]$$

$$(\vartheta_j - \vartheta_k)^2 = \frac{1}{L} \sum_{q,q'} \vartheta(q)\vartheta(q') e^{i(q+q')j} \left[1 - e^{-iq(j-k)}\right]\left[1 - e^{-iq'(j-k)}\right]$$

In the sum over nearest neighbours $\sum_{\langle jk \rangle}$ $j-k = 1$

Therefore, neglecting the constant in the hamiltonian we have

$$-\beta\mathcal{H} = \frac{K}{2} \sum_{\langle jk \rangle} (\vartheta_j - \vartheta_k)^2 = \frac{K}{2} \sum_j (\vartheta_j - \vartheta_{j-1})^2 = \qquad (5)$$

$$= \frac{1}{L}\frac{K}{2} \sum_{qq'} \vartheta(q)\vartheta(q') \sum_j e^{i(q+q')j} (1 - e^{-iq})(1 - e^{-iq'})$$

Since

$$\frac{1}{L}\sum_j e^{i(q+q')j} = \delta_{q',-q}$$

we can rewrite (5) as

$$-\beta\mathcal{H} = \frac{K}{2} \sum_q \vartheta(q)\vartheta(-q) |1 - e^{-iq}|^2 \qquad (6)$$

Using (4) we finally get

$$-\beta\mathcal{H} = \frac{1}{2} \sum_q K(q) |\vartheta(q)|^2$$

with

$$K(q) = K|1 - e^{-iq}|^2 = 2K(1 - \cos q)$$

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for $q \ll 1$ (wavelength >> lattice constant)

$$K(q) \simeq Kq^2$$

1d) For arbitrary number of dimensions d we can label each site by a vector index $\vec{r}$

$$j = \vec{r} = (r_1, r_2, \ldots, r_d) = \sum_{\alpha=1}^{d} r_\alpha \vec{e}_\alpha$$

where $\vec{e}_\alpha$ are unit vectors along the axis.

Again

$$\vartheta(\vec{q}) = L^{-d/2} \sum_{\vec{r}} \vartheta_{\vec{r}}\, e^{-i\vec{q}\vec{r}}$$

$$\vartheta_{\vec{r}} = L^{-d-2} \sum_{\vec{q}} \vartheta(\vec{q})\, e^{i\vec{q}\vec{r}}$$

Periodic boundary conditions lead to

$$\vec{q} = \sum_{\alpha=1}^{d} \vec{e}_\alpha q_\alpha \qquad q_\alpha = \frac{2\pi n_\alpha}{L}, \quad n_\alpha = 0, \pm 1, \pm 2, \ldots, \pm \frac{L}{2}$$

Now the sum over nearest neighbours can be written as

$$\sum_{\langle \vec{r}, \vec{r}' \rangle} = \sum_{\vec{r}} \sum_{\alpha} \delta_{\vec{r}', \vec{r}+\vec{e}_\alpha}$$

For instance

$$\sum_{\langle jk \rangle} (\vartheta_j - \vartheta_k)^2 = \frac{1}{L^d} \sum_{\vec{q},\vec{q}'} \vartheta(\vec{q})\vartheta(\vec{q}') \sum_{\vec{r},\alpha} e^{i(\vec{q}+\vec{q}')\vec{r}} (1-e^{-i\vec{q}\vec{e}_\alpha})(1-e^{-i\vec{q}'}$$

Summation over $\vec{r}$ again gives a δ-function and finally we have

$$-\beta\mathcal{H} = \frac{1}{2} \sum_{\vec{q}} K(\vec{q})\, |\vartheta(\vec{q})|^2$$

with

$$K(\vec{q}) = 2K \sum_{\alpha=1}^{d} (1 - \cos q_\alpha) \underset{q_\alpha \to 0}{\approx} Kq^2$$

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1e) $$Z = \Pi_{\vec{q}} \int \exp\left\{ -\frac{1}{2} \cdot |\vartheta(\vec{q})|^2 K(\vec{q}) \right\} d\vartheta(\vec{q}) = \qquad (7)$$

$$= \Pi_{\vec{q}} \sqrt{\frac{2\pi}{K(\vec{q})}}$$

since the normal modes are independent

$$\ln Z = \text{const} - \frac{1}{2} \sum_{\vec{q}} \ln K(\vec{q})$$

Taking into account that, the density of states in q-space is $(L/2\pi)^d$ we can write the free energy in the limit $L \to \infty$ as

$$F = \frac{T}{2} L^d \int \frac{d\vec{q}}{(2\pi)^d} \ln K(q)$$

Since $K \sim 1/T$ at $T \to \infty$ we can neglect all of the T-independent factors in $K(q)$. As a result

$$F = -\frac{T}{2} (\ln T + \text{const}) L^d$$

Specific heat per site is therefore

$$C = -T \frac{\partial^2 F}{\partial T^2} \frac{1}{L^d} = \frac{1}{2}$$

1f) $$\vartheta_{\vec{x}} - \vartheta_{\vec{0}} = \frac{1}{L^{d/2}} \sum_{\vec{q}} \vartheta(\vec{q}) \left\{ e^{i\vec{q}\vec{x}} - 1 \right\}$$

Therefore

$$\langle e^{i(\vartheta_{\vec{x}} - \vartheta_{\vec{0}})} \rangle = \frac{1}{Z} \int \Pi_{\vec{q}} d\vartheta(\vec{q}) \exp\left\{ -\frac{K(\vec{q})}{2} \vartheta(\vec{q}) \vartheta(-\vec{q}) + \right.$$

$$\left. + \frac{i}{2L^{d/2}} \left[\vartheta(q)(e^{i\vec{q}\vec{x}} - 1) + \vartheta(-\vec{q})(e^{-i\vec{q}\vec{x}} - 1) \right] \right. \qquad (8)$$

where Z is given by (7). Rewriting (8) as

$$\frac{1}{Z} \int \Pi_{q} d\vartheta(q) \exp\left\{ -\frac{K(q)}{2} \left[\vartheta(q) - i\frac{(e^{-i\vec{q}\vec{x}} - 1)}{L^{d/2} K(q)} \right] \left[\vartheta(-q) - \frac{i(e^{iqx} - 1)}{L^{d/2} K(q)} \right] \right\} \times$$

$$\times \exp\left\{ -\frac{|e^{iqx} - 1|^2}{\quad} \right\}$$

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we see that the integral cancells with Z and only the last exp{ } is left:

$$\langle e^{i(\vartheta_{\vec{x}}-\vartheta_{\vec{0}})}\rangle = \exp\left\{-\sum_{\vec{q}}\frac{|e^{i\vec{q}\vec{x}}-1|^2}{2K(\vec{q})L^d}\right\} =$$

$$= \exp\left\{-\int\frac{d\vec{q}}{(2\pi)^d}\,\frac{1-\cos(\vec{q},\vec{x})}{2K(q)}\right\}$$

$\boxed{d=1}$

$$\langle e^{i(\vartheta_x-\vartheta_0)}\rangle = \exp\left\{-\frac{1}{4\pi K}\int_{-\pi}^{\pi}\frac{1-\cos q|x|}{1-\cos q}\,dq\right\}$$

$$= \exp\left[-\frac{|x|}{2K}\right]$$

The integral increases $\sim |x|$. The correlation function tends to zero at $|x|\to\infty$

$\boxed{d=2}$

$$\int\frac{d^2q}{(2\pi)^2}\,\frac{1-\cos(\vec{q}\,\vec{x})}{Kq^2} = \int\frac{q\,dq\,d\varphi}{(2\pi)^2Kq^2}\left[1-\cos(|q||x|\cos\varphi)\right] =$$

$$= \int_0^{\pi}\frac{dq}{\pi Kq}\left[1-J_0(q|x|)\right] = \quad (J_0 \text{ is a Bessel function,}$$

$$= \frac{1}{\pi K}\int^{\pi|x|}\frac{dq}{q}\left\{1-J_0(q)\right\} = \frac{1}{\pi K}\ln|x| + \text{const}$$

Therefore

$$\langle e^{i(\vartheta_x-\vartheta_0)}\rangle \propto e^{-\frac{1}{2\pi K}\ln|x|} = |x|^{-\frac{1}{2\pi K}} \xrightarrow[x\to\infty]{} 0$$

$\boxed{d\geqslant 3}$ The integral

$$I = \int\frac{d\vec{q}}{(2\pi)^d}\,\frac{1-\cos(\vec{q}\,\vec{x})}{K(\vec{q})} \quad \text{converges at } q\to 0$$

Therefore $I \xrightarrow[|x|\to\infty]{} C/K$ (C is a number)

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and

$$\langle e^{i(\vartheta_x-\vartheta_0)}\rangle \xrightarrow[x\to\infty]{} \exp\left(-\frac{c}{2K}\right) \xrightarrow[K\to\infty]{} 1-\frac{c}{2K}$$

correlation does not disappear at $|x|\to\infty$

1g) $d>2$. For large K

$$\langle e^{i(\vartheta_x-\vartheta_0)}\rangle = 1-\int\frac{d\vec{q}}{(2\pi)^d}\,\frac{1-\cos qx}{K(q)}$$

Similarily

$$\langle e^{i\vartheta_x}\rangle = 1-\frac{1}{2}\int\frac{d\vec{q}}{(2\pi)^d}\,\frac{1}{K(q)}$$

Consider now the transverse succeptibility

$$\chi_t = \int\langle S_x S_0\rangle_c\, d^dx = \int\langle e^{i(\vartheta_x-\vartheta_0)}\rangle_c\, d^dx$$

where the irreducable correlator $\langle\ \rangle_c$ means

$$\langle e^{i(\vartheta_x-\vartheta_0)}\rangle_c = \langle S_x S_0\rangle - \langle S_x\rangle\langle S_0\rangle =$$
$$= \langle e^{i(\vartheta_x-\vartheta_0)}\rangle - \langle e^{i\vartheta_x}\rangle\langle e^{i\vartheta_0}\rangle =$$

Hence

$$\langle S_x S_0\rangle_c = \int\frac{d\vec{q}}{(2\pi)^d}\,\frac{\cos\vec{q}\vec{x}}{K(\vec{q})} \simeq \frac{1}{K}\int\frac{d\vec{q}}{(2\pi)^d q^2}\cos\vec{q}\vec{x} \propto \frac{1}{|x|^{d-2}K}$$

For χ_t we therefore have

$$\chi_t \propto K^{-1}L^d\,L^{2-d} = L^2/K$$

1h) For $d=2$ $\langle S_x\rangle=0$ - no order

$$\langle\vec{S}_x\vec{S}_0\rangle_c = \langle\vec{S}_x\vec{S}_0\rangle = |x|^{-1/2\pi K}$$

$\chi_t \sim \int d^2x\ |x|^{-1/2\pi K}$ — converges for $|x|\to\infty$ if

$\frac{1}{2\pi K}>2$ or $K<\frac{1}{4\pi}=K_c$

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$K < K_c$ $\quad \chi_t$ is L-independent

$K > K_c$ $\quad \chi_t \propto L^{2-2\,K_c/K}$

2a) The equation of the surface is

$$x_d - h(x_1, \ldots, x_{d-1}) = 0$$

$$dA = \frac{1}{\cos\alpha}\, dx_1 \ldots dx_{d-1}$$

where α is the angle of the deviation of the normal to the surface from the d-direction. This normal has components

$$\left(-\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_{d-1}}, 1\right) \Big/ \sqrt{1+(\nabla h)^2} = \vec{n}$$

(we need denominator for $n^2=1$).

$$\cos\alpha = n_d = \left[1+(\nabla h)^2\right]^{-1/2} \simeq 1 - \frac{1}{2}(\nabla h)^2$$

Therefore

$$Z = \int \mathcal{D} h(\vec{x})\, e^{-\beta\sigma\int dA} =$$

$$= \int \mathcal{D} h(\vec{x})\, e^{-\beta\sigma\int \frac{(\nabla h)^2}{2} d^{d-1}x}$$

2b) $h(\vec{x}) = \left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}} \int h(\vec{q})\, e^{i\vec{q}\vec{x}}\, d^{d-1}\vec{q}$

$$Z = \int \mathcal{D} h(\vec{q}) \exp\left[-\frac{1}{2}\beta\sigma \int d^{d-1}\, |h(\vec{q})|^2\right]$$

2c) $h(\vec{x}) \to h(\vec{x}) + h_0$ does not change the energy of a configuration

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Substituting $h_0 x$ by a slowly varying function $h_0(x)$ should change the energy only slightly $q \to$

2d) $$h(x) - h(x') = \left(\frac{1}{2\pi}\right)^{\frac{d-1}{2}} \int h(q) \left(e^{i\vec{q}\vec{x}} - e^{i\vec{q}\vec{x}'}\right) d^d q$$

$$\left(h(x) - h(x')\right)^2 = \int |h(\vec{q})|^2 2 \{1 - \cos q(x - x')\} d^{d-1} q$$

Similarily to 1f) we get

$$\left\langle \left(h(x) - h(x')\right)^2 \right\rangle = \frac{1}{\beta\sigma} \frac{2}{(2\pi)^{d-1}} \int d^{d-1} q \quad \frac{1 - \cos \vec{q}(\vec{x} - \vec{x}')}{|q|^2}$$

2e) Dimension d now corresponds to d-1 in problem 1. Therefore

$\boxed{d \geq 4}$ $$\langle (h - h')^2 \rangle \xrightarrow[|x - x'| \to \infty]{} \text{const}$$

$$\langle h h' \rangle_c \propto \frac{1}{|x - x'|}$$

$\boxed{d = 3}$ $$\langle (h - h')^2 \rangle \approx \frac{1}{\pi\beta\sigma} \ln |x - x'|$$

$\boxed{d = 2}$ $$\langle (h - h')^2 \rangle \propto |x - x'|$$

$$\Big\} \xrightarrow[|x - x'| \to \infty]{} \infty$$

2f) We can estimate $(\nabla h)^2$ as $\dfrac{\langle (h(\vec{x}) - h(\vec{x}'))^2 \rangle}{|x - x'|^2} = R$

$$R \propto \begin{cases} |x - x'|^{1-d} & d < 3 \\ |x - x'|^{-2} & d > 3 \end{cases}$$

Therefore $(\nabla h)^2$ can be made small by coarse-graining provided $d > 1$

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1. *Fluctuations Around a Tricritical Point:* As seen in problem set # 2, the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x}[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 + vm^6],$$

with $u = 0$ and $v > 0$ describes a tricritical point.

(a) Calculate the heat capacity singularity as $t \to 0$ by the saddle point approximation.

(b) Include both longitudinal and transverse fluctuations by looking at

$$\vec{m}(\mathbf{x}) = \left(\overline{m} + \phi_\ell(\mathbf{x})\right)\hat{e}_\ell + \sum_{\alpha=2}^{n} \phi_t^\alpha(\mathbf{x})\hat{e}_\alpha,$$

and expanding $\beta\mathcal{H}$ to quadratic order in ϕ.

(c) Calculate the longitudinal and transverse correlation functions.

(d) Compute the first correction to the saddle point free energy from fluctuations.

(e) Find the fluctuation correction to the heat capacity.

(f) By comparing the results from parts (a) and (e) obtain a Ginzburg criterion, and the upper critical dimension for validity of mean-field theory (i.e. the behavior calculated in PS#2) at a tricritical point.

(g) A generalized multicritical point is described by replacing the term vm^6 with $u_{2n}m^{2n}$. Use simple power counting to find the upper critical dimension of this multicritical point.

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2. *Scaling in Fluids:* Near the liquid–gas critical point, the free energy is assumed to take the scaling form $F/N = t^{2-\alpha}g(\delta\rho/t^\beta)$, where $t = |T - T_c|/T_c$ is the reduced temperature, and $\delta\rho = \rho - \rho_c$ measures deviations from the critical point density. The leading singular behavior of any thermodynamic parameter $Q(t, \delta\rho)$ is of the form t^x on approaching the critical point along the isochore $\rho = \rho_c$; or $\delta\rho^y$ for a path along the isotherm $T = T_c$. Find the exponents x and y for the following quantities:

(a) The internal energy per particle $\langle H\rangle/N$, and the entropy per particle $s = S/N$.

(b) The heat capacities $C_V = T\partial s/\partial T\mid_V$, or $C_P = T\partial s/\partial T\mid_P$.

(c) The isothermal compressibility $\kappa_T = \partial\rho/\partial P\mid_T/\rho$, and the thermal expansion coefficient $\alpha = \partial V/\partial T\mid_P/V$.

(d) Sketch the behavior of the latent heat per particle, L, on the coexistence curve for $T < T_c$, and find its singularity as a function of t.

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Suggested Reading: Stanley, Chapters 2, 7, 11, and 12.

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8.33~~4~~ Problem Set 4, Solutions March '92

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1) $$\beta\mathcal{H} = \int d^dx \left[\frac{1}{2}K\vec{\nabla}\vec{m}\cdot\vec{\nabla}\vec{m} + \frac{1}{2}t\,\vec{m}\cdot\vec{m} + v(\vec{m}\cdot\vec{m})^3\right]$$

a) Saddle point: Minimum of $\beta\mathcal{H}$ is $\nabla m = 0$, $t\vec{m} + 6v\vec{m}(\vec{m}\vec{m})^2 =$

$t>0$, $\vec{m} = 0$; $t<0$, $\vec{m} = \overline{m}\,\hat{e}_l$, $\overline{m} = \left(\frac{|t|}{6v}\right)^{1/4}$

Free energy/Volume $= 0,\ t\geq 0$; $= \frac{1}{2}t\overline{m}^2 + v\overline{m}^6 = -\frac{1}{3}\frac{|t|^{3/2}}{(6v)^{1/2}}$, $t<0$

$C = -T_c\frac{d^2f}{dt^2} = 0,\ t>0$; $= \frac{1}{4}\frac{T_c}{(6v)^{1/2}}|t|^{-\frac{1}{2}}$, $t<0$.

b) $\vec{m} = (\overline{m}+\phi_l)\hat{e}_l + \sum_2^n \phi_t^\alpha \hat{e}_\alpha$, $\hat{e}_l, \hat{e}_\alpha$ orthonormal vectors in $\vec{m}$ space

Linear terms in $\beta\mathcal{H}$ are zero because $\vec{m} = \overline{m}\,\hat{e}_l$ is a minimum

Quadratic terms are $\int d^dx\left[\frac{1}{2}K(\nabla\phi_l)^2 + \frac{1}{2}K\sum(\nabla\phi_t^\alpha)^2 + V\right]$

$V = \frac{1}{2}t\phi_l^2 + \frac{1}{2}t\sum\phi_t^{\alpha 2}$ + quadratic term in $v(\overline{m}^2 + 2\overline{m}\phi_l + \phi_l^2 + \phi_t^2)^3$

i.e $V = \frac{1}{2}t\phi_l^2 + \frac{1}{2}t\sum\phi_t^{\alpha 2} + 3v\overline{m}^4(\phi_l^2+\phi_t^2) + 3v\overline{m}^2\cdot 4\overline{m}^2\phi_l^2$

$= \frac{1}{2}\phi_l^2(t+30v\overline{m}^4) + \frac{1}{2}\sum\phi_t^{\alpha 2}(t+6v\overline{m}^4)$

$t>0$; $V = \frac{1}{2}t(\phi_l^2 + \sum\phi_t^{\alpha 2})$

$t<0$; $V = \frac{1}{2}\phi_l^2\, 4|t|$ [No ϕ_t^2 term because of broken rotational symmetry in $\vec{m}$ space]

c) Weight $e^{-\frac{1}{2}\int d^dx\{K(\nabla\phi)^2 + s\phi^2\}}$ gives correlation function

$C(x) = \langle\phi(x)\phi(0)\rangle \propto \int\frac{e^{iq\cdot x}d^dq}{Kq^2+s}$, where $s\propto|t|$ for $t>0$, ϕ_t and ϕ_l
$t<0$, ϕ_l
and $s=0$ for $t<0$, ϕ_t

When $s=0$ and $d>2$, $C(x)\sim\frac{1}{|x|^{d-2}}$. Case $d\leq 2$ discussed in problem set 3.

When $s\propto|t|$ and $|x|\ll|t|^{-\frac{1}{2}}$, $C(x)\sim 1/|x|^{d-2}$, for $|x|\gg|t|^{-\frac{1}{2}}$, $C(x)\sim e^{-|x|t^{1/2}}$

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d) $\int \mathcal{D}\phi \, e^{-\frac{1}{2}\int dx \left(K(\nabla\phi)^2 + s\phi^2\right)} = \int \prod d\phi_q \, e^{-\frac{1}{2}\sum \phi_q \phi_{-q}(Kq^2+s)}$

$\propto \prod_q (Kq^2+s)^{-\frac{1}{2}}$ so $\ln Z = -\frac{1}{2}\sum_q \ln(Kq^2+s)$

Contribution to free energy density from fluctuations is

$$\frac{n}{2}\int \frac{d^dq}{(2\pi)^d} \ln(Kq^2+t) \quad , \quad t>0$$

$$\frac{1}{2}\int \frac{d^dq}{(2\pi)^d} \ln(Kq^2+4|t|) \quad , \quad t<0$$

e) $C = -T\frac{d^2F}{dT^2}$ so $C_{\text{fluctuation}} \propto \int \frac{d^dq}{(Kq^2+|t|)^2}$.

For $d>4$ this converges at low q as $t\to 0$ so $\to$ finite contribution and mean field contribution dominates for $t<0$

For $d<4$, C converges at high q and (with $q = \tilde{q}\sqrt{\frac{|t|}{K}}$)

$$C_{FL} \propto \frac{|t|^{d/2-2}}{K^{d/2}}$$

f) Compare to $C_{\text{mean field}} \propto \frac{|t|^{-\frac{1}{2}}}{v^{1/2}}$. As $t\to 0$, mean field dominates

provided $\frac{d}{2}-2 > -\frac{1}{2}$ i.e. $d>3$.

For $d<3$, will ~~still~~ have $C_{FL} \ll C_{\text{mean field}}$ if $|t|^{\frac{3-d}{2}} \gg v^{1/2}/K^{d/2}$

i.e. $|t|^{3-d} \gg v/K^d$ so far enough from critical point still expect fluctuations to be small.

g) With term m^{2n} will have $\bar{m} \propto |t|^{\frac{1}{2n-2}}$, $F \propto |t|^{\frac{n}{n-1}}$,

$C_{\text{mean field}} \propto |t|^{\frac{n}{n-1}-2}$ while still $C_{FL} \propto |t|^{d/2-2}$

so upper critical dimension is $d = \frac{2n}{n-1}$

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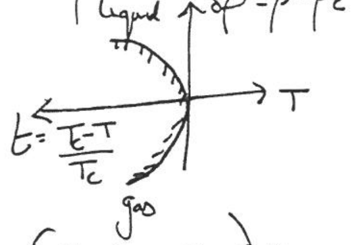
2) If $Q(t, \delta\rho) \propto t^{x_Q} g_Q\left(\frac{\delta\rho}{t^\beta}\right)$, then as $\lambda \to \infty$ must have $g(\lambda) \propto \lambda^{\frac{x_Q}{\beta}}$

so that $Q(0, \delta\rho) \propto (\delta\rho)^{\frac{x_Q}{\beta}}$ ie $y_Q = \frac{x_Q}{\beta}$ in each case



We have $\frac{F}{N} = f = t^{2-\alpha} g_F\left(\frac{\delta\rho}{t^\beta}\right)$

For $T < T_c$, $\frac{\partial}{\partial T} = -\frac{1}{T_c}\frac{\partial}{\partial t}$

Coexistence curve has upper and lower branches with $\delta\rho_\pm / t^\beta = \lambda_\pm$

a) $s = -\frac{\partial f}{\partial T} = \frac{1}{T_c} t^{1-\alpha} g_s\left(\frac{\delta\rho}{t^\beta}\right)$; $g_s(\lambda) = (2-\alpha) g_F(\lambda) - \beta\lambda g_F'(\lambda)$

and $x_s = 1-\alpha$

$\frac{\langle H \rangle}{N} = f + sT \approx sT_c$ so $x_H = 1-\alpha$

b,c) $C_V = T\left(\frac{\partial s}{\partial T}\right)_{\delta\rho} \approx -\left(\frac{\partial s}{\partial t}\right)_{\delta\rho} = \frac{1}{T_c} t^{-\alpha}\left\{\beta\lambda g_s' - (1-\alpha) g_s\right\}$; $x_C = -\alpha$

$P = -\frac{\partial F}{\partial V} = \rho_c^2 \frac{\partial f}{\partial \delta\rho} = \rho_c^2 t^{2-\alpha-\beta} g_F'$

$\left(\frac{\partial P}{\partial \delta\rho}\right)_T = \rho_c^2 t^{2-\alpha-2\beta} g_F''$; $K_T = \frac{1}{\rho_c} 1 / \left(\frac{\partial P}{\partial \delta\rho}\right)_t$ so $x_{K_T} = 2\beta + \alpha - 2$

$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P = \frac{1}{\rho_c}\left(\frac{\partial P}{\partial T}\right)_\rho \Big/ \left(\frac{\partial P}{\partial \rho}\right)_T \propto \frac{t^{1-\alpha-\beta}}{t^{2-\alpha-2\beta}} g_\alpha$; $x_\alpha = \beta - 1$

Also $C_P - C_V = T\left(\frac{\partial s}{\partial \rho}\right)_T \left(\frac{\partial \rho}{\partial T}\right)_P \propto t^{1-\alpha-\beta} t^{\beta-1} = t^{-\alpha}$ so $x_{C_P} = x_{C_V} = -\alpha$

d) On coexistence curve $TS = t^{1-\alpha} g_s(\lambda_\pm)$

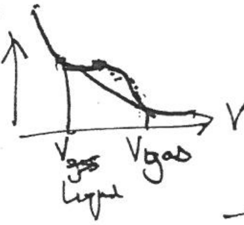
so $L = T(S_+ - S_-) \propto t^{1-\alpha}$

[Coexistence implies $P_+ = P_-$ and $\mu_+ = \left(F + \frac{P}{\rho}\right)_+ = \mu_-$

ie $g_F'(\lambda_+) = g_F'(\lambda_-)$ and $\mu = f + \rho\frac{\partial f}{\partial \rho} = \rho_c \frac{\partial f}{\partial \delta\rho} + \underbrace{f + \delta\rho \frac{\partial f}{\partial \delta\rho}}_{t^{2-\alpha}\left(g_F + \frac{\delta\rho}{t^\beta} g_F'\right)}$

so $g_F(\lambda_+) + \lambda_+ g_F'(\lambda_+) = g_F(\lambda_-) + \lambda_- g_F'(\lambda_-)$

Expressed in terms of $\frac{1}{\lambda}$ at fixed t, this is just the 'Maxwell' construction on the free energy as a function of V curve.



We see this is all consistent with homogeneous form of F.]

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8.334 PS. #4 Solutions

1. $$\beta\mathcal{H} = \int d^dx \left[\frac{K}{2}(\vec\nabla\vec m\cdot\vec\nabla\vec m) + \frac{t}{2}(\vec m\cdot\vec m) + v(\vec m\cdot\vec m)^3 \right]$$

1a) Saddle point. Minimum of $\beta\mathcal{H}$:

1) $\vec\nabla\vec m$ $\quad \vec m = \vec e_L\,\bar m$ where e_L is a unit vector

2) $t\vec m + 6v\vec m(\vec m\cdot\vec m)^2 = 0$

$t\bar m + 6v\bar m^5 = 0$

Hence

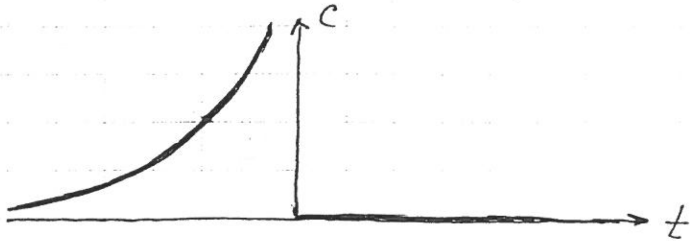
$$\bar m = \begin{cases} 0 & \text{for } t>0 \\ \left(-\frac{t}{6v}\right)^{1/4} & \text{for } t<0 \end{cases} \qquad (1)$$

Free energy of a unit volume equals to

$$\cancel{f} \qquad f = \frac{t}{2}\bar m^2 + v\bar m^6 = \begin{cases} 0 & \text{for } t>0 \\ -\frac{1}{2}\frac{|t|^{3/2}}{(6v)^{1/2}} + \frac{1}{6}\frac{|t|^{3/2}}{(6v)^{1/2}} = -\frac{|t|^{3/2}}{3(6v)^{1/2}} & t<0 \end{cases}$$

Therefore for the specific heat C we have

$$C = C_{s.p} = -T\frac{\partial^2 f}{\partial t^2} \simeq -T_c\frac{\partial^2 f}{\partial t^2} = \begin{cases} 0 & t>0 \\ \frac{T_c}{4}(-6vt)^{-1/2} & t<0 \end{cases}$$



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1b) $\vec{m} = \bar{m}\vec{e}_L + \phi_L \vec{e}_L + \sum_{\alpha=2}^{n} \vec{e}_\alpha \phi_t^\alpha$

Here $\vec{e}_\alpha$ together with $\vec{e}_L$ form an orthonormal set of n vectors in $\vec{m}$ space $\vec{e}_L\vec{e}_\alpha = 0$; $\vec{e}_\alpha\vec{e}_\beta = \delta_{\alpha\beta}$

$\beta\mathcal{H}$ is a function of ϕ_L and ϕ_t^α. Since $\bar{m}\vec{e}_L$ is a minimum, there are no linear terms in the expansion of $\beta\mathcal{H}$ over ϕ.

$$(\nabla\vec{m})^2 \Rightarrow (\nabla\phi_L)^2 + \sum_{\alpha=2}^{n}(\nabla\phi_t^\alpha)^2$$

$$\cancel{t}(\vec{m})^2 \Rightarrow \phi_L^2 + \sum_{\alpha=2}^{n}(\phi_t^\alpha)^2$$

$$(\vec{m})^6 = (\vec{m}^2)^3 = \left(\bar{m}^2 + 2\bar{m}\phi_L + \phi_L^2 + \sum_\alpha (\phi_t^\alpha)^2\right)^3 \Rightarrow$$

$$\Rightarrow 12\bar{m}^4\phi_L^2 + 3\bar{m}^4\left(\sum_{\alpha=2}^{n}(\phi_t^\alpha)^2 + \phi_L^2\right)$$

Therefore

$$\beta\mathcal{H}(\phi_L, \phi_t^\alpha) = \beta\mathcal{H}(0,0) + \int d^dx \left[\frac{K}{2}\left\{(\nabla\phi_L)^2 + \sum_{\alpha=2}^{n}(\nabla\phi_t^\alpha)^2\right\} + \right.$$

$$+ \frac{t}{2}\left\{\phi_L^2 + \sum_{\alpha=2}^{n}(\phi_t^\alpha)^2\right\} +$$

$$+ v\left\{\phi_L^2 \cdot 15\bar{m}^4 + 3\bar{m}^4\sum_{\alpha=2}^{n}(\phi_t^\alpha)^2\right\} =$$

$$= \beta\mathcal{H}(0,0) + \int d^dx \left[\frac{K}{2}\left\{(\nabla\phi_L)^2 + \sum_{\alpha=2}^{n}(\nabla\phi_t^\alpha)^2\right\} + \frac{\phi_L^2}{2}(t + 30\bar{m}^4) + \right.$$

$$\left. + \sum_{\alpha=2}^{n}\frac{(\phi_t^\alpha)^2}{2}(t + 6\bar{m}^4)\right]$$

Substituting $\bar{m}$ from (1) we have

$$\beta\mathcal{H}(\phi_L, \phi_t^\alpha) - \beta\mathcal{H}(0,0) = \int d^dx \left[\frac{K}{2}\left\{(\nabla\phi_L)^2 + \sum_\alpha (\nabla\phi_t^\alpha)^2\right\} + \begin{cases} \frac{t}{2}\left\{\phi_L^2 + \sum_\alpha (\phi_t^\alpha)^2\right\} & t> \\ -2t\phi_L^2 & t<< \end{cases}\right.$$

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$\beta\mathcal{H}$ does not depend on ϕ_t^α at $t<0$ due to the rotational symmetry

1c) For both signs of t $\beta\mathcal{H}$ can be written as

$$\beta\mathcal{H}(\phi_L, \phi_t^\alpha) = \beta\mathcal{H}(0,0) + \beta\mathcal{H}_L(\phi_L) + \sum_{\alpha=2}^{n} \beta\mathcal{H}_t(\phi_t^\alpha)$$

where as we have seen

$$\beta\mathcal{H}_{L,t}(\phi) = \frac{K}{2}\int d^dx\left[(\nabla\phi)^2 + \frac{\phi_t^2}{\xi_{\delta_t}^2}\right]$$

with

$$\frac{1}{\xi_{\delta_L}^2} = \begin{cases} t/K & \text{for} \quad t>0 \\ -2t/K & \text{for} \quad t<0 \end{cases}$$

$$\frac{1}{\xi_{\delta_t}^2} = \begin{cases} t/K & \text{for} \quad t>0 \\ 0 & \text{for} \quad t<0 \end{cases}$$

Since in this harmonic approximation $\beta\mathcal{H}$ turned out to be a sum of hamiltonians of different fluctuation modes ϕ_L, ϕ_t^α, the modes are independen of each other:

$$\langle \phi_L \phi_t^\alpha \rangle = 0 \qquad\qquad \langle \phi_t^\alpha \phi_t^\gamma \rangle \propto \delta_{\alpha\gamma}$$

To determine the correlation functions $\langle\phi_L(\vec{0})\phi_L(\vec{x})\rangle$ and $\langle\phi_t^\alpha(\vec{0})\phi_t^\alpha(\vec{x})\rangle$ we have to average the product $\phi(\vec{0})\phi(\vec{x})$ with a weight

$$\exp\left\{-\frac{K}{2}\int d^dx\left[(\nabla\phi)^2 + \frac{\phi^2}{\xi_\delta^2}\right]\right.$$

As it was shown in lectures

$$\langle\phi(\vec{0})\phi(\vec{x})\rangle = \int\frac{d^dq}{(2\pi)^d}\langle|\phi_{\vec{q}}|^2\rangle e^{i\vec{q}\vec{x}} = \frac{1}{K}\int\frac{d^dq}{(2\pi)^d}\frac{e^{i\vec{q}\vec{x}}}{q^2+\xi_\delta^{-2}}$$

or

$$\langle\phi(0)\phi(\vec{x})\rangle = \frac{1}{K} I_d(|x|)$$

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$$-\nabla^2 I_d + \frac{1}{\xi^2} I_d = \delta^d(\vec{x})$$

It is known that

$$\nabla^2\left(\frac{1}{|x|^{d-2}}\right) = - S_d\, \delta^d(\vec{x})\, |d-2|$$

where S_d is the "area" of a hypersphere with a unit radii in d-dimensional space:

$$S_1 = 2; \quad S_2 = 2\pi; \quad S_3 = 4\pi$$

(In a general case $S_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ where $\Gamma(z)$ is the Γ-function

(It can be checked strightforwardly that $\nabla^2(|x|^{2-d}) = 0$ for $|x| \neq 0$. To determine the prefactor of the δ-function one can integrate the equation over $\vec{x}$ within a hypersphere with an infinitesimal radii by parts)

Therefore
$$I_d = \frac{\exp(-|x|/\xi)}{S_d\, |x|^{d-2}\, |d-2|}$$

As a result for $t > 0$

$$\langle \phi_l(\vec{0})\, \phi_l(\vec{x}) \rangle = \langle \phi_t^\alpha(\vec{0})\, \phi_t^\alpha(\vec{x}) \rangle = \frac{\exp\left[-|x|\sqrt{t/K}\,\right]}{K S_d\, |x|^{d-2}\, |d-2|}$$

while for $t < 0$

$$\langle \phi_l(\vec{0})\phi_l(\vec{x}) \rangle = \frac{\exp\left[-|x|\sqrt{4|t|/K}\,\right]}{K S_d\, |d-2|\, |x|^{d-2}}$$

$$\langle \phi_t^\alpha(\vec{0})\, \phi_t^\gamma(\vec{x}) \rangle = \frac{\delta_{\alpha\gamma}}{K S_d\, |d-2|\, |x|^{d-2}}$$

For $d=2$ we have to substitute for $\left(|d-2|\, |x|^{d-2}\right)^{-1}$

$\ln(\xi/|x|)$ if $|x| < \xi$

const if $|x| > \xi$

1d) Partition function can be written as

$$Z = \exp\left\{-\beta\mathcal{H}(\phi_L, \phi_t^\alpha)\right\} = e^{-\beta\mathcal{H}(0,0)}\, e^{-\beta\mathcal{H}_L}\, e^{-(n-1)\beta\mathcal{H}_t}$$

For a LG hamiltonian

$$e^{-\beta\mathcal{H}(\phi)} = \int \mathcal{D}\phi(\vec{x}) \exp\left\{-\frac{K}{2}\int\left[(\nabla\phi)^2 + \xi^{-2}\phi^2\right] d^dx\right\}$$

$$= \int \prod_q d\phi_q \exp\left\{-\frac{K}{2}\sum_q \phi_q\phi_q^*(q^2+\xi^{-2})\right\} =$$

$$= \prod_q \left[K(q^2+\xi^{-2})\right]^{-1/2} = \exp\left\{-\frac{1}{2}\sum_q \ln(Kq^2+K\xi^{-2})\right\}$$

and the free energy equals to

$$\beta\mathcal{H} = \frac{1}{2}\int\frac{d^dq}{(2\pi)^d} \ln(Kq^2+K\xi^{-2})$$

In our case

$$\beta F = \beta\mathcal{H} = \beta\mathcal{H}(0,0) + \begin{cases} \frac{n}{2}\int\frac{d^dq}{(2\pi)^d}\ln(Kq^2+t) & \text{for } t>0 \\ \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\ln(Kq^2+4|t|) & \text{for } t<0 \end{cases}$$

We are not taking into account the contributions of transverse modes at $t<0$ since these contributions are independent of t and therefore of T. The first term is a saddle point free energy

1e) $C = -T\frac{d^2F}{dT^2}$ ~~$\propto \int\frac{d^dq}{(2\pi)^d}$~~

Hence

$$C - C_{s.p.} \propto \begin{cases} n\int\frac{d^dq}{(2\pi)^d}(Kq^2+t)^{-2} & \text{for } t>0 \\ 16\int\frac{d^dq}{(2\pi)^d}(Kq^2+4t)^{-2} & \text{for } t<0 \end{cases}$$

For $d>4$ the integrals are dominated by large q contributions and $C - C_{s.p.}$ is finite i.e. the mean field contribution dominates ($C_{s.p} \propto |t|^{-1/2} \to \infty$)

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For $d<4$ the fluctuation contribution to the specific heat is proportional to

$$C_{fl} = C - C_{s.p} \propto K^{-d/2} |t|^{\frac{d}{2}-2}$$

1f) $C_{s.p.} \propto \frac{1}{\sqrt{v|t|}}$ hence $\frac{C_{fl}}{C_{s.p.}} \propto |t|^{\frac{d-3}{2}} \frac{1}{\sqrt{K^d v^{-1}}}$

Therefore meanfield contribution $C_{s.p.}$ dominates at $t \to$ provided

$$d>3$$

For $d<3$ the mean field result is still valid far enough from the critical point $t=0$

$$|t|^{3-d} \gg \frac{v}{K^d} \quad \text{or} \quad |t| \gg \left(\frac{v}{K^d}\right)^{1/(3-d)}$$

For

$$|t| < \left(v/K^d\right)^{1/(3-d)}$$

the fluctuation contribution to the specific heat dominates.

$d=3$ is the upper critical dimension

1g) With the term $U_{2n} m^{2n}$ instead of $v m^6$ we will have

$$\overline{m} \propto |t|^{\frac{1}{2n-2}} \qquad F_{s.p.} \propto |t|^{\frac{n}{n-1}} \qquad C_{s.p.} \propto |t|^{\frac{n}{n-1}-2} = |t|^{\frac{2-}{n-}}$$

At the same time $C_{fl} \propto |t|^{\frac{d}{2}-2}$ for any n.

Therefor the upper critical dimension is determined by the equation $\frac{d}{2} - 2 = \frac{n}{n-1} - 2$

or $\quad d = \frac{2n}{n-1}$

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2. Any thermodynamic parameter $Q(t, \delta\rho)$ can be written in the form

$$Q(t, \delta\rho) = t^{x_Q} g_Q\left(\frac{\delta\rho}{t^\beta}\right)$$

For Q to be independent of t for large enough $\delta\rho$ we have to require

$$g_Q(\lambda) \xrightarrow[\lambda\to\infty]{} \propto \lambda^{x_Q/\beta}$$

Hence, provided $Q(0, \delta\rho) \propto (\delta\rho)^{y_Q}$,

$$y_Q = \frac{1}{\beta} x_Q$$

2a) $f = \frac{F}{N} = t^{2-\alpha} g\left(\frac{\delta\rho}{t^\beta}\right)$ free energy per particle

Since $t = \frac{T_c - T}{T_c} = 1 - \frac{T}{T_c}$ $\quad \frac{\partial}{\partial T} = -\frac{1}{T_c}\frac{\partial}{\partial t}$ (for $T < T_c$)

Entropy

$$s = -\frac{\partial f}{\partial T} = \frac{1}{T_c}\frac{\partial f}{\partial t} = \frac{t^{1-\alpha}}{T_c} g_s\left(\frac{\delta\rho}{t^\beta}\right)$$

where

$$g_s(\lambda) = (2-\alpha) g(\lambda) - \beta\lambda \frac{dg(\lambda)}{d\lambda}$$

It means that

$$x_s = 1 - \alpha \qquad y_s = \frac{x_s}{\beta} = \frac{1-\alpha}{\beta}$$

$$f = \frac{\langle \mathcal{H} \rangle}{N} - Ts \Rightarrow \frac{\langle \mathcal{H} \rangle}{N} = f + sT = sT_c(1 + O(t))$$

Therefore

$$x_{\mathcal{H}} = 1 - \alpha \qquad y_s = \frac{1-\alpha}{\beta}$$

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2b) $C_V = T\frac{\partial S}{\partial T}\Big)_{\delta\rho} = -\frac{\partial s}{\partial t}\Big)_{\delta\rho}\ (1+O(t))$

Therefore $C_V = \frac{1}{T_c}t^{-\alpha}g_{c_V}\left(\frac{\delta\rho}{t^\beta}\right)$ or $x_{c_V} = -\alpha$ $\quad y_{c_V} = -\frac{\alpha}{\beta}$

(For $g_{c_V}(\lambda)$- function we have

$$g_{c_V}(\lambda) = \beta\lambda g_s'(\lambda) - (1-\alpha)g_s(\lambda) =$$
$$= (\alpha-2)(1-\alpha)g(\lambda) + \beta\lambda g'(\lambda)(2\alpha-3) + \beta^2\lambda^2(1-\alpha)g''(\lambda)$$

The preassure P is determined as $P = -\frac{\partial F}{\partial V} = -\frac{\partial f}{\partial(V/N)} = \rho^2\frac{\partial f}{\partial\rho}$ since the volume per particle V/N equals to $1/\rho$.

$\rho = \rho_c + \delta\rho$, $\delta\rho \ll \rho_c$. Hence

$$P = \rho_c^2\frac{\partial f}{\partial\delta\rho} = \rho_c^2 t^{2-\alpha-\beta}g'\left(\frac{\delta\rho}{t^\beta}\right) \qquad (2)$$

We have to determine the dependance $\delta\rho(t)$ at $P = const$
Thermodynamic identity

$$\frac{\partial\delta\rho}{\partial t}\Big)_P = -\frac{(\partial P/\partial t)_\rho}{(\partial P/\partial\delta\rho)_t}$$

According to (2) at $\delta\rho \ll t^\beta$

$$P \propto t^{2-\alpha-\beta}\left(1 + A\frac{\delta\rho}{t^\beta}\right) \Rightarrow \frac{\partial P}{\partial t}\Big)_\rho \propto t^{1-\alpha-\beta}$$
$$\frac{\partial P}{\partial\rho}\Big)_t \propto t^{2-\alpha-2\beta}$$
$$\Rightarrow \frac{\partial\delta\rho}{\partial t}\Big)_P \propto t^{\beta-}$$

At $\delta\rho \gg t^\beta$

$$P \propto \delta\rho^{(2-\alpha-\beta)/\beta}\left(1 + B\frac{t}{\delta\rho^{1/\beta}}\right) \Rightarrow \begin{array}{l}(\partial P/\partial t)_\rho \propto \delta\rho^{(1-\alpha-\beta)/\beta}\\ (\partial P/\partial\rho)_t \propto \delta\rho^{(2-\alpha-2\beta)/\beta}\end{array} \Rightarrow \frac{\partial\delta\rho}{\partial t}\Big)_P \propto \delta\rho^{-}$$

We have used the fact that there shoul be no singularities in P as a function of $\delta\rho$ if $\delta\rho \ll t^{\beta}$ and as a function of t if $t \ll \delta\rho^{1/\beta}$

$$\left.\frac{\partial \rho}{\partial t}\right)_P \propto t^{\beta-1} \Rightarrow \rho \propto t^{\beta}$$

$$\left.\frac{\partial \rho}{\partial t}\right)_P \propto \delta\rho^{\frac{1-\beta}{\beta}} \Rightarrow t \propto \rho^{1/\beta} \quad \text{or} \quad \rho \propto t^{\beta}$$

Therefore for $P = \text{const}$ $\delta\rho \propto t^{\beta} + \dots$, $s \propto t^{1-\alpha}$

$$C_p \propto t^{-\alpha} \qquad x_{c_p} = x_{c_v} = -\alpha \qquad y_{c_p} = y_{c_v} = -\frac{\alpha}{\beta}$$

2c) $$k_T = \frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{1}{\rho_c}\left[\frac{\partial P}{\partial \rho}\right]_t^{-1} = \frac{1}{\rho_c^3 t^{2-\alpha-2\beta} g''\left(\frac{\delta\rho}{t^{\beta}}\right)}$$

Therefore $$k_T = \frac{1}{\rho_c^3} t^{2\beta+\alpha-2} g_k\left(\frac{\delta\rho}{t^{\beta}}\right)$$

with $g_k(\lambda) = 1/g''(\lambda)$

As a result $$x_K = 2\beta + \alpha - 2 \qquad y_K = 2 + \frac{\alpha-2}{\beta}$$

$$\alpha = \left.\frac{\partial V}{\partial T}\right)_P \frac{1}{V} = -\frac{1}{T_c}\left.\frac{\partial \rho^{-1}}{\partial t}\right)_P \rho = \frac{1}{\rho_c}\left.\frac{\partial \rho}{\partial t}\right)_P$$

Since $(\partial\rho/\partial t)_P \propto t^{1-\alpha}$

$$x_{\alpha} = \beta - 1 \qquad y_{\alpha} = 1 - \frac{1}{\beta}$$

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2d) $\delta\rho_+ = \rho - \frac{1}{v_+}$ $\delta\rho_- = -\rho + \frac{1}{v_-}$

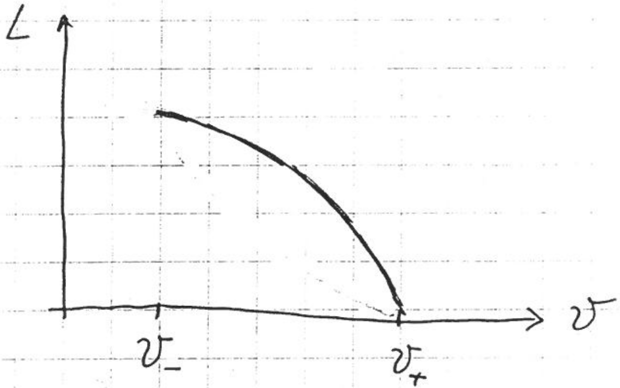
where $v_\pm$ is a volume per particle in a gas (~~+~~) and liquid phases

$$S_\pm T = t^{1-\alpha} g_s\left(\frac{v_\pm \delta\rho_\pm}{t^\beta}\right)$$

The latent heat L equals to

$$L = T(S_+ - S_-) = t^{1-\alpha}\left[g_s\left(\frac{v_+\delta\rho_+}{t^\beta}\right) - g_s\left(\frac{v_-\delta\rho_-}{t^\beta}\right)\right]$$

$$x_L = 1-\alpha$$



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8.334: Statistical Mechanics II Problem Set # 5 Due: **3/9/92**

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Susceptibilities

1. *Transverse susceptibility:* An n–component magnetization field $\vec{m}(\mathbf{x})$ is coupled to an external field $\vec{h}$ through a term $-\int d^d\mathbf{x}\ \vec{h}.\vec{m}(\mathbf{x})$ in the Hamiltonian $\beta\mathcal{H}$. If $\beta\mathcal{H}$ for $\vec{h}=0$ is invariant under rotations of $\vec{m}(\mathbf{x})$; then the free energy density ($f=-\ln Z/V$) can only depend on the absolute value of $\vec{h}$; i.e. $f(\vec{h})=f(h)$ where h is magnitude of $\vec{h}$.

(a) Show that $m_\alpha = \langle\int d^d\mathbf{x} m_\alpha(\mathbf{x})\rangle/V = -h_\alpha f'(h)/h$.

(b) Relate the susceptibility tensor $\chi_{\alpha\beta}=\partial m_\alpha/\partial h_\beta$ to $f"(h)$, $\vec{m}$ and $\vec{h}$.

(c) Show that the transverse and longitudinal susceptibilities are given by $\chi_t = m/h$ and $\chi_\ell = -f"(h)$; where m is the magnitude of $\vec{m}$.

(d) Conclude that χ_t diverges as $\vec{h}\to 0$ whenever there is a spontaneous magnetization. Is there any similar reason for χ_ℓ to diverge?

****

2. *Longitudinal Susceptibility:* In fact, due to fluctuations, the longitudinal susceptibility does diverge in dimensions $d<4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau-Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d\mathbf{x}[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2] \quad ,$$

describing an n–component magnetization vector $\vec{m}(\mathbf{x})$, in the ordered phase for $t<0$.

(a) Let $\vec{m}(\mathbf{x}) = \big(\overline{m}+\phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

(b) Regard the quadratic terms in ϕ_ℓ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_ℓ and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U = 4u\overline{m}\int d^d\mathbf{x}\phi_\ell(\mathbf{x})\vec{\phi}_t(\mathbf{x})^2.$$

Write U in Fourier space in terms of $\phi_\ell(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

(c) Calculate the bare expectation values $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle_0$ and $\langle\phi_t^\alpha(\mathbf{q})\phi_t^\beta(\mathbf{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

(d) Calculate $\langle\vec{\phi}_t(\mathbf{q}_1).\vec{\phi}_t(\mathbf{q}_2)\ \ \vec{\phi}_t(\mathbf{q}_1').\vec{\phi}_t(\mathbf{q}_2')\rangle_0$. Note that $\vec{\phi}_t$ is an $(n-1)$ component vector.

(e) Write down the expression for $\langle \phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second-order in the perturbation U. Note that since U is odd in ϕ_ℓ, only two terms at the second order are non–zero.

(f) Using the form of U in Fourier space, write the correction term as a product of two 4–point expectation values as in (d). Note that only connected terms for the longitudinal 4–point function should be included.

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

(h) Show that for $d < 4$, the correction term diverges as q^{d-4} for $q \to 0$, implying an infinite longitudinal susceptibility.

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Suggested Reading: Parisi, Chapter 10.

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8.334 ## Solutions to Problem Set 5 March 92

1) $F(\vec{h}) = F(h)$. $\langle m_\alpha \rangle = -\frac{\partial f}{\partial h_\alpha}$, $\langle m_\alpha m_\beta \rangle_c = \chi_{\alpha\beta} = -\frac{\partial^2 f}{\partial h_\alpha \partial h_\beta}$

$\frac{\partial h}{\partial h_\alpha} = \frac{h_\alpha}{h}$ so $\langle m_\alpha \rangle = -\frac{h_\alpha}{h} f'(h)$

$$\chi_{\alpha\beta} = -\frac{\delta_{\alpha\beta}}{h} f'(h) + \frac{h_\alpha h_\beta}{h^3} f'(h) - \frac{h_\alpha h_\beta}{h^2} f''(h)$$

If $h_\alpha = e_\alpha h$, e_α a unit vector, $\langle m_\alpha \rangle = e_\alpha m$, $m = -f'(h)$

$$\chi_{\alpha\beta} = (\delta_{\alpha\beta} - e_\alpha e_\beta)\frac{m}{h} - e_\alpha e_\beta f''(h)$$

so $\chi_t = \frac{m}{h}$, $\chi_l = -f''(h)$ $\Big[\chi_{\alpha\beta} e_\beta = \chi_l e_\alpha$
$\chi_{\alpha\beta} n_\beta = \chi_t n_\beta + e_\alpha n_\alpha = 0\Big]$

If as $h \to 0$, $m \not\to 0$, $f'(0) \neq 0$ and $\chi_t \to \infty$ as $h \to 0$

But there is no reason why $f''(0)$ should not be finite

[e.g. in mean field approx to L.G. $\beta\mathcal{H} = -|t|\frac{m^2}{2} + um^4 - mh$,
$h = 4um^3 - m|t|$, $\frac{dh}{dm} = 12um^2 = 3|t|$ when $h = 0$ so $f''(0) = -\frac{1}{3|t|}$]

2) a) $4u\bar{m}^2 = |t|$. Linear terms in ϕ are zero since $\bar{m}$ gives minimum.

Dropping term independent of ϕ,

$$\beta\mathcal{H} = \int d^dx \Big[\Big\{\frac{K}{2}(\nabla\phi_l)^2 + |t|\phi_l^2\Big\} + \Big\{\frac{K}{2}(\nabla\vec{\phi}_t)^2\Big\} + 4u\bar{m}\Big\{\phi_l^3 + \phi_l \vec{\phi}_t\cdot\vec{\phi}_t\Big\} + u\Big\{\phi_l^4 + 2\phi_l^2\vec{\phi}_t\cdot\vec{\phi}_t + (\vec{\phi}_t\cdot\vec{\phi}_t)^2\Big\}\Big]$$

b) $$U = 4u\bar{m}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^2q_2}{(2\pi)^d}\phi_l(-\vec{q}_1 - \vec{q}_2)\vec{\phi}_t(q_1)\cdot\vec{\phi}_t(q_2)$$

c) $$\langle \phi_l(q)\phi_l(q')\rangle_0 = \frac{(2\pi)^d \delta^d(\vec{q}+\vec{q}')}{Kq^2 + 2|t|}, \quad \langle \phi_t^\alpha(q)\phi_t^\beta(q')\rangle_0 = \frac{(2\pi)^d\delta^d(\vec{q}+\vec{q}')\delta_{\alpha\beta}}{Kq^2}$$

$\chi_l = \frac{1}{Kq^2 + 2|t|}$, $\chi_t = \frac{1}{Kq^2}$

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d) $$\left\langle \vec{\phi}_t(q_1)\cdot\vec{\phi}_t(q_2)\,\vec{\phi}_t(q_1')\cdot\vec{\phi}_t(q_2')\right\rangle_0 = \left\langle \phi_{t\alpha}(1)\phi_{t\alpha}(2)\phi_{t\beta}(1')\phi_{t\beta}(2')\right\rangle_0$$

$$= \left\langle \phi_{t\alpha}(1)\,\phi_{t\alpha}(2)\right\rangle_0\left\langle \phi_{t\beta}(1')\,\phi_{t\beta}(2')\right\rangle_0 + \left\langle \phi_{t\alpha}(1)\phi_{t\beta}(1')\right\rangle_0\left\langle \phi_{t\alpha}(2)\,\phi_{t\beta}(2')\right\rangle_0$$

$$+\left\langle \phi_{t\alpha}(1)\,\phi_{t\beta}(2')\right\rangle_0\left\langle \phi_{t\alpha}(2)\,\phi_{t\beta}(1')\right\rangle_0$$

$$= (2\pi)^{2d}\,(n-1)^2\,\frac{\delta(q_1+q_2)}{Kq_1^2}\,\frac{\delta(q_1'+q_2')}{Kq_1'^2} + (2\pi)^{2d}\,(n-1)\,\frac{\delta(q_1+q_1')\,\delta(q_2+q_2')}{Kq_1^2\;Kq_2^2}$$

$$+(2\pi)^{2d}\,(n-1)\,\frac{\delta(q_1+q_2')\,\delta(q_2+q_1')}{Kq_1^2Kq_2^2} \qquad \begin{bmatrix}\delta_{\alpha\alpha}\delta_{\beta\beta} = (n-1)^2\\ \delta_{\alpha\beta}\,\delta_{\alpha\beta} = (n-1)\end{bmatrix}$$

e) $$\left\langle \phi_L(q)\,\phi_L(q')\right\rangle = \frac{\left\langle \phi_L(q)\,\phi_L(q')\,e^{-U}\right\rangle_0}{\left\langle e^{-U}\right\rangle_0} = \left\langle \phi_L(q)\phi_L(q')\right\rangle_0 + \frac{1}{2}\left\{\left\langle \phi_L(q)\phi_L(q')\,UU\right\rangle_0 - \langle\phi\phi\rangle_0\langle UU\rangle_0\right\}$$

to second order in U, using fact that $\langle \text{odd number of } \phi\rangle_0 = 0$ and that U has 3 ϕ's. So to this order,

f) $$\left\langle \phi_L(q)\,\phi_L(q')\right\rangle - \left\langle \phi_L(q)\,\phi_L(q')\right\rangle_0 = \cancel{\frac{1}{2}(4u\bar{m})^2\int\left\langle \phi_L(q)\phi_L(q')\phi_L(-q_1-q_2)\,\phi_t(q_1)\cdot\phi_t\;\phi_L(q_1'-q_2')\,\phi_t(q_1')\cdot\phi_t(q_2')\right\rangle_0 \frac{d^dq_1\,d^dq_2}{(2\pi)^{2d}}}$$

$$= \frac{1}{2}(4u\bar{m})^2\int\left\langle \phi_L(q)\,\phi_L(q')\,\phi_L(-q_1-q_2)\,\vec{\phi}_t(q_1)\cdot\vec{\phi}_t(q_2)\,\phi_L(-q_1'-q_2')\,\vec{\phi}_t(q_1')\cdot\vec{\phi}_t(q_2')\right\rangle_0 \frac{d^dq_1\,d^dq_2\,d^dq_1'\,d^dq_2'}{(2\pi)^{4d}}$$

$$-\frac{1}{2}(4u\bar{m})^2\int\left\langle \phi_L\,\phi_L\right\rangle_0\left\langle \phi_L\;\;\phi_t\cdot\phi_t\;\;\phi_L\,\phi_t\cdot\phi_t\right\rangle_0 \frac{d^dq_1}{(2\pi)^d}\cdots$$

In calculating $\langle\;\;\rangle_0$, use fact that Gaussian distribution of ϕ_L is independent of distribution of $\vec{\phi}_t$ so $\langle\cdots\rangle_0$ factorizes and we get

$$\left\langle \phi_L(q)\,\phi_L(q')\right\rangle - \left\langle \phi_L(q)\,\phi_L(q')\right\rangle_0$$

$$= \frac{1}{2}(4u\bar{m})^2\int\left[\left\langle \phi_L(q)\,\phi_L(q')\,\phi_L(-q_1-q_2)\,\phi_L(-q_1'-q_2')\right\rangle_0 - \left\langle \phi_L(q)\phi_L(q')\right\rangle_0\left\langle \phi_L(-q_1-q_2)\,\phi_L(-q_1'-q_2')\right\rangle_0\right]$$

$$\times\left[\left\langle \vec{\phi}_t(q_1)\cdot\vec{\phi}_t(q_2)\;\vec{\phi}_t(q_1')\cdot\vec{\phi}_t(q_2')\right\rangle_0\right] \frac{d^dq_1\cdots d^dq_2'}{(2\pi)^{4d}}$$

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g) Dropping the term $\propto (n-1)^2$ [which has a factor $\delta(q_1+q_2)\,\delta(q_1'+q_2')$ and so a factor $\delta(q)\,\delta(q')$ and so only contributes to $\langle \phi_l(0)\,\phi_l(0)\rangle$], ~~and using~~ the longitudinal and transverse 4-point functions are each the sum of two terms. All four products give equal contribution to integral, using $q_1' \leftrightarrow q_2'$ and $q_1, q_2 \leftrightarrow q_1', q_2'$. So we get

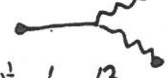
$$4 \times \frac{1}{2} \times (4u\bar{m})^2 (n-1) \int \frac{\delta(q_1+q_1')\,\delta(q_2+q_2')}{Kq_1^2 \quad Kq_2^2} \frac{\delta(q-q_1-q_2)\,\delta(q'-q_1'-q_2')}{Kq^2+2|t| \quad Kq'^2+2|t|} \underline{d^dq_1 \cdots d^dq_2'}$$

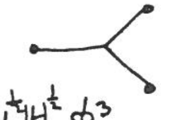
$$= \frac{2(n-1)(4u\bar{m})^2}{Kq^2+2|t| \quad Kq'^2+2|t|} (2\pi)^d \delta(q+q') \int \frac{d^dk}{(2\pi)^d} \frac{1}{Kk^2} \frac{1}{K(k-q)^2}$$

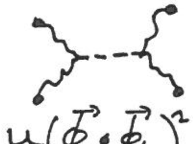
h) The integral over k might diverge for $\vec{k}=0$ or $\vec{k}=\vec{q}$. But if we put $\vec{q}=0$,
If $d>2$, both these integrals converge. Also for $d<4$,
integral will diverge at $\vec{k}=0$ for $d<4$.
integral converges at large $\vec{k}$ so we can replace upper limit by ∞
and then writing $\vec{q} = q\hat{e}$, $\vec{k} = q\vec{k}'$, integral becomes $\frac{q^{d-4}}{K^2}\int \frac{d^dk'}{(2\pi)^d} \frac{1}{k'^2(k'-\hat{e})^2}$
and so diverges like $\frac{1}{q^{4-d}}$ as $q \to 0$.
For $d>4$, we can simply put $\vec{q}=0$ and integral $\to \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2(k^2)^2}$ as $\vec{q} \to 0$
which is dependent on behaviour at large k ie on short distance details.

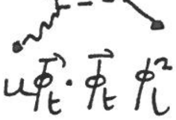
Comments

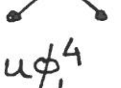
$4u\bar{m} = 2|t|^{\frac{1}{2}} u^{\frac{1}{2}}$ so at fixed t if we expand in powers of u we ought to go to second order in the ϕ^3 term and first order in the ϕ^4 term. To use diagrams, we can represent all the cubic and quartic terms in $\beta\mathcal{H}$ by vertices


$u^{\frac{1}{2}}|t|^{\frac{1}{2}}\phi_L\phi_t^2$


$u^{\frac{1}{2}}|t|^{\frac{1}{2}}\phi_l^3$


$u(\vec{\phi}_t\cdot\vec{\phi}_t)^2$


$u\vec{\phi}_t\cdot\vec{\phi}_t\,\phi_l^2$


$u\phi_l^4$

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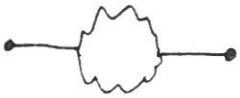
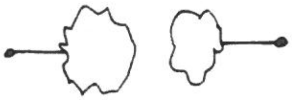
$\langle \phi_t \phi_t \rangle_0$ is represented by •∿∿∿• $\left(\frac{1}{Kq^2}\right)$

$\langle \phi_L \phi_L \rangle_0$ " •——• $\left(\frac{1}{Kq^2+2|t|}\right)$

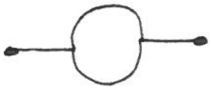
We need all connected diagrams contributing to $\langle \phi_L \phi_L \rangle$ up to order u. Abbreviating by omitting internal dots the different types are



(I)

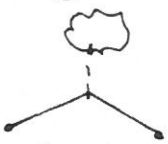


(Ia)



(II)

(IIa)



(III)

(IV)

(I) gives the contribution we have evaluated.

(II) gives a similar contribution but the final integral becomes $\int \frac{d^dk}{(Kk^2+2|t|)\,(K(q-k)^2+2|t|)}$

so does not diverge as $q \to 0$ with $|t|$ fixed.

(Ia) and (IIa) only contribute for $q = q' = 0$, in fact (Ia) is the term $\propto (n-1)^2$ we dropped in part (g). They clearly represent contributions to $\langle \phi_L \rangle \langle \phi_L \rangle$ and do not contribute to $\langle \phi_L \phi_L \rangle_c$. Note that $\langle \phi_L \rangle \neq 0$ ie the mean value $\overline{m}$ should now be shifted by $\langle \phi_L \rangle$.

(III) and (IV) give contributions like $\frac{1}{(Kq^2+|t|)^2} \; u \int \frac{d^dk}{Kk^2}$

This represents the expansion in powers of u of an order u modification to $|t|$ ie a shift in the critical temperature of order u.

Of course we know that for $d<4$ the perturbation expansion parameter is really $u/|t|^{\frac{4-d}{2}}$ and we have to use R.G. arguments to get reliable results.

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8.334: Statistical Mechanics II Problem Set # 5 Due: 3/8/93

Susceptibilities

1. *Transverse susceptibility:* An n–component magnetization field $\vec{m}(\mathrm{x})$ is coupled to an external field $\vec{h}$ through a term $-\int d^d\mathrm{x}\ \vec{h}\cdot\vec{m}(\mathrm{x})$ in the Hamiltonian $\beta\mathcal{H}$. If $\beta\mathcal{H}$ for $\vec{h}=0$ is invariant under rotations of $\vec{m}(\mathrm{x})$; then the free energy density $(f=-\ln Z/V)$ can only depend on the absolute value of $\vec{h}$; i.e. $f(\vec{h})=f(h)$ where $h=|\vec{h}|$.

(a) Show that $m_\alpha=\langle\int d^d\mathrm{x} m_\alpha(\mathrm{x})\rangle/V=-h_\alpha f'(h)/h$.

(b) Relate the susceptibility tensor $\chi_{\alpha\beta}=\partial m_\alpha/\partial h_\beta$ to $f''(h)$, $\vec{m}$ and $\vec{h}$.

(c) Show that the transverse and longitudinal susceptibilities are given by $\chi_t=m/h$ and $\chi_\ell=-f''(h)$; where m is the magnitude of $\vec{m}$.

(d) Conclude that χ_t diverges as $\vec{h}\to 0$ whenever there is a spontaneous magnetization. Is there any similar reason for χ_ℓ to diverge?

****

2. *Longitudinal Susceptibility:* In fact, due to fluctuations, the longitudinal susceptibility does diverge in dimensions $d<4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau-Ginzburg Hamiltonian:

$$\beta\mathcal{H}=\int d^d\mathrm{x}\left[\frac{K}{2}(\nabla\vec{m})^2+\frac{t}{2}\vec{m}^2+u(\vec{m}^2)^2\right],$$

describing an n–component magnetization vector $\vec{m}(\mathrm{x})$, in the ordered phase for $t<0$.

(a) Let $\vec{m}(\mathrm{x})=\big(\overline{m}+\phi_\ell(\mathrm{x})\big)\hat{e}_\ell+\vec{\phi}_t(\mathrm{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

(b) Regard the quadratic terms in ϕ_ℓ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_ℓ and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U=4u\overline{m}\int d^d\mathrm{x}\phi_\ell(\mathrm{x})\vec{\phi}_t(\mathrm{x})^2.$$

Write U in Fourier space in terms of $\phi_\ell(\mathrm{q})$ and $\vec{\phi}_t(\mathrm{q})$.

(c) Calculate the bare expectation values $\langle\phi_\ell(\mathrm{q})\phi_\ell(\mathrm{q}')\rangle_0$ and $\langle\phi_{t,\alpha}(\mathrm{q})\phi_{t,\beta}(\mathrm{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathrm{q})_0$ and $\chi_t(\mathrm{q})_0$.

(d) Calculate $\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\ \ \vec{\phi}_t(\mathbf{q}'_1)\cdot\vec{\phi}_t(\mathbf{q}'_2)\rangle_0$. Note that $\vec{\phi}_t$ is an $(n-1)$ component vector.

(e) Write down the expression for $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second-order in the perturbation U. Note that since U is odd in ϕ_ℓ, only two terms at the second order are non–zero.

(f) Using the form of U in Fourier space, write the correction term as a product of two 4–point expectation values as in (d). Note that only connected terms for the longitudinal 4–point function should be included.

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

(h) Show that for $d < 4$, the correction term diverges as q^{d-4} for $q \to 0$, implying an infinite longitudinal susceptibility.

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Suggested Reading: Parisi, Chapter 10.

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8.334 Problem Set #5 Solutions

1a) $$\beta \mathcal{H}(h) = \beta \mathcal{H}(0) - \int d^d\vec{x}\, \vec{h}\cdot\vec{m}(\vec{x}) \qquad (1)$$

By definition the average of any operator Θ is

$$\langle \Theta \rangle = \frac{\int \mathcal{D}\vec{m}(\vec{x}) \exp(-\beta\mathcal{H})\, \Theta}{\int \mathcal{D}\vec{m}(\vec{x}) \exp(-\beta\mathcal{H})} \qquad (2)$$

Therefore

$$\left\langle \int d^d\vec{x}\, m_\alpha(\vec{x}) \right\rangle = \frac{1}{Z} \int d\vec{x}\, m_\alpha(\vec{x})\, \mathcal{D}\vec{m}(\vec{y}) \exp(-\beta\mathcal{H}) \qquad (3)$$

where

$$Z = \int \mathcal{D}\vec{m}(\vec{y}) \exp(-\beta\mathcal{H})$$

is the partition function. Using (1) we can rewrite (3) as

$$\left\langle \int d^d\vec{x}\, m_\alpha(\vec{x}) \right\rangle = \frac{1}{Z}\frac{\partial}{\partial h_\alpha} \int \mathcal{D}\vec{m}(\vec{x}) \exp(-\beta\mathcal{H}) =$$

$$= \frac{1}{Z}\frac{\partial Z}{\partial h_\alpha} = \frac{\partial}{\partial h_\alpha} \ln Z = -\frac{\partial}{\partial h_\alpha}(Vf) \qquad (4)$$

Since the free energy per unit volume f depends only on the magnitude of the magnetic field h

$$\frac{\partial f(h)}{\partial h_\alpha} = \frac{\partial h}{\partial h_\alpha}\frac{\partial f}{\partial h} = \frac{\partial \sqrt{\sum_{\gamma=1}^{n} h_\gamma^2}}{\partial h_\alpha}\frac{\partial f}{\partial h} = \frac{h_\alpha}{\sqrt{\sum_{\gamma=1}^{n} h_\gamma^2}} f'(h) = \frac{h_\alpha}{h} f'(h,$$

Therefore, from (4) we get

$$\frac{1}{V}\left\langle \int d^d\vec{x}\, m_\alpha(\vec{x}) \right\rangle = -\frac{h_\alpha}{h} f'(h) \qquad (5)$$

1b) $\chi_{\alpha\beta} = \frac{\partial \langle m_\alpha \rangle}{\partial h_\beta}$ (we can drop $\int \frac{d^d x}{V}$ since $\langle m_\alpha(\vec{x}$ does not depend on $\vec{x}$)

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From (5) we get

$$\chi_{\alpha\beta} = -\frac{\partial}{\partial h_\beta}\left[\frac{h_\alpha}{h} f'(h)\right] = -\frac{\partial h_\alpha}{\partial h_\beta} h^{-1} f'(h) - h_\alpha \frac{\partial}{\partial h_\beta}\left[h^{-1} f'(h)\right]$$

$$= -\frac{\partial h_\alpha}{\partial h_\beta} h^{-1} f'(h) - h_\alpha h_\beta \frac{\partial}{\partial h}\left[h^{-1} f'(h)\right]$$

Since h_α are independent $\partial h_\alpha / \partial h_\beta = \delta_{\alpha\beta}$, and

$$\chi_{\alpha\beta} = -\delta_{\alpha\beta}\, h^{-1} f'(h) + h_\alpha h_\beta \left[h^{-3} f'(h) - h^{-2} f''(h)\right]$$

or

$$\chi_{\alpha\beta} = -\left(\delta_{\alpha\beta} - \frac{h_\alpha h_\beta}{h^2}\right)\frac{f'(h)}{h} - \frac{h_\alpha h_\beta}{h^2} f''(h) \qquad (6)$$

Using (5) we can relate $f'(h)$ with $m = \sqrt{\sum_{\alpha=1}^{n} \langle m_\alpha \rangle^2}$

$$m = \frac{|f'(h)|}{h}\sqrt{\sum_\alpha h_\alpha^2} = |f'(h)| = -f'(h)$$

As a result

$$\chi_{\alpha\beta} = \left(\delta_{\alpha\beta} - \frac{h_\alpha h_\beta}{h^2}\right)\frac{m}{h} - \frac{h_\alpha h_\beta}{h^2} f''(h) \qquad (7)$$

1c) By definition of the transverse and longitudinal succeptibilities χ_t and χ_ℓ from (7) we have

$$\chi_t = \frac{m}{h} \qquad \chi_\ell = -f''(h) \qquad (8)$$

1d) From (8) it is clear that for $h \to 0$ $\chi_t \to \infty$ provided there is a spontaneous magnetization and $\langle \vec{m} \rangle \neq 0$ at $h=0$. On the other hand, there is no reason why $f''(0)$ should be infinite. For instance, in the mean field approximation for the L.G. hamiltonian

$$\beta \mathcal{H} = -\frac{|t|}{2} m^2 + um3 - mh$$

$h = 4um^3 - m|t|$, $dm/dh = (dh/dm)^{-1} = (12um^2)^{-1}$

At $h=0$ $um^2 = -\frac{1}{4}|t|$ and $\chi_\ell = \frac{dm}{dh} = \frac{-1}{3|t|}$ is finite

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2a) $$\beta\mathcal{H} = \int d^d\vec{x}\left\{\frac{K}{2}(\vec{\nabla}\phi_\ell)^2 + \frac{t}{2}(\overline{m}+\phi_\ell)^2 + \vec{\phi}_t^2 + u\left[(\overline{m}+\phi_\ell)^2 + \vec{\phi}_t^2\right]^2 + \frac{K}{2}(\vec{\nabla}\vec{\phi}_t)^2 + \frac{t}{2}\vec{\phi}_t^2\right\}$$

Linear in ϕ terms vanish since $\overline{m}$ gives a minimum to $\beta\mathcal{H}$.

$$\beta\mathcal{H}(\vec{\phi}) - \beta\mathcal{H}(\vec{0}) = \int d^d\vec{x}\left\{\frac{K}{2}\left[(\vec{\nabla}\phi_\ell)^2 + (\vec{\nabla}\vec{\phi}_t)^2\right] + \left[\frac{t}{2} + 4u\overline{m}^2\right]\phi_\ell^2 \right.$$
$$\frac{t}{2} + 2u\overline{m}^2\vec{\phi}_t^2 + 4u\overline{m}(\phi_\ell^3 + \phi_\ell\vec{\phi}_t^2) +$$
$$\left. + u\left[\phi_\ell^4 + (\vec{\phi}_t^2)^2 + 2\phi^2(\vec{\phi})^2\right]\right\}$$

Since $\overline{m}^2 = |t|/4u$ and $|t| = -t$

$$\beta\mathcal{H}(\vec{\phi}) - \beta\mathcal{H}(\vec{0}) = \int d^d\vec{x}\left\{\frac{K}{2}\left[(\nabla\phi_\ell)^2 + (\vec{\nabla}\vec{\phi})^2\right] + \frac{|t|}{2}(\phi_\ell^2 + \phi_t^2) \right.$$
$$\left. + 4um(\phi_\ell^3 + \phi_\ell\phi_t^2) + u\left[\phi_\ell^4 + (\phi_t^2)^2 + 2\phi^2(\vec{\phi})^2\right]\right\}$$

2b) $$V = 4u\overline{m}\int\frac{d^d\vec{q}_1}{(2\pi)^d}\frac{d^d\vec{q}_2}{(2\pi)^d}\phi_\ell(-\vec{q}_1 - \vec{q}_2)\phi_t(\vec{q}_1)\phi_t(\vec{q}_2) \qquad (9)$$

2c) $$\langle\phi_\ell(q_1)\phi_\ell(q_2)\rangle = (2\pi)^d\frac{\delta(\vec{q}_1 + \vec{q}_2)}{Kq_1^2 + |t|}$$
$$(10)$$
$$\langle\phi_t^\alpha(q_1)\phi_t^\beta(q_2)\rangle = (2\pi)^d\frac{\delta(\vec{q}_1 + \vec{q}_2)\delta_{\alpha\beta}}{Kq_1^2}$$

This follows directely from

$$\mathcal{H}_0 = \int d^d\vec{x}\left\{\frac{K}{2}\left[(\vec{\nabla}\phi_\ell)^2 + (\vec{\nabla}\vec{\phi}_t)^2\right] + \frac{|t|}{2}|\phi_\ell|^2\right\}$$

As a result

$$\chi_\ell = \frac{1}{Kq^2 + |t|} \qquad \chi_t = \frac{1}{Kq^2}$$

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2d) For an averaging with a Gaussian weight we have

$$\langle[\vec\Phi_t(\vec q_1)\cdot\vec\Phi_t(\vec q_2)][\vec\Phi_t(\vec q_1{}')\,\vec\Phi_t(\vec q_2{}')]\rangle_0=\langle\phi_{t\alpha}(\vec q_1)\phi_{t\alpha}(\vec q_2)\phi_{t\beta}(\vec q_1{}')\phi_{t\beta}(\vec q_2{}')\rangle$$

$$=\langle\phi_{t\alpha}(\vec q_1)\phi_{t\alpha}(\vec q_2)\rangle_0\cdot\langle\phi_{t\beta}(\vec q_1{}')\phi_{t\beta}(\vec q_2{}')\rangle_0+\langle\phi_{t\alpha}(\vec q_1)\phi_{t\beta}(\vec q_1{}')\rangle_0\cdot\langle\phi_{t\alpha}(\vec q_2)\phi_{t\beta}(\vec q_2{}')\rangle$$

$$+\langle\phi_{t\alpha}(\vec q_1)\cdot\phi_{t\beta}(\vec q_2{}')\rangle_0\cdot\langle\phi_{t\alpha}(\vec q_2)\phi_{t\beta}(\vec q_1{}')\rangle_0$$

where summation over α and over β is assumed $1\leq\alpha,\beta\leq n-1$. Substituting (10) gives

$$\langle\phi_{t\alpha}(\vec q_1)\phi_{t\alpha}(\vec q_2)\phi_{t\beta}(\vec q_1{}')\phi_{t\beta}(\vec q_2{}')\rangle=\frac{(2\pi)^{2d}}{K^2}\left\{(n-1)^2\,\frac{\delta(\vec q_1+\vec q_2)\delta(\vec q_1{}'+\vec q_2{}')}{q_1^2\,q_1'^2}+\right.$$

$$\left.+(n-1)\left[\frac{\delta(\vec q_1+\vec q_1{}')\,\delta(\vec q_2+\vec q_2{}')}{q_1^2\,q_2^2}+\frac{\delta(\vec q_1+\vec q_2{}')\,\delta(\vec q_1{}'+\vec q_2)}{q_1^2\,q_2^2}\right]\right\}\quad(11)$$

since $\delta_{\alpha\alpha}\delta_{\beta\beta}=(n-1)^2$, while $\delta_{\alpha\beta}\delta_{\alpha\beta}=n-1$

2e) $$\langle\phi_\ell(q)\,\phi_\ell(\vec q\,')\rangle=\frac{\langle\phi_\ell(\vec q)\phi_\ell(\vec q\,')\exp(-U)\rangle_0}{\langle\exp(-U)\rangle_0}$$

Expanding $\exp(-U)=1-U+\frac{U^2}{2}-\ldots$ and using the facts that U is qubic in ϕ and $\langle\text{odd number of }\phi\text{-s}\rangle_0=$ we get

$$\langle\phi_\ell(\vec q)\,\phi_\ell(\vec q\,')\rangle=\frac{\langle\phi_\ell(\vec q)\,\phi_\ell(\vec q\,')(1+\frac{1}{2}U^2)\rangle_0}{1+\frac{1}{2}\langle U^2\rangle_0}=$$

$$=\langle\phi_\ell(\vec q)\phi_\ell(\vec q\,')\rangle_0+\frac{1}{2}\left[\langle\phi_\ell(\vec q)\phi_\ell(\vec q\,')U^2\rangle_0-\langle\phi_\ell(\vec q)\phi_\ell(\vec q\,')\rangle_0\langle U^2\rangle_0\right.$$

2f) Substituting (9) we get for the second order correction

$$\langle\phi_\ell(\vec q)\phi_\ell(\vec q\,')\rangle-\langle\phi_\ell(\vec q)\,\phi_\ell(\vec q\,')\rangle_0=$$

$$=\frac{1}{2}(4u\bar m)^2\int\frac{dq_1\,dq_2\,dq_1'\,dq_2'}{(2\pi)^{4d}}\langle\phi_\ell(\vec q)\phi_\ell(\vec q\,')\phi_\ell(-\vec q_1-\vec q_2)[\phi_t(\vec q_1)\cdot\phi_t(\vec q_2)]\,*$$

$$*\,\phi_\ell(-\vec q_1{}'-\vec q_2{}')[\vec\Phi_t(\vec q_1{}')\cdot\vec\Phi_t(\vec q_2{}')]\rangle_0-\frac{1}{2}\langle\phi_\ell(\vec q)\,\phi_\ell(\vec q\,')\rangle_0\langle U^2\rangle_0$$

Note that (1) terms like $\langle \phi_\ell \phi_\ell \rangle_0 \langle U^2 \rangle$ cancell each other

(2) $\langle \phi_\ell \phi_t \rangle_0 = 0$

As a result there are only two terms left proportional to $\langle \phi_\ell(q)\phi_\ell(-\vec{q}_1-\vec{q}_2)\rangle_0$ and to $\langle \phi_\ell(\vec{q})\phi_\ell(-\vec{q}_1{}'-\vec{q}_2{}')\rangle_0$:

$$\langle \phi_\ell(\vec{q})\phi_\ell(\vec{q}\,')\rangle - \langle \phi_\ell(\vec{q})\phi_\ell(\vec{q}\,')\rangle_0 = 8(u\bar{m})^2 \int \frac{d\vec{q}_1\, d\vec{q}_2\, d\vec{q}_1{}'\, d\vec{q}_2{}'}{(2\pi)^{4d}} \times$$

$$\times \Big\{ \langle \phi_\ell(q)\phi_\ell(-\vec{q}_1-\vec{q}_2)\rangle_0 \cdot \langle \phi_\ell(\vec{q}\,')\phi_\ell(-\vec{q}_1{}'-\vec{q}_1{}')\rangle_0 \cdot \langle [\vec{\phi}_t(\vec{q}_1)\vec{\phi}_t(\vec{q}_2)][\vec{\phi}_t(\vec{q}_1{}')\vec{\phi}_t(\vec{q}_2{}')]\rangle_0 +$$

$$+ \langle \phi_\ell(q)\phi_\ell(-\vec{q}_1{}'-\vec{q}_2{}')\rangle_0 \cdot \langle \phi_\ell(\vec{q}\,')\phi_\ell(-\vec{q}_1-\vec{q}_2)\rangle_0 \langle [\phi_t(\vec{q}_1)\phi_t(\vec{q}_2)][\phi_t(\vec{q}_1{}')\phi_t(\vec{q}_2)]\rangle_0 \Big\}$$

Using (11) and (10) and also substituting $u^2\bar{m}^2 = \frac{u^2|t|}{4u} = u|t|/4$ we get

$$\langle \phi_\ell(\vec{q})\phi_\ell(\vec{q}\,')\rangle - \langle \phi_\ell(\vec{q})\phi_\ell(\vec{q}\,')\rangle_0 = 2u|t| \int \frac{d\vec{q}_1\, d\vec{q}_2\, d\vec{q}_1{}'\, d\vec{q}_2{}'}{(2\pi)^{4d}} \cdot \frac{(2\pi)^{6d}}{K^2} \times$$

$$\times \Big\{ (n-1)^2 \frac{\delta(\vec{q}_1+\vec{q}_2)\,\delta(\vec{q}_1{}'+\vec{q}_2{}')}{q_1^2 (q_1')^2} + (n-1) \frac{\delta(\vec{q}_1+\vec{q}_1{}')\,\delta(\vec{q}_2+\vec{q}_2{}') + \delta(\vec{q}_1+\vec{q}_2{}')\,\delta(\vec{q}_2+\vec{q}_1{}')}{q_1^2 q_2^2} \Big\}$$

$$\times \frac{\delta(\vec{q}-\vec{q}_1-\vec{q}_2)\,\delta(\vec{q}\,'-\vec{q}_1{}'-\vec{q}_2{}') + \delta(\vec{q}-\vec{q}_1{}'-\vec{q}_2{}')\,\delta(\vec{q}\,'-\vec{q}_1-\vec{q}_2)}{(Kq^2+|t|)(Kq_1'^2+|t|)} \Big\} =$$

$$= \frac{2u|t| \cdot (2\pi)^{2d}}{K^2(Kq^2+|t|)(Kq'^2+|t|)} \Big\{ (n-1)^2 \delta(\vec{q})\delta(\vec{q}\,') \int \frac{d\vec{q}_1\, d\vec{q}_1{}'}{q_1^2 q_1'^2} +$$

$$2(n-1)\,\delta(\vec{q}+\vec{q}\,') \int \frac{d^d\vec{q}_1}{(\vec{q}+\vec{q}_1)^2 q_1^2} \Big\}$$

2g). Neglecting the discontinuous first term we have

$$\langle \phi_\ell(\vec{q})\phi_\ell(\vec{q}\,')\rangle = \delta(\vec{q}+\vec{q}\,') \Big\{ \frac{(2\pi)^d}{Kq^2+|t|} + \frac{4u|t|(n-1)}{(Kq^2+|t|)^2} (2\pi)^{2d} \int \frac{d^d\vec{q}_1}{K^2\vec{q}_1^{\,2}(\vec{q}+\vec{q}_1)^2}$$

2h) The correction evidently diverges as q^{4-d} for $q \to 0$ and $d<4$ implying infinite longitudional succeptibility

$$\chi_\ell(q) = (Kq^2+|t|)^{-1} + (2\pi)^d \cdot 4u|t|(n-1)(Kq^2+|t|)^{-2} \int \frac{d^d\vec{q}_1}{K^2 q_1^2 (q+q_1)^2}$$

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ε–Expansions

1. *Long-Range Interactions* between spins can be described by adding a term

$$\int d^d\mathbf{x} \int d^d\mathbf{y} J(|\mathbf{x}-\mathbf{y}|)\vec{m}(\mathbf{x}).\vec{m}(\mathbf{y})$$

to the usual Landau-Ginzburg Hamiltonian.

(a) Show that for $J(r) \propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\beta\mathcal{H} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{t + K_2 q^2 + K_\sigma q^\sigma + \cdots}{2} \vec{m}(\mathbf{q}).\vec{m}(-\mathbf{q}) +$$
$$u \int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}} \vec{m}(\mathbf{q}_1).\vec{m}(\mathbf{q}_2)\vec{m}(\mathbf{q}_3).\vec{m}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \quad .$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν, η, and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

(e) For $\sigma < 2$, use a perturbation expansion in u to construct the recursion relations for (t, K_σ, u) as in lectures.

(f) For $d < 2\sigma$, calculate the critical exponents ν and η to first order in $\varepsilon = d - 2\sigma$.
(See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).)

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!

****

2. *This problem is optional. Do as much of it as you like.* Given the exponents

$$\nu = \frac{1}{2} + \frac{(n+2)}{4(n+8)}\varepsilon + \frac{(n+2)(n^2+23n+60)}{8(n+8)^3}\varepsilon^2 \quad ,$$
$$\eta = \frac{(n+2)}{2(n+8)^2}\varepsilon^2 \quad ,$$

use scaling relations to obtain ε–expansions for α, β, γ, δ and Δ. Make a table of the results obtained by setting $\varepsilon = 1,\ 2$ for $n = 1,\ 2$ and 3; and compare to the best estimates of these exponents that you can find by other sources (series, experiments, etc.; quoted in Ma, Stanley, or elsewhere).

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Suggested reading: Ma, Chapters VII and IX, Stanley, Chapter 8.

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8.334 Problem Set 6, Solutions March 92

1) a) With $m(x) = \int \frac{d^d q}{(2\pi)^d} m(q) e^{iq\cdot x}$,

$$\int K(x-y)\, m(x)\cdot m(y)\, d^dx\, d^dy = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d} m(q_1)\cdot m(q_2) e^{iq_1\cdot x} e^{i(q_2+q_1)\cdot y} K(x)\, d^dx\, d^dy$$

[Changing x to $x+y$] $= \int \frac{d^dq}{(2\pi)^d} m(q)\cdot m(-q) \tilde{K}(q)$

with $\tilde{K}(q) = \int K(x) e^{iq\cdot x} d^dx$

For $K(x) \propto |x|^{-d-\sigma}$, $\tilde{K}(q) \propto \int_a^\infty \frac{dr}{r^{\sigma+1}} F(qr)$

where $F(qr)$ is the integral over angles of $e^{iq\cdot x}$ and a is the short-distance cut-off. For qr small, $F(qr) = F_0 + F_2 (qr)^2 + F_4 (qr)^4 \cdots$

[For qr large, $F \sim \frac{\sin(qr+\delta_d)}{(qr)^{\frac{d-1}{2}}}$ and integral converges at large r, certainly for $\sigma > 0$ –

If $2n < \sigma < 2n+2$, $\tilde{K}(q) = \int_0^\infty \frac{dr}{r^{\sigma+1}} \left\{ F(qr) - F_0 - F_2 q^2 r^2 \cdots - F_{2n} q^{2n} r^{2n} \right\}$

$$- \int_0^a \frac{dr}{r^{\sigma+1}} \left\{ F_{2n+2} (qr)^{2n+2} + \cdots \right\} + \int_a^\infty \frac{dr}{r^{\sigma+1}} \left\{ F_0 + F_2 (qr)^2 + \cdots + F_{2n} (qr)^{2n} \right\}$$

The first integral now converges at $r=0$ and ∞ and so $\propto q^\sigma$ $(qr = r')$, the other two give the analytic function of q, $\frac{1}{a^\sigma}\sum_m F_{2m} \frac{q^{2m} a^{2m}}{\sigma - 2m}$

If $\sigma = 2n$, terms like $q^\sigma \ln q$ will appear.

We see that the extra term in the L-G $\beta\mathcal{H}$ is of form $\frac{1}{2}\int \frac{d^dq}{(2\pi)^d} K_\sigma q^\sigma m(q)\cdot m$

b) With $\beta\mathcal{H} = \frac{1}{2}\int_0^\Lambda K(q)\, m(q)\cdot m(-q) \frac{d^dq}{(2\pi)^d}$, integrating out $m(q)$, $|q| > \frac{\Lambda}{b}$

gives $Z = Z_> \int \mathcal{D}m(q) \exp \int_0^{\frac{\Lambda}{b}} K(q)\, m(q)\cdot m(-q) \frac{d^dq}{(2\pi)^d}$

$$\propto \int \mathcal{D}m(q) \exp \int_0^\Lambda K\left(\frac{q'}{b}\right) z^2 m'(q')\cdot m'(-q')\, b^{-d} \frac{d^dq'}{(2\pi)^d}$$

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where $m\left(\frac{q}{b}\right) = Z m'(q)$ ie coarse graining and renormalization

sent $K \to K'$ where $K'(q) = Z^2 b^{-d} K\left(\frac{q}{b}\right)$

With $K(q) = t + K_2 q^2 + K_\sigma q^\sigma$,

$$t' = Z^2 b^{-d} t, \quad K_2' = Z^2 b^{-d-2} K_2, \quad K_\sigma' = Z^2 b^{-d-\sigma} K_\sigma$$

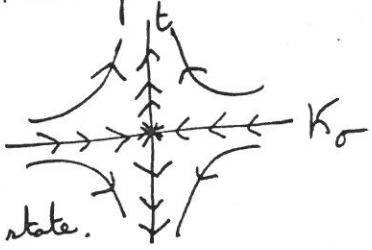
After coarse-graining reduced the volume of the system by b^d

(clearly true in position space R.G. In q-space when we spread $|q| < \frac{\Lambda}{b}$ out over $|q| < \Lambda$ we reduced the density of points in q space by b^d which is same as reducing volume in position space. So if $f(K)$ is free energy density we have $f(K) = b^{-d} f(K')$ (since $\ln Z = \ln Z'$)

Choose Z so that $K_2' = K_2$ ie $Z^2 = b^{d+2}$.

Then $t' = b^2 t$, $K_\sigma' = b^{2-\sigma} K_\sigma$. Fixed point is $t^* = K_\sigma^* = 0$

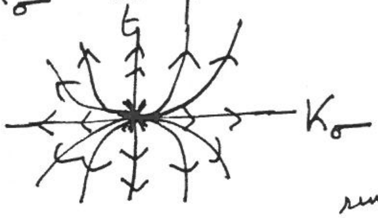
and R.G. flow in t, K_σ for $\sigma > 2$ is



K_σ direction is an irrelevant eigendirection

$t = 0$ and any value of K_σ correspond to critical state.

If $\sigma < 2$, flow is

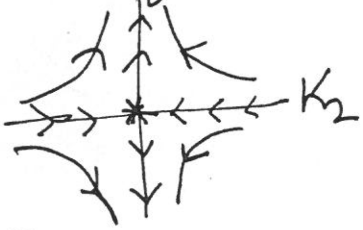


which gives us no information about phase transition since all flows are away from fixed point

c) Instead for $0 < \sigma < 2$ choose Z so that $K_\sigma' = K_\sigma$

ie $Z^2 = b^{d+\sigma}$, $t' = b^\sigma t$, $K_2' = b^{\sigma-2} K_2$

Fixed point is $t^* = K_2^* = 0$ and flow is



This time K_2 is irrelevant direction.

Fixed Hamiltonians have $K^*(q) = \begin{cases} K_2 q^2 & \text{for } \sigma > 2 \\ K_\sigma q^\sigma & \text{for } \sigma < 2 \end{cases}$

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i.e $\beta \mathcal{H}^* \propto \begin{cases} \int |\nabla m(x)|^2 \, d^d x & \text{for } \sigma > 2 \\ \int \frac{m(x)\cdot m(y)}{|x-y|^{d+\sigma}} \, d^d x \, d^d y & \text{for } \sigma < 2 \end{cases}$

d) Since correlation length scales by b i.e $\xi(K) = b\,\xi(K')$

we have $\xi(t) = \begin{cases} b\,\xi(b^2 t) & \sigma > 2 \\ b\,\xi(b^\sigma t) & \sigma < 2 \end{cases}$ $\quad \therefore \xi \propto t^{-\frac{1}{2}}$, $\quad \xi \propto t^{-\frac{1}{\sigma}}$

i.e $\nu = \frac{1}{2}$ $\sigma > 2$, $\nu = \frac{1}{\sigma}$ $\sigma < 2$.

For η, we have directly at $t = 0$ $\langle m(x)\, m(0) \rangle \propto \int \frac{d^d q \, e^{iq\cdot x}}{q^\sigma}$ for $\sigma < 2$

$\propto \frac{1}{|x|^{d-\sigma}}$, $\eta = 2 - \sigma$ (integral is dominated for large $|x|$ by singularity at $q = 0$)

[Or, using R.G., $\langle m(x_1)\, m(x_2) \rangle_K \propto \int m(q_1) m(q_2) e^{i(q_1 x_1 + q_2 x_2)} dq_1 dq_2 \, \mathcal{D}m \, e^{-\frac{1}{2}\int K(q) m(q) m(-q) dq}$

$$= \int Z^2 m'(q_1 b) m'(q_2 b) e^{i(q_1 x_1 + q_2 x_2)} d^d q_1 \, d^d q_2 \, \mathcal{D}m' \, e^{-\frac{1}{2}\int Z^2 K(q) m'(qb) m'(-qb) dq}$$

$$= \int Z^2 b^{-2d} m'(q_1) m'(q_2) e^{i(q_1 \frac{x_1}{b} + q_2 \frac{x_2}{b})} dq_1 dq_2 \, \mathcal{D}m' \, e^{-\frac{1}{2}\int K'(q) m'(q) m'(-q) dq}$$

$= Z^2 b^{-2d} \left\langle m\left(\frac{x_1}{b}\right) m\left(\frac{x_2}{b}\right) \right\rangle_{K'}$. So for $\sigma < 2$, at fixed point $K' = K$,

$Z^2 = b^{d+\sigma}$ and $\langle m(x)\, m(0) \rangle^* = b^{-d+\sigma} \left\langle m\left(\frac{x}{b}\right) m(0) \right\rangle^*$

so $\langle m(x)\, m(0) \rangle^* \propto 1/|x|^{d-\sigma}$]

To find γ, we introduce a magnetic field term into $\beta\mathcal{H}$

which is $h\int m(x)\, d^d x = h\, m(q=0) = z h m'(q=0)$

so $h' = zh$ and $f(t, h) = b^{-d} f(t', h')$

$= b^{-d} f(b^\sigma t, b^{\frac{d+\sigma}{2}} h)$

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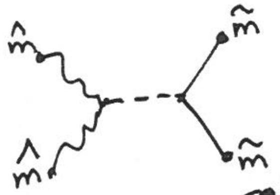
Since $\chi \propto \frac{\partial^2 f}{\partial h^2}$, $\chi(t) = b^{\sigma}\chi(b^{\sigma}t) \propto t^{-1}$, $\gamma = -1$

Term in $\beta\mathcal{H}$ proportional to $u\int m^4 d^{3d}q$ becomes $u\int z^4 m'^4 \frac{d^{3d}q}{b^{3d}}$
so $u' = uz^4 b^{-3d} = u(b^{d+\sigma})^2 b^{-3d} = u b^{2\sigma-d}$
so if $d>2\sigma$, u flows to $u=0$ and u direction is irrelevant
and generalized Gaussian results are valid for $\begin{cases}\sigma<2, & d>4\\ \sigma<d/2, & d<4\end{cases}$

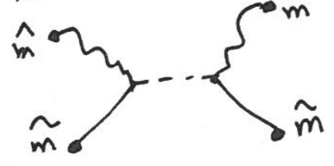
e) As in Gaussian case, perturbation terms in $\beta\tilde{\mathcal{H}}(\tilde m)$
are calculated from $-\ln\langle e^{-U(\tilde m,\hat m)}\rangle = \langle U\rangle - \frac{1}{2}\{\langle U^2\rangle - \langle U\rangle^2\} - -$
where $\tilde m(q) = m(q)$, $q<\frac{\Lambda}{b}$, $\hat m(q) = m(q)$, $q>\frac{\Lambda}{b}$
and $\langle\ \rangle$ means average over $\hat m$ with weight $e^{-\frac{1}{2}\int K\hat m\hat m}$
ie, in diagrams contract $\hat m$ with $\hat m$ 〰 to get $\frac{1}{t+Kq^{\sigma}}$

We need vertices in U,



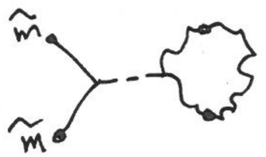
(2u)

and



(4u)

First order term in $\beta\tilde{\mathcal{H}}$ is



+

which gives $(2un+4u)\int \tilde m(q)\cdot\tilde m(-q)\frac{d^dq}{(2\pi)^d}\int_{\frac{\Lambda}{b}}^{\Lambda}\frac{d^dq_1/(2\pi)^d}{(t+Kq_1^{\sigma})}$

and so $\frac{\tilde t}{2} = \frac{t}{2} + 2u(n+2)\int_{\frac{\Lambda}{b}}^{\Lambda}\frac{d^dk/(2\pi)^d}{t+Kk^{\sigma}}$

and after rescaling and renormalizing [ie $m(q) = zm'(qb)$]

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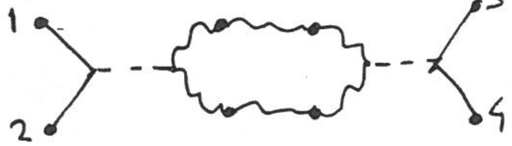
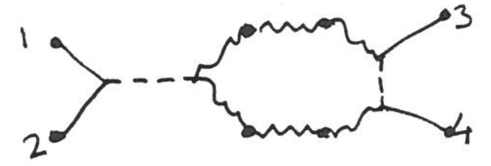
$$t' = \left[t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \cdots \right] Z^2 b^{-d}$$

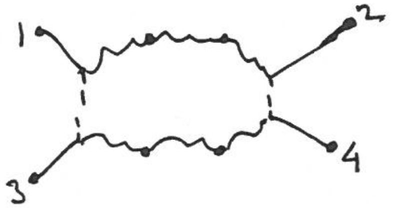
Important contribution to $\tilde{u}$ comes from



$$-\left(\frac{1}{2}(2u)^2 2n\right) \tilde{m}_1\cdot\tilde{m}_2\,\tilde{m}_3\cdot\tilde{m}_4$$



$$-\left(\frac{1}{2}\cdot 2\,.\,(2u)(4u)2\right) \tilde{m}_1\cdot\tilde{m}_2\,\tilde{m}_3\cdot\tilde{m}_4$$



$$-\left(\frac{1}{2}\,(4u)^2\,2\right) \tilde{m}_1\cdot\tilde{m}_2\,\tilde{m}_3\cdot\tilde{m}_4$$

and for q_1, q_2, q_3, q_4 all $\to 0$ we get

$$-(4n+32)u^2\int\frac{d^dq_1\,d^dq_2\,d^dq_3}{(2\pi)^{3d}}\,\tilde{m}(q_1)\cdot\tilde{m}(q_2)\,\tilde{m}(q_3)\cdot\tilde{m}(q_4) \times \int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{1}{(t+Kk^{\sigma})^2}$$

and so $u' = b^{-3d}Z^4\left[u - 4(n+8)u^2\int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{1}{(t+K_\sigma k^{\sigma})^2}\right]$

Corrections to K_σ are of order u^2 and will be of order ϵ^2 when u is of order $\epsilon = 2\sigma - d$ so $K'_\sigma = K_\sigma$ determines $Z^2 = b^{d+\sigma}$

Now put $b = e^{l} \sim 1+l$ to get differential recursion relations

$$\frac{dt}{dl} = \sigma t + 4u(n+2)\frac{K_d\Lambda^d}{t+K_\sigma\Lambda^\sigma}$$

$$\frac{du}{dl} = (2\sigma-d)u - 4(n+8)u^2\,\frac{K_d\Lambda^d}{(t+K_\sigma\Lambda^\sigma)^2}$$

$O(n)$ fixed point will be for t^*, u^* of order $\epsilon = 2\sigma - d$

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$$u^* = \frac{\epsilon}{4(n+8)}\frac{K_\sigma^2}{K_{d=2\sigma}} \qquad t^* = -\epsilon\frac{(n+2)}{(n+8)}\frac{K_\sigma}{\sigma}\Lambda^\sigma$$

and then linearizing the recursion relations,

$$\frac{d}{dl}\delta t = \sigma\,\delta t + 4(n+2)\frac{K_d\Lambda^d}{t^*+K_\sigma\Lambda^\sigma}\delta u - 4u^*\frac{(n+2)K_d\Lambda^d}{(t^*+K_\sigma\Lambda^\sigma)^2}\delta t$$

$$\frac{d}{dl}\delta u = (2\sigma-d)\delta u - 8(n+8)u^*\delta u\frac{K_d\Lambda^d}{(t^*+K_\sigma\Lambda^\sigma)^2} + O(\epsilon^2)\,\delta t$$

which become
$$\frac{d}{dl}\delta t = \left[\sigma - \epsilon\frac{(n+2)}{(n+8)}\right]\delta t + \left[4(n+2)\frac{K_{2\sigma}\Lambda^\sigma}{K_\sigma}\right]\delta u$$

$$\frac{d}{dl}\delta u = \left[\epsilon - 2\epsilon\right]\delta u$$

so u is irrelevant and $y_t = \sigma - \frac{n+2}{n+8}\epsilon$

$$\nu = \frac{1}{y_t} = \frac{1}{\sigma}\left\{1 + \frac{n+2}{n+8}\frac{\epsilon}{\sigma}\right\}$$

Since K_σ is only affected by u to order u^2, Z is unchanged and η has previous value $2-\sigma$

g) $J(r) = e^{-r/a} \Rightarrow \tilde{K}(q)$ a series $K_0 + K_2 q^2 + \cdots\cdot$ so including a short range interaction like this just changes $t, K_2, \cdots$ by fixed amounts. No change in critical behaviour.

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8.334 Problem Set 6, Solutions March 92

2) Scaling relations are

i) $\alpha = 2 - d\nu \Rightarrow \alpha = \frac{4-n}{2(n+8)}\epsilon - \frac{(n+2)^2(n+28)}{4(n+8)^3}\epsilon^2$

ii) $\gamma = \nu(2-\eta) \Rightarrow \gamma = 1 + \frac{(n+2)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2$

iii) $\beta = \frac{2-\alpha-\gamma}{2} \Rightarrow \beta = \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \frac{(n+2)(2n+1)}{2(n+8)^3}\epsilon^2$

iv) $\Delta = \beta + \gamma \Rightarrow \Delta = \frac{3}{2} + \frac{(n-1)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+26n+54)}{4(n+8)^3}\epsilon^2$

v) $\delta = \frac{\Delta}{\beta} \Rightarrow \delta = 3 + \epsilon + \frac{(n^2+14n+60)}{2(n+8)^2}\epsilon^2$

	ν	η	α	γ	β	Δ	δ
$\epsilon=1$, $n=1$ $d=3$, Ising	0.6265	0.0185	0.0771	1.2438	0.3395	1.5833	4.4630
$\epsilon=1$, $n=2$ $d=3$, XY	0.6383	0.0200	0.0200	1.3000	0.3600	1.6600	4.4600
$\epsilon=1$, $n=3$ $d=3$, Heisenberg	0.6756	0.0207	−0.1001	1.3465	0.3899	1.7233	4.4587
$\epsilon=2$, $n=1$ $d=2$, Ising [EXACT]	0.8395 [1]	0.0741 [0.25]	−0.0247 [0]	1.6420 [1.75]	0.1913 [0.125]	1.8333	6.8518 [15]
$\epsilon=2$, $n=2$ $d=2$, XY	0.9200	0.0800	−0.2800	1.8000	0.6600	2.0400	6.8400
$\epsilon=2$, $n=3$ $d=2$ Heisenberg	0.9864	0.0826	−0.4914	1.9316	0.2799	2.2115	6.8347

TURN OVER

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Stanley Table 8.3 (p127) has high temperature series results for FCC Lattices with D dimensional spins

The $d=2$ Heisenberg system has no finite temperature phase transition since there is no ordered phase in two dimensions with continuous symmetry.

[The $d=2$ XY model has an exact solution (see J.M. KOSTERLITZ, J. Phys. C, $\underline{7}$, p1046) with very special behaviour (Kosterlitz-Thouless) ξ diverges exponentially, $\xi \propto \exp b t^{-\frac{1}{2}}$.
In terms of this correlation length, $\chi \propto \xi^{2-\eta}$, $\eta = 0.25$ for $t>0$ and $\chi = \infty$, $t<0$. At $t=0$, $m \propto h^{1/\delta}$, $\delta = 15$
Also $C_h \propto \xi^{\tilde{\alpha}}$, $\tilde{\alpha} = -2$.]

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Cubic Anisotropy

Consider the modified Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + v\sum_{i=1}^{n} m_i^4 \right] \quad ,$$

for an n–component vector $\vec{m}(\mathbf{x}) = (m_1, m_2, \cdots, m_n)$. The "cubic anisotropy" term, $\sum_{i=1}^{n} m_i^4$, breaks the full rotational symmetry and selects specific directions as discussed in problem 1.

1. *Mean-Field Theory:*

(a) For a fixed magnitude $|\vec{m}|$; what directions in the n component magnetization space are selected for $v > 0$ and for $v < 0$? What is the degeneracy of easy magnetization axes in each case?

(b) What are the restrictions on u and v for $\beta\mathcal{H}$ to have finite minima? Sketch these regions of stability in the (u, v) plane.

(c) In general higher order terms (e.g. $u_6(\vec{m}^2)^3$ with $u_6 > 0$) are present and ensure stability in the regions not allowed in part (b); (as in case of the tricritical point discussed in problem set 2). With such terms in mind, sketch the phase diagram in the (t, v) plane for $u > 0$; clearly identifying the phases, and order of the transition lines.

(d) Are there any Goldstone modes in the ordered phases?

****

2. *ε–Expansion:*

(a) By looking at diagrams in a second order perturbation expansion in both u and v; show that the recursion relations for these couplings are

$$\begin{cases} \frac{du}{d\ell} = \varepsilon u - 4C\left[\,(n+8)u^2 + 6uv\,\right] \\ \frac{dv}{d\ell} = \varepsilon v - 4C\left[\,12uv + 9v^2\,\right] \end{cases} \quad ,$$

where $C = K_d\Lambda^d/(t + K\Lambda^2)^2 \approx K_4/K^2$; is approximately a constant.

(b) Find all fixed points in the (u, v) plane, and draw the flow patterns for $n < 4$ and $n > 4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case.

(c) Find the recursion relation for the reduced temperature t, and calculate the exponent ν at the stable fixed points for $n < 4$ and $n > 4$.

(d) Is the region of stability in the (u, v) plane calculated in part 1(b) enhanced or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about the nature of the phase transition for a small negative v and $n > 4$?

(e) Draw schematic phase diagrams in the (t, v) plane $(u > 0)$ for $n > 4$ and $n < 4$, identifying the ordered phases. Are there Goldstone modes in any of these phases close to the phase transition?

****

Suggested Reading: Amit, Part II, Chapter 4.

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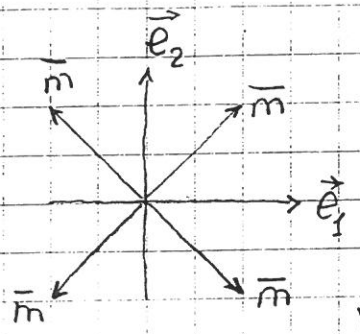
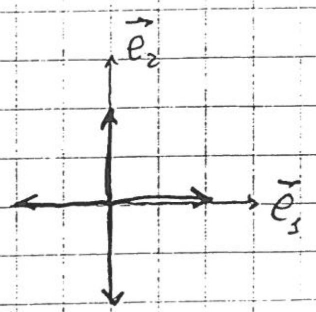
8.334 Problem Set #7

$$\beta \mathcal{H} = \int d^d x \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 + u (m^2)^2 + v \sum_{i=1}^{n} m_i^4 \right]$$

1a)



(n=2 example)



V>0
diagonal order

V<0
cubic axis order

V>0 $\quad \bar{m}$ avoids the axis $\vec{e}_i$

$$\vec{\bar{m}} = \frac{|\bar{m}|}{\sqrt{n}} (\pm 1, \pm 1, \pm 1, \dots, \pm 1) \quad \text{2n-fold degenerate}$$

V<0 $\quad \vec{\bar{m}}$ points along an n-dimensional cube axis

$$\vec{\bar{m}} = \pm \bar{m} e_i \text{ for some } i \quad \text{2n-fold degenerate}$$

we always can choose $\vec{\bar{m}} = \pm |\bar{m}| \vec{e}_1$

1b) $$\frac{\beta \mathcal{H}_{min}}{V} = \frac{t}{2} \bar{m}^2 + u \bar{m}^4 + \frac{V}{n} \bar{m}^2 \qquad V>0$$

$$\frac{\beta \mathcal{H}_{min}}{V} = \frac{t}{2} \bar{m}^2 + u \bar{m}^4 + V \bar{m}^2 \qquad V<0$$

It means that for stability we need

$$u + \frac{V}{n} > 0 \qquad \text{for } V>0$$

$$u + V > 0 \qquad \text{for } V<0$$

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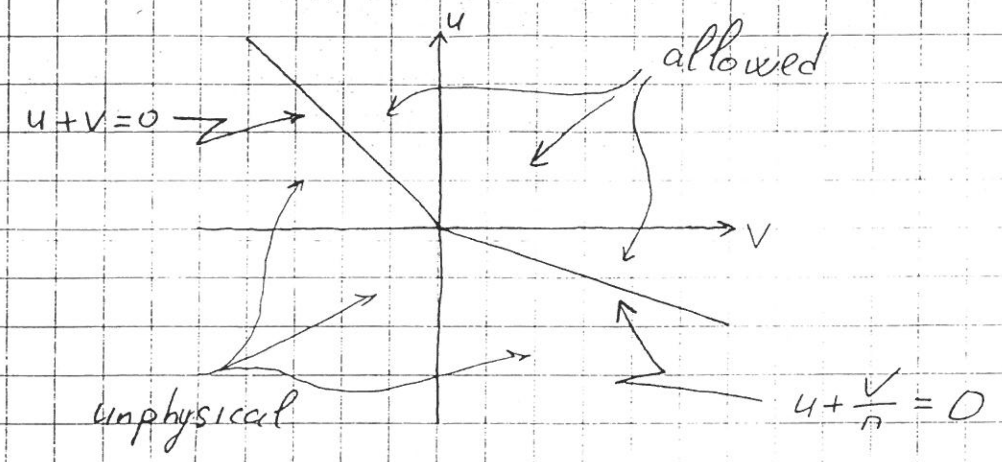
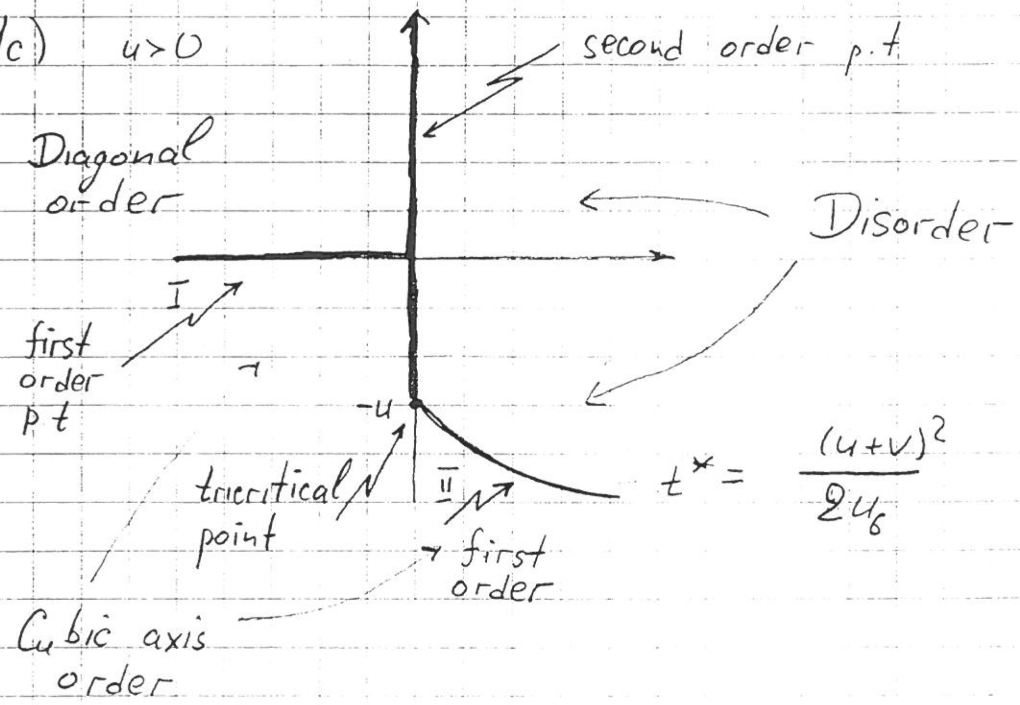


1c)



Line II for the first order phase transition:

$u+v<0$ $$-\frac{\beta\mathcal{H}}{V} = \frac{t}{2}\bar{m}^2 + (u+v)\bar{m}^4 + u_6\bar{m}^6$$

min $$t + 4(u+v)\bar{m}^2 + 6\bar{m}^4 u_6 = 0$$

nonzero absolute min $$t + 2(u+v)\bar{m}^2 + 2\bar{m}^4 u_6 = 0$$

$$\Rightarrow \bar{m}^2 = -\frac{(u+v)^2}{2u_6}$$

$$t^* = +\frac{(u+v)^2}{u_6} - \frac{(u+v)^2}{2u_6} = \frac{(u+v)^2}{2u_6}$$

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1d) There are no Goldstone modes in the ordered phase because here the symmetry is discrete rather than continous. The bare propagator for the transverce modes is

$$\langle \phi_t(q)\, \phi_t(-q) \rangle = \frac{1}{q^2 + \frac{tv}{K(u+v)}}$$

So the modes become massless only when $v=0$

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2a)
$$\beta\mathcal{H} = \int \frac{d^dq}{(2\pi)^d}\,\frac{1}{2}(t+Kq^2)|m(q)|^2 +$$
$$+\int \frac{d^dq_1\,d^dq_2\,d^dq_3}{(2\pi)^{3d}}\Big[u m_i(q_1) m_i(q_2) m_j(q_3) m_j(q_4) +$$
$$+ v\, m_i(q_1) m_i(q_2) m_i(q_3) m_i(q_4)\Big]$$

We assume summation over repeating indeses and put
$$q_4 = -q_1 - q_2 - q_3$$

$t' = b^{-d} z^2 \tilde{t}$ $\quad$ $K' = b^{-d-2} z^2 \tilde{K}$ $\quad$ in analogy with PS#6

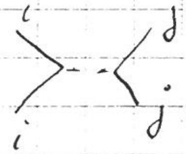
$u' = b^{-3d} z^4 \tilde{u}$ $\quad$ $v' = b^{-3d} z^4 \tilde{v}$

we choose $K' = K$ i.e $z^2 = b^{d+2}$

also $b = e^{\delta l}$ $\quad$ $\varepsilon = 4-d$

$t' = (1+2\delta l)\tilde{t}$ $\quad$ $u' = (1+\varepsilon\delta l)\tilde{u}$ $\quad$ $v' = (1+\varepsilon\delta l)\tilde{v}$

Both u^* and v^* are $O(\varepsilon)$

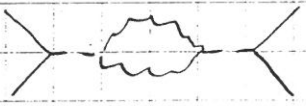
$u =$ 

$v =$ 

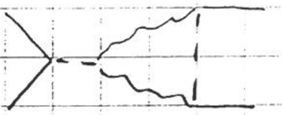
At $O(\varepsilon^2)$ in u and v we have one momentum shell integration which gives a factor
$$C\delta l = \frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l \qquad C \approx const$$

Now we have to calculate the difference between $\tilde{u}$ and u and $\tilde{v}$ and v

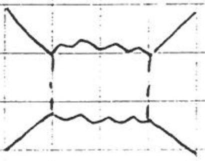
Contributions to $\tilde{u}$



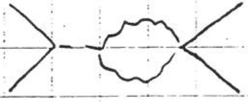
$$\frac{u^2}{2!}\,\frac{4\times4}{2}\,n = 4nu^2$$



$$\frac{u^2}{2!}\,\frac{4\times4\times2\times2}{2} = 16u^2$$



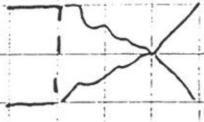
$$\frac{u^2}{2!}\,\frac{4\times4\times2\times2}{2!} = 16u^2$$



$$uv\,\frac{4\times4\times3}{2} = 24uv$$

Contributions to $\tilde{v}$

$$\frac{v^2}{2!}\,\frac{4\times4\times3\times3}{2} = 36v^2$$



$$uv\,\frac{4\times4\times3\times2}{2} = 48uv$$

$$\tilde{u} = u - 4c\delta\ell\left[(n+8)u^2 + 6uv\right] \qquad \tilde{v} = v - 4c\delta\ell(9v^2 + 12uv)$$

$$\frac{du}{d\ell} = \frac{u'-u}{\delta\ell} = \varepsilon u - 4C\left[(n+8)u^2+6uv\right] \qquad \frac{dv}{d\ell} = \varepsilon v - 4C(9v^2+12uv)$$

We can rescale these variables $u = \frac{u^+}{4c}$ $\quad v = \frac{v^+}{4c}$
As a result

$$\frac{du}{d\ell} = \varepsilon u - \left[(n+8)u^2+6uv\right] \qquad \frac{dv}{d\ell} = \varepsilon v - \left[12uv + 9v^2\right]$$

2b) Fixed points $\left\{\varepsilon - \left[(n+8)u^* + 6v^*\right]\right\}u^* = 0$

$$\left\{\varepsilon - (12u^* + 9v^*)\right\}v^* = 0$$

(1) Gaussian fixed point $u^* = v^* = 0$
(2) Ising fixed point $u^* = 0 \quad v^* = \varepsilon/9$
(3) Heizenberg fixed point $u^* = \varepsilon/(n+8) \quad v^* = 0$
(4) Cubic fixed point $u^* = \varepsilon/3n \quad v^* = (n-4)\varepsilon/9n$

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Near a fixed point $u = u^* + \Delta u$ $\quad v = v^* + \Delta v$ we can rewrite the recursion relations

$$\frac{du}{dl} = f_u(u,v) \qquad \frac{dv}{dl} = f_v(u,v)$$

as

$$\frac{du}{dl} = \frac{\partial f_u}{\partial u}\Big|_{u^*,v^*} \Delta u + \frac{\partial f_u}{\partial v}\Big|_{u^*v^*} \Delta v$$

$$\frac{dv}{dl} = \frac{\partial f_v}{\partial u}\Big|_{u^*,v^*} \Delta u + \frac{\partial f_v}{\partial v}\Big|_{u^*v^*} \Delta v$$

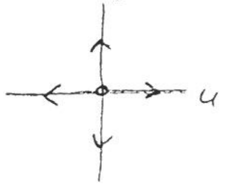
To determine the stability of a fixed point we have to diagonalize the matrix

$$A = \begin{pmatrix} \frac{\partial f_u}{\partial u} & \frac{\partial f_u}{\partial v} \\ \frac{\partial f_v}{\partial u} & \frac{\partial f_v}{\partial v} \end{pmatrix}\Bigg|_{\substack{u=u^* \\ v=v^*}} = \begin{pmatrix} \varepsilon - 2(n+8)u^* - 6v^* & -6u^* \\ -12v^* & \varepsilon - 12u^* - 18v^* \end{pmatrix}$$

each positive eigenvalue corresponds to an unstable direction

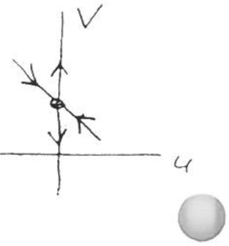
each negative "————————" a stable directio.

(1) $u^* = v^* = 0$ $\qquad A = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$ $\quad \lambda_1 = \lambda_2 = \varepsilon$



This is doubly unstable point for $\varepsilon > 0$

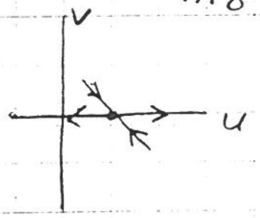
(2) $u^* = 0 \quad v^* = \varepsilon/9$ $\qquad A = \begin{bmatrix} \varepsilon/3 & 0 \\ -4\varepsilon/3 & -\varepsilon \end{bmatrix}$ $\quad \lambda_1 = \frac{\varepsilon}{3},\ \lambda_2 = -\varepsilon$



This point has one stable one unstable direction
For $u=0$ the system decouples into n noninteracting 1 component Ising models

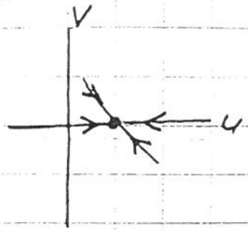
(3) $v^*=0 \quad u^*=\frac{\varepsilon}{n+8}$ $\quad A=\begin{bmatrix}-\varepsilon & -6\varepsilon/(n+8)\\ 0 & \frac{n-4}{n+8}\varepsilon\end{bmatrix}$

$\lambda_1=-\varepsilon \quad \lambda_2=\frac{n-4}{n+8}\varepsilon$



$n>4$

one stable and one unstable direction



$n<4$

both directions are stable and this fixed point determines critical behavour for $n<4$

(4) $u^*=\frac{\varepsilon}{3n} \qquad v^*=\frac{n-4}{9n}\varepsilon$

$$A=\begin{pmatrix}-(n+8)/3 & -2\\ -4(n-4)/3 & 4-n\end{pmatrix}\cdot\frac{\varepsilon}{n}$$

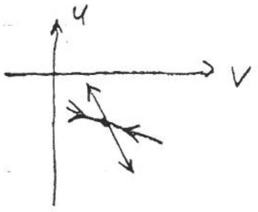
$\lambda=\frac{\varepsilon}{n}x \qquad x^2+\frac{4n-4}{3}x+\frac{(n+8)(n-4)}{3}-\frac{8(n-4)}{3}=0$

$x^2+4\,\frac{n-1}{3}x+\frac{n(n-4)}{3}=0 \qquad x_{1,2}=\frac{-2}{3}(n-1)\pm\sqrt{\frac{4(n-1)^2}{9}-\frac{n^2-4}{3}}$

$= -\frac{2}{3}(n-1)\pm\frac{n+2}{3}$

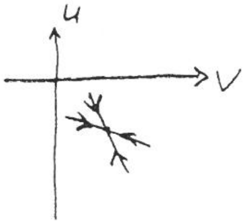
$x_1=-\frac{n}{3}+\frac{4}{3}=\frac{4-n}{3} \qquad x_2=-n$

$\lambda_1=\varepsilon\,\frac{4-n}{n} \qquad \lambda_2=-\varepsilon$



$n<4$

one stable one unstable directions



$n>4$

this is a stable fixed point and it determines the critical behavour for $n>4$

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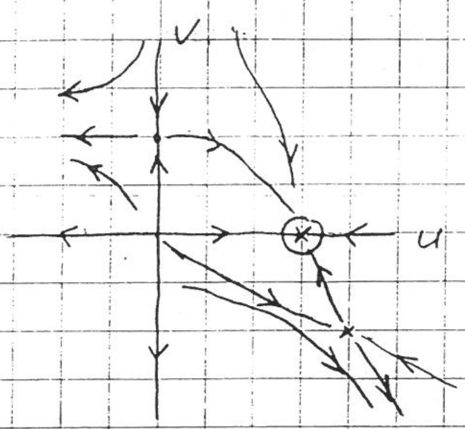
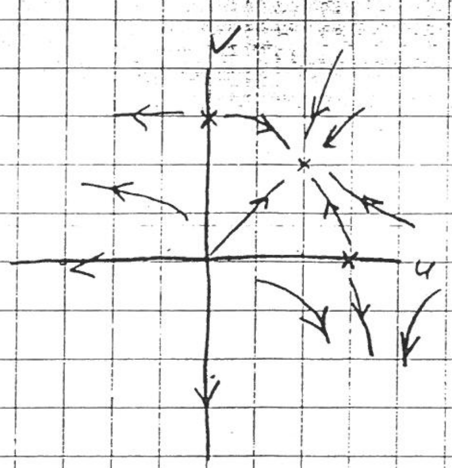
As a result



$n<4$

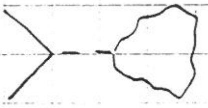


$n>4$

$n>4$: v^* is finite

$n<4$: $v^*=0$ so that the cubic term is irrelevant i.e. fluctuations restore full rotational symmetry

2c) In linear in ε order the following diagrams contribute to $\tilde{t}$:

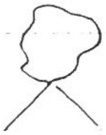


$2nC\delta\ell u$

$$C = \frac{K_d \Lambda^d}{t + K\Lambda^2}$$

$4C\delta\ell u$

$$\tilde{t} = t - 4C\delta\ell\left[(n+2)u + 3v\right]$$



$6C\delta\ell u$

$$t' = (1+2\delta\ell)\tilde{t}$$

$$\frac{dt}{d\ell} = 2t + 4\left[(n+2)u + 3v\right]C$$

$$y_t = \frac{d}{dt}\frac{dt}{d\ell}\Big)_* = 2 - 4C\left[(n+2)u^* + 3v^*\right]$$

$n<4$ $\quad y_t = 2 - \frac{(n+2)\varepsilon}{n+8} + O(\varepsilon^2)$ $\qquad$ $n>4$ $\quad y_t = 2 - \frac{2(n-1)\varepsilon}{3n} + O(\varepsilon^2)$

$\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{(n+2)}{4(n+8)}\varepsilon$ $\qquad$ $\nu = \frac{1}{2} + \frac{(n-1)\varepsilon}{6n}$

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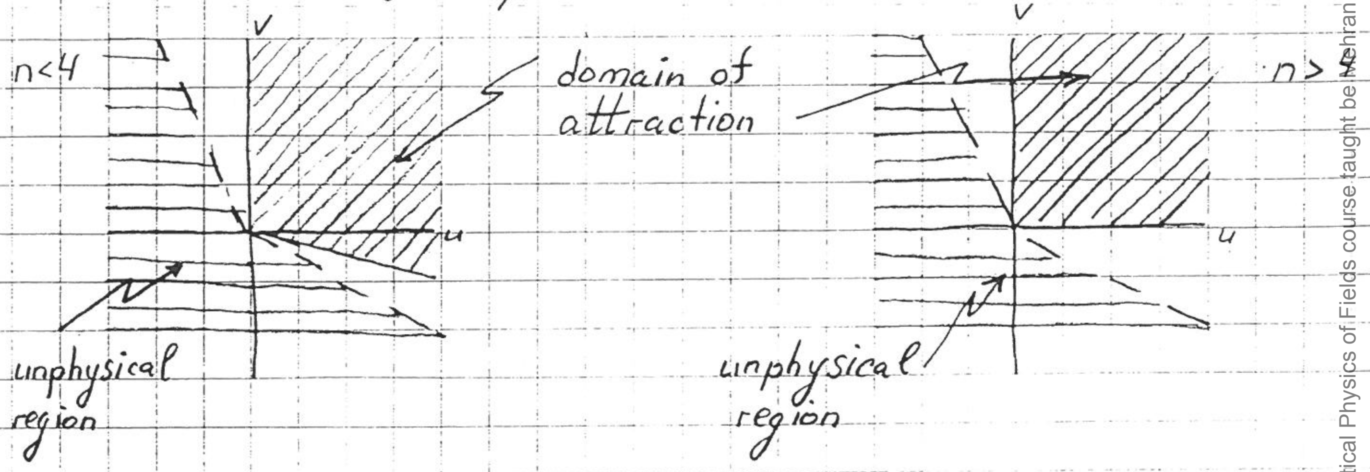
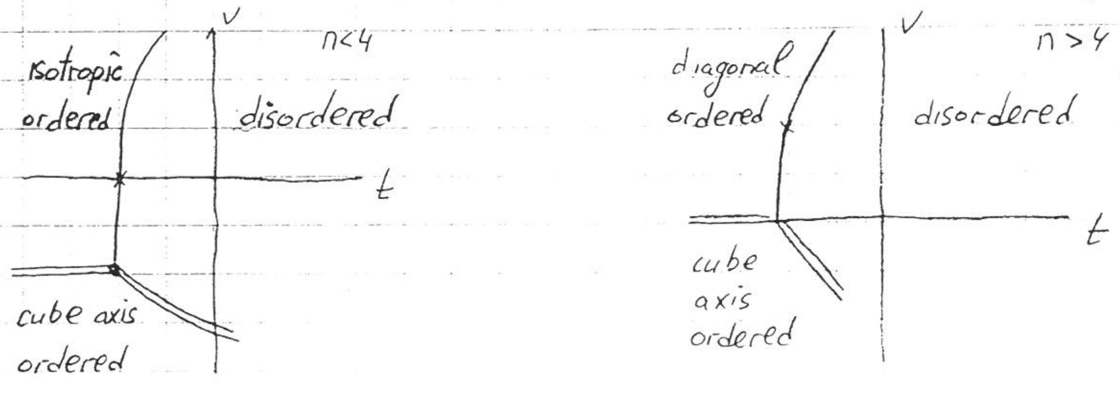
2d) All fixed points are within the allowed region calculated in 1b. Therefore the fluctuations decrease the region of stability. The domain of attraction of the stable fixe point is



Flows not originally within this domain of attraction will flow into the "unphysical" region - a signal of a fluctuation driven first order phase transition

2e) From the recursion relation for t 2c we have

$$t^* = -\frac{1}{2}\left[(n+2)u^* + 3v^*\right] \propto -\varepsilon \quad \text{(after changing variables)}$$



For $n>4$ there are no Goldstone modes

For $n<4$ there <u>are</u> Goldstone modes near the second order phase transition since v is renormalized to zero and $\xi_T^{-2} = tv/K(u+v) \to 0$ (see problem 1)

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Position-Space Renormalization–Group

1. *Cumulant Method:* Apply Niemeijer and van Leeuwen's first order cumulant expansion to the Ising model on a square lattice with $-\beta\mathcal{H} = K\sum_{<ij>}\sigma_i\sigma_j$, by following these steps:

(a) For an RG with $b = 2$, divide the bonds into intra–cell components ($\beta\mathcal{H}_0$); and inter–cell components (βU).

(b) For each cell α define a renormalized spin $\sigma'_\alpha = \text{sign}(\sigma_1^\alpha + \sigma_2^\alpha + \sigma_3^\alpha + \sigma_4^\alpha)$. This choice becomes ambiguous for configurations such that $\sum_i \sigma_i^\alpha = 0$. Distribute the weight of these configurations equally between $\sigma'_\alpha = +1$ and -1 (i.e. put a factor of 1/2 in addition to the Boltzmann weight). Make a table for all possible configurations of a cell, the internal probability $\exp(-\beta\mathcal{H}_0)$, and the weights contributing to σ'_α.

(c) Express $\langle\beta U\rangle_0$ in terms of the cell spins σ'_α; and hence obtain the recursion relation $K'(K)$.

(d) Find the fixed point K^* and the thermal eigenvalue y_t.

(e) In the presence of a small magnetic field $(\beta U)_2 = h\sum_i \sigma_i$ find the recursion relation for h; and calculate the magnetic eigenvalue y_h at the fixed point.

(f) Compare K^*, y_t and y_h to their exact values.

****

2. *Migdal–Kadanoff method:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, on sites i of a hyper-cubic lattice, interacting with their nearest-neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K\sum_{<ij>}\delta_{s_i,s_j} \ .$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Indentify all fixed points and note their stability.

(b) Write down the recursion relation $K'(K)$ in d–dimensions for $b = 2$, using the Migdal–Kadanoff bond moving scheme.

(c) By considering the stability of the fixed points at zero and infinity coupling, prove the existence of a non–trivial fixed point at finite K^* for $d > 1$.

(d) For $d = 2$ obtain K^* and y_t for $q = 3$.

****

Suggested Reading: Ma, Section VIII.4, and Parisi, Chapter 7.

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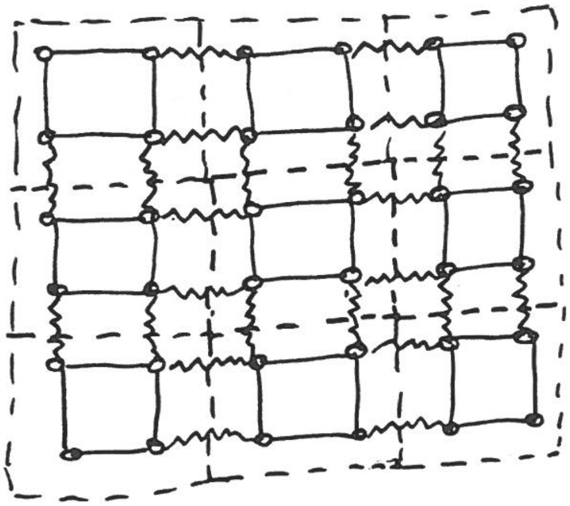
8.334 Problem Set 8, Solutions Apr 92

1) a)



$$-\beta H'(\sigma') \approx \ln Z_0' - \langle \beta U \rangle_0'$$

where averages are over states consistent with given set of σ'

b) i) Configurations with $\sigma' = +1$

$\begin{matrix}+&+\\+&+\end{matrix}$ Weight e^{4K} ; $\begin{matrix}-&+\\+&+\end{matrix}$ 4 times, Weight 1

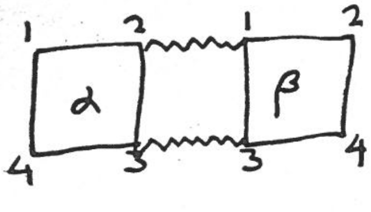
$\begin{matrix}+&+\\-&-\end{matrix}$ 4 times, Weight $\frac{1}{2}\times 1$; $\begin{matrix}+&-\\-&+\end{matrix}$ 2 times, Weight $\frac{1}{2}\times e^{-4K}$

ii) Configurations with $\sigma' = -1$

$\begin{matrix}-&-\\-&-\end{matrix}$ Weight e^{4K} ; $\begin{matrix}+&-\\-&-\end{matrix}$ 4 times, Weight 1

$\begin{matrix}+&+\\-&-\end{matrix}$ 4 times, Weight $\frac{1}{2}\times 1$; $\begin{matrix}+&-\\-&+\end{matrix}$ 2 times, Weight $\frac{1}{2}\times e^{-4K}$

c) Z_0' for one cell is $e^{4K} + 6 + e^{-4K}$ whether $\sigma' = \pm 1$ (sum weights) so does not contribute to relation for $K'(K)$



Contribution to $\langle -\beta U \rangle_0'$ from each pair of nearest neighbor cells, α, β is

$$K \langle \sigma_{\alpha 2}\sigma_{\beta 1} + \sigma_{\alpha 3}\sigma_{\beta 3} \rangle_0' = 2K \langle \sigma_{\alpha 1} \rangle_0' \langle \sigma_{\beta 1} \rangle_0'$$

since Z_0' distribution of weights does not correlate cells.

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If $\sigma_\alpha' = +1$, $\langle \sigma_{\alpha 1} \rangle_0' = \dfrac{e^{4K} + (3-1) + \frac{1}{2}(2-2) + \frac{1}{2}e^{-4K}(1-1)}{e^{4K} + 4 + \frac{1}{2}\cdot 4 + \frac{1}{2}\cdot e^{-4K}\cdot 2}$

Clearly if $\sigma_\alpha' = -1$, $\langle \sigma_{\alpha 1} \rangle_0'$ changes sign so $\langle \sigma_{\alpha 1} \rangle_0' = \sigma_\alpha' \dfrac{e^{4K}+2}{e^{4K}+6+e^{-4K}}$

So $-\beta H' = \sum_{\langle \alpha\beta \rangle} \sigma_\alpha' \sigma_\beta' K'$, $\quad K' = 2K\left(\dfrac{e^{4K}+2}{e^{4K}+6+e^{-4K}}\right)^2$

d) For $K' = K = K^*$, let $e^{4K^*} = x$, $\dfrac{x+2}{x+6+1/x} = \dfrac{1}{\sqrt{2}}$

$x^2(\sqrt{2}-1) - x(6-2\sqrt{2}) - 1 = 0 \Rightarrow K^* = 0.52$

Near $K = K^*$, $\delta K' = \left(\dfrac{\partial K'}{\partial K}\right)^* \delta K = 2^{y_t}\delta K$ $(b=2)$

$$2^{y_t} = 2\times\frac{1}{2} + 2K\times\frac{1}{2}\times 2\times\left[\frac{4e^{4K}}{e^{4K}+2} - \frac{4e^{4K}-4e^{-4K}}{e^{4K}+6+e^{-4K}}\right]$$

$$= 1 + 8K\left(\frac{x}{2+x} - \frac{x-1/x}{x+6+1/x}\right) \Rightarrow y_t = 1.006$$

e) Since $h\sum_{\alpha,i}\langle \sigma_{\alpha i} \rangle_0' = 4h\sum_\alpha \sigma_\alpha' \dfrac{e^{4K}+2}{e^{4K}+6+e^{-4K}}$

so $h' = 4h\dfrac{e^{4K}+2}{e^{4K}+6+e^{-4K}}$ and $2^{y_h} = 4\cdot\dfrac{1}{\sqrt{2}} = 2^{3/2}$,

$y_h = 1.5$

F) Exact values are $K^* = 0.44$, $y_t = 1$, $y_h = 1.875$

y_t looks good but this is obviously an accident.

2d) $d=2$, $q=3$; $x = e^{K^*}$, $x = \dfrac{2+x^4}{1+2x^2}$, $x=2$, $K^* = 0.69$

$2^{y_t} = \left(\dfrac{\partial K'}{\partial K}\right)^* = \dfrac{\partial x'}{\partial x} = \dfrac{16}{9}$, $y_t = 0.83$ $\quad$ [Exact values $K^* = 1.005$, $y_t = 1.2$]

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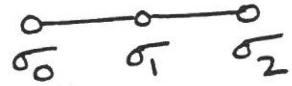
2 a) 

$$\sum_{\sigma_1=1}^{q} e^{K(\delta_{\sigma_0\sigma_1}+\delta_{\sigma_1\sigma_2})}$$

$= (q-1)+e^{2K}$ if $\sigma_0=\sigma_2$; $(q-2)+2e^{K}$ if $\sigma_0\neq\sigma_2$

$= e^{A+K'\delta_{\sigma_0\sigma_2}}$ with $e^{K'} = \dfrac{q-1+e^{2K}}{q-2+2e^{K}}$

Fixed points $e^{K'}=e^{K}=x$, $x=\dfrac{q-1+x^2}{q-2+2x}$; $x=1, 1-q$

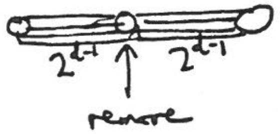
so only $K^*=0$ is finite fixed point.

For $K\to\infty$, $e^{K'}\sim\frac{1}{2}e^{K}$, $K'=K-\ln 2$ so $K^*=\infty$ is an unstable fixed point.

For $K\to 0$, $e^{K'}\approx\dfrac{1+\frac{1}{q}(e^{2K}-1)}{1+\frac{2}{q}(e^{K}-1)}\sim\dfrac{1+\frac{2K}{q}+\frac{2K^2}{q}}{1+\frac{2K}{q}+\frac{K^2}{q}}\sim 1+\dfrac{K^2}{q}$

so $K'\sim\dfrac{K^2}{q}$ and $K^*=0$ is stable

b) Moving bonds strengthens remaining bonds by factor 2^{d-1}. Follow by a one-dimensional RG transformation



[We have $2\times 2^{d-1}=2^{d}$ bonds on each edge of new lattice, which gives same total number as on old lattice]

Thus $e^{K'} = \dfrac{q-1+e^{2^{d}K}}{q-2+2\exp(2^{d-1}K)}$

c) $K=0\to K'=0$. Near $K^*=0$, from (a), $K'\sim\frac{1}{q}2^{2d-2}K^2$ so $K^*=0$ is stable.

As $K\to\infty$, $e^{K'}\sim\frac{1}{2}e^{(2^{d}-\frac{1}{2}2^{d-1})K}$

so $K'=2^{d-1}K-\ln 2$ and $K^*=\infty$ is stable for $d>1$

There must be a finite K^* with flow away from it ie a critical point

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Review Problems

The midterm quiz will take place on Friday April 3, in room 13–3101 from 3:00 to 5:00pm. The material covered includes all topics discussed up to section IV of the syllabus. It will be a closed book exam, but you may bring a two–sided sheet of formulas if you wish. The following problems, taken form previous exams, are provided to help you review the material. *This is not a problem set* and you are not expected to turn in any solutions.

1. *Superfluid with impurities:* The order parameter for superfluid He^4 is a complex number $\Psi(\mathbf{x})$. He^3 impurities interact with this order parameter, so that in the presence of a concentration fluctuation, $c(\mathbf{x})$, of He^3 impurities, the system has the following Landau–Ginzburg energy

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}|\nabla\Psi|^2 + \frac{t}{2}|\Psi|^2 + u|\Psi|^4 + v|\Psi|^6 + \frac{c(\mathbf{x})^2}{2\sigma^2} - \gamma c(\mathbf{x})|\Psi|^2\right] ,$$

with u, v, and γ positive.

(a) The He^3 concentration fluctuations can be integrated out of the partition function, i.e.

$$Z = \int \mathcal{D}\Psi(\mathbf{x})\mathcal{D}c(\mathbf{x})e^{-\beta\mathcal{H}[\Psi,c]} = \int \mathcal{D}\Psi(\mathbf{x})e^{-\beta\mathcal{H}_{\text{eff}}[\Psi]}.$$

Assuming $-\infty < c(x) < \infty$, perform the integrations, and obtain $\beta\mathcal{H}_{\text{eff}}\,[\Psi]$.

(b) By examining $\beta\mathcal{H}_{\text{eff}}\,[\Psi]$ in the saddle point approximation show that for concentrations $\sigma > \sigma^*$ the transition becomes discontinuous, and identify σ^*.

(c) Write down the conditions that determine the position $(\bar{t}, \bar{\sigma})$ of the discontinuous transition, and the magnitude, $\overline{\Psi}$, of the discontinuity.

(d) The discontinuous transition is accompanied by a jump in $\langle\Psi\rangle$. How does this jump vanish as $\sigma \to \sigma^*$?

(e) Sketch the phase boundary in the (t, σ) plane and indicate how its two segments join at σ^*.

(f) Show that $\langle c(\mathbf{x}) - \gamma\sigma^2|\Psi(\mathbf{x})|^2\rangle = 0$.

(j) Show that $\langle c(\mathbf{x})c(\mathbf{y})\rangle = \gamma^2\sigma^4\langle|\Psi(\mathbf{x})|^2|\Psi(\mathbf{y})|^2\rangle$ for $x \neq y$.

(h) Qualitatively discuss how $\langle c(\mathbf{x})c(0)\rangle$ decays with x in the disordered phase.

(i) Qualitatively discuss how $\langle c(\mathbf{x})c(0)\rangle$ decays to its asymptotic value in the ordered phase.

****

2. *Renormalization:* Starting with

$$\beta\mathcal{H}=\int d^d\mathbf{x}\left[\frac{t}{2}m^2+\frac{K}{2}(\nabla m)^2-hm+\frac{L}{2}(\nabla^2\phi)^2+v\nabla m.\nabla\phi\right],$$

coupling two one component fields m and ϕ:

(a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\mathbf{q})$ and $\phi(\mathbf{q})$.

(b) Construct a renormalization group transformation as in class, by rescaling distances such that $\mathbf{q}'=b\mathbf{q}$; and the fields such that $m'(\mathbf{q}')=\tilde{m}(\mathbf{q})/z$ and $\phi'(\mathbf{q}')=\tilde{\phi}(\mathbf{q})/y$. Do not evaluate the integrals that just contribute a constant additive term.

(c) There is a fixed point such that $K'=K$ and $L'=L$. Find y_t, y_h and y_v at this fixed point.

(d) The singular part of the free energy has a scaling from $f(t,h,v)=t^{2-\alpha}g(h/t^{\Delta},v/t^{\omega})$ for t,h,v close to zero. Find α,Δ, and ω.

(e) There is another fixed point such that $t'=t$ and $L'=L$. What are the relevant operators at this fixed point, and how do they scale?

For the remainder of this problem set $h=0$, and treat $V=\int d^d\mathbf{x}v\nabla m.\nabla\phi$ as a small perturbation.

(f) Starting with the Hamiltonian in (a) calculate the bare expectation values $\langle m(\mathbf{q})m(\mathbf{q}')\rangle_0$ and $\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi^0_m(q)$, and $\chi^0_\phi(q)$.

(g) Calculate $\langle\phi(\mathbf{q})\phi(\mathbf{q}')\phi(\mathbf{k})\phi(\mathbf{k}')\rangle_0$.

(h) Write down the expansion for $\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle$ to second–order in the perturbation V, taking advantage of the parity of this term to simplify results.

(i) Using the Fourier–transformed form of V, write the correction in terms of expectation values calculated in parts (f) and (g).

(j) Write down the expression for $\chi_\phi(q)$ to second order in perturbation theory.

****

3. *Ginzburg criterion along the magnetic field direction:* Consider the Hamiltonian

$$\beta\mathcal{H}=\int d^d\mathbf{x}[\frac{K}{2}(\nabla\vec{m})^2+\frac{t}{2}\vec{m}^2+u(\vec{m}^2)^2]\quad,$$

describing an n–component magnetization vector $\vec{m}(\mathbf{x})$, with $u>0$.

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(a) In the saddle point approximation, the free energy is $f = \min\{\Psi(m)\}_m$. Indicate the resulting phase boundary in the (h, t) plane, and label the phases. (h denotes the magnitude of $\vec{h}$.)

(b) Sketch the form of $\Psi(m)$ for $t < 0$ on both sides of the phase boundary, and for $t > 0$ at $h = 0$.

(c) For t and h close to zero, the spontaneous magnetization can be written as $\overline{m} = t^{\beta} g_m(h/t^{\Delta})$. Identify the exponents β and Δ in the saddle point approximation.

For the remainder of this problem set $t = 0$.

(d) Calculate the transverse and longitudinal susceptibilities at a finite h.

(e) Include fluctuations by setting $\vec{m}(\mathbf{x}) = \left(\overline{m} + \phi_\ell(\mathbf{x})\right)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expanding $\beta\mathcal{H}$ to second order in the ϕs. ($\hat{e}_\ell$ is a unit vector parallel to the average magnetization, and $\hat{e}_t$ is perpendicular to it.)

(f) Calculate the longitudinal and transverse correlation lengths.

(g) Calculate the first correction to the free energy from these fluctuations. (The scaling form is sufficient for parts (g) and (h).)

(h) Calculate the first correction to magnetization, and to longitudinal susceptibility from the fluctuations.

(i) By comparing the saddle point value in (d), with the correction calculated in (h), find the upper critical dimension, d_u, for the validity of the saddle point result.

(j) For $d < d_u$ obtain a Ginzburg criterion by finding the field h_G below which fluctuations are important. (You may ignore the numerical coefficients in h_G, but the dependences on K and u are required.)

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Series Expansions and Duality

1. *Triangular Lattice:* Perform a high temperature expansion for the Ising model ($-\beta\mathcal{H} = K\sum_{<ij>}\sigma_i\sigma_j$) on a triangular lattice in powers of $t = \tanh(K)$. Calculate terms up to and including t^5.

****

2. *Potts Model:* Consider Potts spins $s_i = (1, 2, \cdots, q)$; interacting via the Hamiltonian $-\beta\mathcal{H} = K\sum_{<ij>}\delta_{s_i,s_j}$.

(a) To treat this problem graphically at high temperatures, the Boltzmann weight for each bond is written as

$$\exp\left(K\delta_{s_i,s_j}\right) = C(K)\left[1 + T(K)g(s_i,s_j)\right],$$

with $g(s,s') = q\delta_{s,s'} - 1$. Find $C(K)$ and $T(K)$.

(b) Show that

$$\sum_{s=1}^{q} g(s,s') = 0 \ , \ \sum_{s=1}^{q} g(s_1,s)g(s,s_2) = qg(s_1,s_2) \ , \ \text{and} \ \sum_{s,s'}^{q} g(s,s')g(s',s) = q^2(q-1).$$

(c) Use the above results to calculate the free energy, and the correlation function $\langle g(s_m,s_n)\rangle$ for a one–dimensional chain.

(d) Using the above results, calculate the partition function on the square lattice to order of T^4. Also calculate the first term in the low–temperature expansion of this problem.

(e) By comparing the first terms in low–temperature and high–temperature series, find a duality rule for Potts models. Don't worry about higher order graphs, they will work out! Assuming a single transition temperature, find the value of $K_c(q)$.

(f) How do the higher order terms in the high–temperature series for the Potts model differ from that of the Ising model? What is the fundamental difference that sets apart the graphs for $q = 2$? (This is ultimately the reason why only the Ising model is solvable.)

****

3. *Ising model in a field:* Consider the partition function for the Ising model ($\sigma_i = \pm 1$) on a square lattice, *in a magnetic field h*; i.e.

$$Z = \sum_{\{\sigma_i\}}\exp\left[K\sum_{<ij>}\sigma_i\sigma_j + h\sum_i\sigma_i\right].$$

(a) Find the general behavior of the terms in a low–temperature expansion for Z.

(b) Think of a model whose high–temperature series reproduces the generic behavior found in (a); and hence obtain the Hamiltonian, and interactions of the dual model.

****

Optional Problem

4. *Energy:* Consider the Ising model ($\sigma_i = \pm 1$) on a square lattice with $-\beta\mathcal{H} = K\sum_{<ij>}\sigma_i\sigma_j$.

(a) Starting from the duality expression for the free energy, derive a similar relation for the internal energy $U(K) = \langle H\rangle = -\partial \ln Z/\partial \ln K$.

(b) Using (a), calculate the exact value of U at the critical point K_c.

****

Suggested Reading: Feynman, Chapter 5; Stanley, Chapter 9; Parisi, Chapter 4.

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8.334 Solutions to Problem Set 9 April 92

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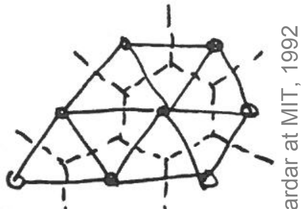
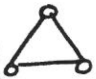
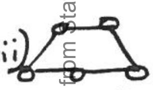


1) Triangular lattice with N sites
has 3N bonds (6 at each site, 2 sites on each bond)
and 2N faces (plaquettes) (6 at each site, 3 sites on each face)
Dual lattice is hexagonal.

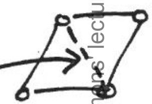
$$Z = \sum_{\{\sigma_i\}} \prod e^{K\sigma_i\sigma_j} = (\cosh K)^{3N} \sum_{\{\sigma_i\}} \prod_{\text{bonds}} (1 + \sigma_i\sigma_j \tanh K)$$

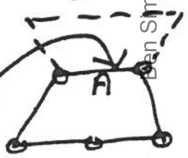
$$= (\cosh K)^{3N} 2^N \sum (\tanh K)^n \times \left[\begin{array}{l}\#\text{ of ways of choosing } n \text{ bonds}\\ \text{with } 0, 2, 4 \text{ or } 6 \text{ bonds at each site}\end{array}\right]$$

since $\operatorname{tr} \sigma_i^r = 0$, r odd ; $= 2$, r even.

Only configurations with $\leqslant 5$ bonds are i)  , ii)  and iii) 
(together with rotations and reflexions)

(i) can be placed in 2N positions.

(ii) There is one position for each placing of diagonal bond 
so there are 3N positions.

iii) There are 2 positions for each placing of side A 
so there are 6N positions.

$$Z = (\cosh K)^{3N} 2^N \left[1 + 2N(\tanh K)^3 + 3N(\tanh K)^4 + 6N(\tanh K)^5 + \cdots\right]$$

2) a) $e^{K\delta_{ss'}} = C(K)\left[1 + T(K)\{q\delta_{ss'} - 1\}\right]$

$$\left.\begin{array}{lrl} s = s' & e^K &= C(K)\left[1 + T(K)(q-1)\right] \\ s \neq s' & 1 &= C(K)\left[1 - T(K)\right] \end{array}\right\} \Rightarrow T = \frac{e^K - 1}{e^K + q - 1} \qquad C = \frac{1}{q}(e^K + q - 1)$$

b) $\sum_{s'} g(s, s') = q - q = 0$

$$\sum_s g(s_1, s) g(s, s_2) = q^2 \sum \delta_{ss_1}\delta_{ss_2} - q\sum\delta_{ss_1} - q\sum\delta_{ss_2} + q$$

$$= q^2 \delta_{s_1 s_2} - q = q\, g(s_1, s_2)$$

$$\sum_{s s'} g(s\, s') g(s' s) = q \sum_s g(s, s) = q^2(q-1)$$

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c) High temperature series is in powers of $T(K)$ since as $K \to 0$, $T \sim \frac{K}{q}$.

Choose set of bonds, each gives factor $T g(s_i, s_j)$.

Since $\sum_{s'} g(s, s') = 0$, must have at least 2 bonds per site (or of course none) otherwise $\sum_{\{s_i\}}$ gives zero.

In one dimension, this means no bonds or all bonds and so

$$Z = [C(K)]^N \sum_{\{s_i\}} 1 = [q C(K)]^N = (e^K + q - 1)^N$$

$$-\frac{\beta F}{N} = \ln(e^K + q - 1)$$

$$\langle g(s_m, s_n) \rangle = \frac{[C(K)]^N}{Z} \sum_{\{s_i\}} g(s_m\, s_n) \prod_i \{1 + T g(s_i, s_{i+1})\}$$

Now only term to give finite power of T comes from (for $n > m$)

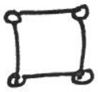
$$\sum_{\{s_i\}} g(s_m\, s_n)\, g(s_m, s_{m+1})\, g(s_{m+1}, s_{m+2}) \cdots g(s_{n-1}, s_n)$$

If we write $\langle s' | G | s \rangle = g(s', s)$, this is $\{\mathrm{tr}\, G^{n-m+1}\} q^{N-(n-m+1)}$

But (b) tells us that $G^2 = qG$ and $\mathrm{tr}\, G = q(q-1)$

so $\mathrm{tr}\, G^{n-m+1} = q^{n-m}\, \mathrm{tr}\, G = q^{n-m+1}(q-1)$

and $\langle g(s_m, s_n) \rangle = \{T(K)\}^{n-m}(q-1)$

d) First term in high temp series for square lattice comes from 

so $Z = [C(K)]^{2N} q^N \left[1 + N T^4 (q-1) + \cdots\right]$

(any closed loop with no intersections give $T^l\, \mathrm{tr}\, G^l = T^l q^l (q-1)$)

At low temp., favored configuration is all spins in same state.

Lowest excitation is obtained by changing one spin to a different state

$$\text{so } Z = e^{2NK}\left[1 + N(q-1)\underset{\substack{\uparrow \\ \text{4 bonds with } s_i \neq s_j}}{e^{-4K}} + \cdots\right]$$

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e) Mapping $e^{-\tilde{K}} = T(K) = \frac{e^{K}-1}{e^{K}+q-1}$ ie $(e^{\tilde{K}}-1)(e^{K}-1) = q$

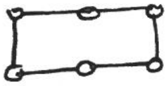
is a possible self-duality map $K \leftrightarrow \tilde{K}$ which sends low temp series into high temp series, at least as far as first term!

A single K_c must have $K_c = \tilde{K}_c$, $K_c = \ln(1+\sqrt{q})$

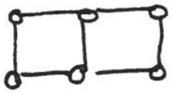
f) Potts model with $q=2$ has bond factors $e^{K}, 1$ so corresponds to Ising model with $2K_{ISING} = K_{POTTS}$.

For $q>2$, in high temp series can have 3 or more bonds per site and in low temp series must count configurations of spins <u>inside</u> islands with $S_i \neq 1$

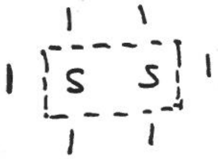
In fact next terms in high temp series come from



and then



[In low temp series we have



and

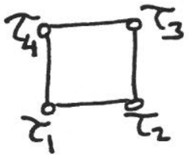
, $S \neq 1$, $S \neq S'$]

so that configurations do not correspond to (even restricted) random walks.

3) a) Low temp series. Almost all spins +1. Islands of spin −1.

Factor $e^{-2KL} \times e^{-2hA}$ where L is length of closed path on dual lattice separating spins −1 and A is area inside closed path

b) To reproduce terms e^{-2hA} in a high temp series we need an interaction for each plaquette. So consider the Z_2 gauge theory in 2 dimensions ie $-\beta H = \tilde{h}\sum_{\text{plaquettes}} \tau_1\tau_2\tau_3\tau_4$



where the τ_a are spin variables on each <u>bond</u>

High T series gives $\sum (\tanh \tilde{h})^{n} \times$ configurations of n plaquettes with each bond on an even # of plaquettes

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This forces us to choose no plaquettes or all plaquettes !

But now include a ~~fact~~ term $\tilde{K} \sum_{\text{bonds}} \tau_a$.

Now we have a factor $\tanh \tilde{K}$ for each <u>selected</u> bond, which must be edge of one selected plaquette. We get islands of plaquettes with a factor $(\tanh \tilde{K})^L (\tanh \tilde{h})^A$ and duality relation is

$$K \sum_{\text{bonds}} \sigma_i \sigma_j + h \sum_{\text{sites}} \sigma_i \longleftrightarrow \tilde{K} \sum_{\text{bonds}} \tau_a + \tilde{h} \sum_{\text{plaquettes}} \tau_a \tau_b \tau_c \tau_d$$

with $e^{-2K} = \tanh \tilde{K}$, $e^{-2h} = \tanh \tilde{h}$.

4) $$Z(K) = e^{2NK} \sum e^{-2KL} \quad \text{(low temp series)}$$

$$= (\cosh K)^{2N} 2^N \sum (\tanh K)^L \quad \text{(high temp series)}$$

Let $\tanh K = e^{-2\tilde{K}}$. Then $Z(K) = (2\cosh^2 K)^N e^{-2N\tilde{K}} Z(\tilde{K})$

$$= (\sinh 2K)^N Z(\tilde{K})$$

where in fact $\sinh 2K \sinh 2\tilde{K} = 1$

a) $$U(K) = -K \frac{\partial}{\partial K} \ln Z(K) = -NK\, 2\frac{\cosh 2K}{\sinh 2K} - K\left(\frac{\partial}{\partial \tilde{K}} \ln Z(\tilde{K})\right) \frac{\partial \tilde{K}}{\partial K}$$

and $$2\frac{\cosh 2K}{\sinh 2K}\, dK + 2\frac{\cosh 2\tilde{K}}{\sinh 2\tilde{K}}\, d\tilde{K} = 0$$

so $$U(K) = -2NK \coth 2K - \frac{K}{\tilde{K}} \tanh 2\tilde{K} \coth 2K\, U(\tilde{K})$$

ie $$U(K) \frac{\tanh 2K}{K} + U(\tilde{K}) \frac{\tanh 2\tilde{K}}{\tilde{K}} = -2N$$

b) At $K = \tilde{K} = K^*$, $\sinh 2K^* = 1$, $\cosh 2K^* = \sqrt{2}$, $e^{2K^*} = \sqrt{2}+1$, $\tanh 2K^* = \frac{1}{\sqrt{2}}$

$$U(K^*) = -\frac{NK^*}{\tanh 2K^*} = -\frac{N}{\sqrt{2}} \ln(\sqrt{2}+1)$$

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8.334: Statistical Mechanics II Problem Set # **10** Due: **4/17/92**

Transfer Matrix Method

A general method for exactly solving one–dimensional (and some higher dimensional) problems with short–range interactions, is to use transfer matrices. This problem set is intended to teach you this technique.

1. *The one–dimensional Ising Model:* Consider a linear closed chain of N Ising spins ($\sigma_i = \pm 1$), with a nearest–neighbor coupling K. The Hamiltonian is

$$-\beta\mathcal{H} = K\big(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \cdots + \sigma_{N-1}\sigma_N + \sigma_N\sigma_1\big) + h\sum_{i=1}^{N}\sigma_i \;.$$

(a) Show that the partition function $Z = \exp(-\beta\mathcal{H})$ can be written as

$$Z = \mathrm{Tr}\left[\langle\sigma_1|T|\sigma_2\rangle\langle\sigma_2|T|\sigma_3\rangle\cdots\langle\sigma_N|T|\sigma_1\rangle\right] \;;$$

where $T \equiv \langle\sigma_i|T|\sigma_j\rangle = \exp\left[K\sigma_i\sigma_j + h(\sigma_i+\sigma_j)/2\right]$ is a 2×2 matrix. Write down the matrix elements explicitly.

(b) Show that $Z = \lambda_+^N + \lambda_-^N \approx \lambda_+^N$ for $N\to\infty$; where $\lambda_+ > \lambda_-$ are eigenvalues of the transfer matrix T. Write down the expression for the free energy per site $\beta f = -\ln Z/N$.

(c) Using the result from (b), express the singular behavior of the free energy as $(K^{-1}, h)\to 0$ in scaling form; and compare to the renormalization–group result obtained in class.

(d) Sketch for $h = 0$, C_h and χ as a function of 'temperature', $T = K^{-1}$, clearly labelling the behaviors as $T\to 0$.

(e) Show that the correlation function $\langle\sigma_i\sigma_{i+\ell}\rangle$ can be written as $\mathrm{Tr}\left[\sigma_z T^\ell \sigma_z T^{N-\ell}\right]/Z$, where σ_z is the Pauli matrix. Hence show that $\langle\sigma_i\sigma_{i+\ell}\rangle \sim (\lambda_-/\lambda_+)^\ell$. (You do not have to explicitly calculate the constant of proportionality.) Calculate the correlation length ξ.

(f) Sketch for $h = 0$, ξ as a function of temperature T. How are the divergences in ξ and χ related? Find the exponent η for this zero temperature phase transition.

****

This procedure can be extended to more general spins and interactions, with a more complex matrix T. Quite generally the largest eigenvalue of the transfer matrix is related to the free energy, while the correlation lengths are obtained from ratios of eigenvalues.

2. *The Potts Model:* In this case the spin s_i on each site takes q values $s_i = (1, 2, \cdots, q)$; and the Hamiltonian is $-\beta\mathcal{H} = K\sum_{i=1}^{N}\delta_{s_i,s_{i+1}} + K\delta_{s_N,s_1}$.

(a) Write down the transfer matrix and diagonalize it. Note that you do not have to solve a q^{th} order secular equation as it is easy to guess the eigenvectors from the symmetry of the matrix.

(b) Calculate the free energy per site; and the correlation length ξ. Compare with results obtained in problem set 9.

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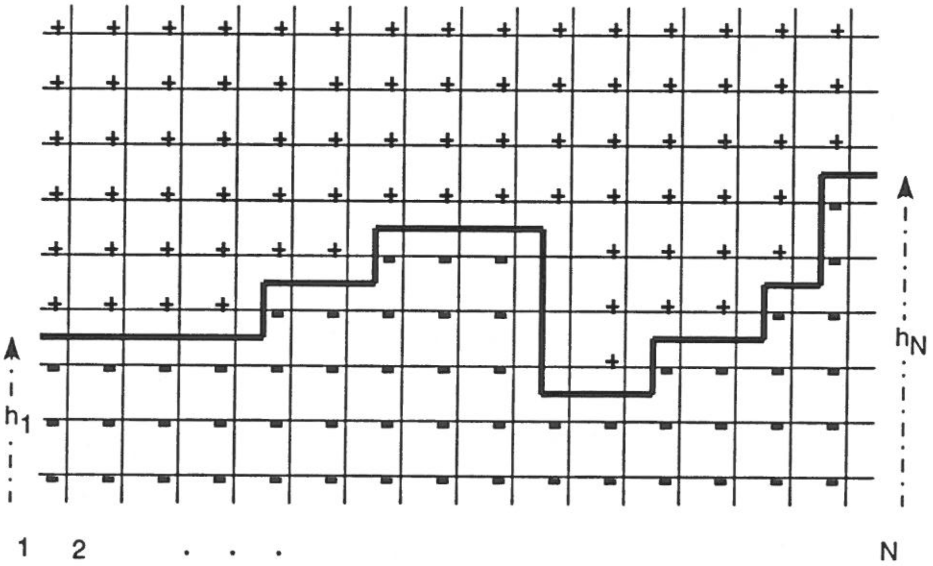
This method is clearly related to the way we calculate sums over random walks in class.

3. *Müller–Hartmann Zittartz estimate* of the interfacial energy of the $d = 2$ Ising model on a square lattice.



(a) Consider an interface on the square lattice with periodic boundary conditions in one direction. Ignoring islands and overhangs, the configurations can be labelled by heights h_n for $1 \leq n \leq L$. Show that for an Ising model of interaction K, the interface Hamiltonian is

$$-\beta\mathcal{H} = -2KL - 2K\sum_{n} |h_{n+1} - h_n| \, .$$

(b) Write down a column–to–column transfer matrix $\langle h|T|h'\rangle$, and diagonalize it.

(c) Obtain the interface free energy using the result in (b), or by any other method.

(d) Find the value of K for which the interfacial free energy vanishes. Does this correspond to the critical point of the original 2d Ising model?

****

Suggested Reading: Huang, Chapter 14; Parisi, Chapter 12; Feynman, Chapter 4.

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8.334 Solutions to Problem Set 10 Apr 92

1 a) $Z = \mathrm{Tr}\, T^N = \sum_{\{\sigma_i\}} \prod_{i=1}^{N} \langle \sigma_i | T | \sigma_{i+1} \rangle$ $[\sigma_{N+1} \equiv \sigma_1]$ $= \sum_{\{\sigma_i\}} e^{-\beta H(\sigma)}$

$$\langle \sigma | T | \sigma' \rangle = \begin{bmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{bmatrix}$$

b) Since Tr is independent of basis and T is symmetric we can choose eigenvectors of T as basis, write $T = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$ and find

$$Z = \lambda_+^N + \lambda_-^N = \lambda_+^N \left(1 + \left\{ \frac{\lambda_-}{\lambda_+} \right\}^N \right)$$

so $\beta f = -\frac{\ln Z}{N} = -\ln \lambda_+ + O(e^{-N/\xi})$ where $\xi^{-1} = \ln \frac{\lambda_+}{\lambda_-} > 0$

c) $\lambda_\pm$ are solutions of $\lambda^2 - \lambda e^K (e^h + e^{-h}) + (e^{2K} - e^{-2K}) = 0$

$\lambda_\pm = e^K \cosh h \pm \sqrt{e^{-2K} + e^{2K} \sinh^2 h}$. We want scaling behaviour as $h \to 0$, $e^{-K} \to 0$. Clearly as $K \to \infty$, a small non-zero value of h will order the spins and give $\beta f \to -K$.

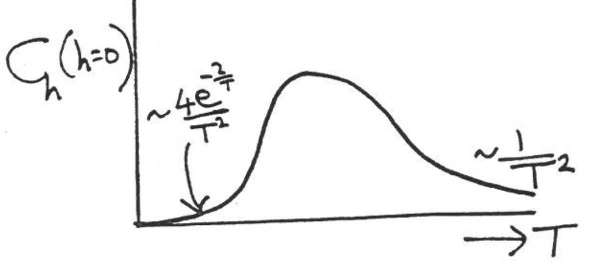
To order h^2 at fixed K, $\lambda_+ = e^K \left[1 + \frac{h^2}{2} + \sqrt{h^2 + e^{-4K}} \right]$

so if $h \sim e^{-2K}$, we can neglect the $\frac{h^2}{2}$ term and get

$\beta f = -K - e^{-2K} \sqrt{1 + (h e^{2K})^2}$. The RG (obtained by comparing T^2 to T) gave for the singular behaviour $\beta f = e^{-2K} g(h e^{2K})$.

d) At $h = 0$, $\beta f = -\ln 2 \cosh K$, $\frac{\partial}{\partial \beta}(\beta f) = -\tanh K$

(with $K = \beta = T^{-1}$) and $C_h = -\frac{\partial}{\partial T}(\tanh K) = \frac{1}{T^2 \cosh^2 1/T}$



[I am using units of energy and temperature so that energy of 2 parallel spins is 1 and of magnetic field so that energy of spin in field H is H. The h in formulae is then β× field]

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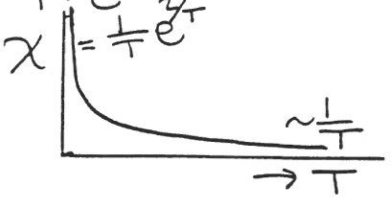
$$m = -\frac{\partial}{\partial h}(\beta f), \quad \chi = \beta \frac{\partial m}{\partial h} = -\beta \frac{\partial^2}{\partial h^2}(\beta f)$$

$$-\beta f = \ln \lambda_+ = K + \ln\left[1 + \frac{h^2}{2} + e^{-2K} + \frac{1}{2}h^2 e^{2K}\right] \quad (\text{as } h \to 0)$$

$$= K + \ln(1+e^{-2K}) + \frac{1}{2}h^2 \frac{1+e^{2K}}{1+e^{-2K}} \text{ so } \chi(h=0) = \beta \frac{1+e^{2K}}{1+e^{-2K}}$$

$$\chi = \frac{1}{T} e^{\frac{2}{T}}$$



e) $$\langle \sigma_i \sigma_{i+l} \rangle = \frac{1}{Z}\sum_{\{\sigma_i\}} e^{-\beta\mathcal{H}(\sigma)} \sigma_i \sigma_{i+l} = \frac{1}{Z}\sum_{\{\sigma_i\}} \langle\sigma_1|T|\sigma_2\rangle \cdots \langle\sigma_N|T|\sigma_1\rangle \sigma_i \sigma_{i+l}$$

$$= \frac{1}{Z} \mathrm{Tr}\, T^{i-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T^{l} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T^{N-i-l+1} = \frac{1}{Z} \mathrm{Tr}\, \sigma_z T^{l} \sigma_z T^{N-l}$$

(using $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$). Now use as basis eigenvectors of T.
In this basis $\sigma_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$ and so

$$\langle \sigma_i \sigma_{i+l} \rangle = \frac{1}{\lambda_+^N + \lambda_-^N} \mathrm{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_+^N a & \lambda_+^l \lambda_-^{N-l} b \\ \lambda_-^l \lambda_+^{N-l} c & \lambda_-^N d \end{pmatrix}$$

$$\approx \mathrm{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & 0 \\ e^{-l/\xi} c & 0 \end{pmatrix} \quad \left(\frac{\lambda_-}{\lambda_+} = e^{-1/\xi}, \text{ drop } e^{-N/\xi} \text{ terms}\right)$$

$$= a^2 + bc\, e^{-l/\xi}.$$

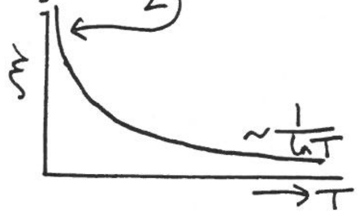
However $\langle \sigma_i \rangle = \frac{1}{Z} \mathrm{tr}\, \sigma_z T^N \approx a$ so correlation $\langle \sigma_i \sigma_{i+l} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+l} \rangle$

$$\simeq bc\, e^{-l/\xi}$$ so correlation length $\xi = 1/\ln\frac{\lambda_+}{\lambda_-}$

f) $h = 0$, $\lambda_\pm = e^{K} \pm e^{-K}$, $\lambda_+/\lambda_- = \coth K$, $\xi = 1/\ln\coth\frac{1}{T}$

$T \to \infty$, $\coth\frac{1}{T} \sim T$, $\xi \sim \frac{1}{\ln T}$; $T \to 0$, $\coth\frac{1}{T} = \frac{e^{1/T} + e^{-\frac{1}{T}}}{e^{1/T} - e^{-\frac{1}{T}}} \sim 1 + 2e^{-\frac{2}{T}}$,

$\xi \sim \frac{1}{2}e^{2/T}$



We see that so far as strongest singularity goes,
as $T \to 0$ $\chi \sim \xi$. $\chi \sim \xi^{2-\eta}$ so $\eta = 1$

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2 a) $e^{-\beta\mathcal{H}(s)} = \prod_{i=1}^{N} \langle S_i | T | S_{i+1} \rangle \quad (S_{N+1} = S_1)$

with $\langle s | T | s' \rangle = \begin{cases} e^K & s = s' \\ 1 & s \neq s' \end{cases}$

Eigenvector x_s, eigenvalue λ satisfy $e^K x_s + \sum_{s' \neq s} x_{s'} = \lambda x_s$

ie $(\lambda - e^K + 1) x_s = \sum_{s'} x_{s'}$ so either $\lambda = e^K - 1$, $\sum x_s = 0$

or $x_s = 1$, $\lambda = e^K - 1 + q$

b) $Z = \mathrm{tr}\, T^N = (e^K - 1 + q)^N + (q-1)(e^K - 1)^N$, $\beta f = -\ln(e^K - 1 + q)$

To calculate correlation length, consider any function $a(s)$ for which the matrix $\langle s' | a | s \rangle = a(s) \delta_{s's}$ becomes A in the basis which makes T diagonal. Then $\langle a(s_i) \rangle = \frac{1}{Z} \mathrm{tr}\, T^N A$

and $\langle a(s_i) b(s_{i+L}) \rangle = \frac{1}{Z} \mathrm{tr}\, A T^L B T^{N-L}$ just as in 1(e).

Again let $\frac{e^K - 1}{e^K - 1 + q} = e^{-1/\xi}$. Then dropping $e^{-N/\xi}$ terms,

$\langle a(s_i) \rangle = A_{11}$ and $\langle a(s_i) b(s_{i+L}) \rangle = A_{11} B_{11} + \sum_{r \neq 1} A_{1r} B_{r1} e^{-L/\xi}$

so $\langle a b \rangle - \langle a \rangle \langle b \rangle = \text{const} \times e^{-L/\xi}$ and ξ is correlation length.

ie $\frac{1}{\xi} = \ln\left(1 + \frac{q}{e^K - 1}\right)$ [cf Problem Set 9, 2(c)]

3 a) Energy of interface $= 2K \times$ number of bonds crossed by interface

$= \text{(from picture) } 2K \times \left[L + \sum_{n=1}^{L} |h_{n+1} - h_n| \right]$

b) $e^{-\beta\mathcal{H}(h_1 \cdots h_L)} = \prod_{n=1}^{L} \langle h_n | T | h_{N+1} \rangle$, $\langle h | T | h' \rangle = e^{-2K} e^{-2K|h' - h|}$

Eigenvalues and eigenvectors are given by $e^{-2K} \sum_{h'} e^{-2K|h-h'|} x(h') = \lambda x(h)$

Because of dependence of T only on $h - h'$, eigenvectors are $x(h) = e^{iqh}$,

$\lambda_q = e^{-2K} \sum_{h=-\infty}^{\infty} e^{-2K|h|} e^{iqh} = e^{-2K} \left\{ 1 + 2\mathrm{Re} \frac{e^{-2K} e^{iq}}{1 - e^{-2K} e^{iq}} \right\}$

c) Clearly this is greatest when $q = 0$, $\lambda_{max} = e^{-2K} \frac{1 + e^{-2K}}{1 - e^{-2K}}$

$\frac{\beta F}{L} = -\ln \lambda_{max} = 2K + \ln \tanh K$

d) $F = 0$ when $e^{-2K} + e^{-4K} = 1 - e^{-2K}$, $(e^{-2K} + 1)^2 = 2$, $e^{-2K} = \sqrt{2} - 1$

$e^{2K} = \sqrt{2} + 1$ which is critical point of 2d Ising model.

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Anisotropy

1. *Random Walks:* Consider the ensemble of all random walks on a square lattice starting at the origin (0,0). Each walk has a weight of $t_x^{\ell_x} . t_y^{\ell_y}$, where ℓ_x and ℓ_y are the number of steps taken along the x and y directions respectively.

(a) Calculate the total weight, $W(x,y)$, of all walks terminating at (x,y). Show that W is well defined only for $\bar{t} = (t_x + t_y)/2 < t_c = 1/4$.

(b) What is the shape of a curve, $W(x,y)$ =constant, for large x and y?

(c) How does the average number of steps $\langle \ell \rangle = \langle \ell_x + \ell_y \rangle$ diverge as $\bar{t}$ approaches t_c?

****

2. *The Ising Model:* Consider the anisotropic Ising model on a square lattice with a Hamiltonian

$$-\beta\mathcal{H} = \sum_{x,y} \left(K_x \sigma_{x,y}\sigma_{x+1,y} + K_y \sigma_{x,y}\sigma_{x,y+1} \right);$$

i.e. with bonds of different strengths along the x and y directions.

(a) By following the method presented in class calculate the free energy for this model. You do not need to write down every step of the derivation. Just describe the steps that need to be modified due to anisotropy; and calculate the final answer for $\ln Z/N$.

(b) Find the critical boundary in the (K_x, K_y) plane from the singularity of the free energy. Show that it coincides with the condition $K_x = \tilde{K}_y$; where $\tilde{K}$ indicates the standard dual interaction to K.

(c) Find the singular part of $\ln Z/N$; and comment on how anisotropy affects critical behavior in the exponent and amplitude ratios.

****

3. *Landau Theory:* Consider an n-component magnetization $\vec{m}(\mathbf{x})$ in d-dimensions.

(a) Using problems (1) and (2) as a guide, generalize the standard Landau–Ginzburg Hamiltonian to include the effects of anisotropy.

(b) Are such anisotropies "relevant"?

(c) In La_2CuO_4, the Cu atoms are arranged on the sites of a square lattice in planes and the planes are then stacked together. Each Cu atom carries a "spin", which we assume to be classical, and can point along any direction in space. There is a very strong antiferromagnetic interaction in each plane. There is also a very weak interplane interaction that prefers to align successive layers. Sketch the low–temperature magnetic phase, and indicate to what universality class the order–disorder transition belongs?

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Suggested reading: Parisi, Chapter 4; Feynman, Chapter 5; Huang, Chapter 15.

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8.334 Solutions to Problem Set 11 Apr 92

1) a) Consider $[t_x e^{iq_x} + t_x e^{-iq_x} + t_y e^{iq_y} + t_y e^{-iq_y}]^L = W_L(q_x, q_y)$

Each term in the expansion of this product into 4^L terms corresponds to a RW with L steps and so $\sum_{L=0}^{\infty} W_L(q_x, q_y) = \sum_{x,y} W(x,y) e^{i(q_x x + q_y y)}$

so $W(x,y) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dq_x dq_y \frac{e^{-i(q_x x + q_y y)}}{1 - 2t_x \cos q_x - 2t_y \cos q_y}$

Well defined if denominator does not vanish ie if $2t_x + 2t_y < 1$

$\bar{t} = \frac{t_x + t_y}{2} < \frac{1}{4} = t_c$

b) For large x, y, important contributions come from small q_x, q_y

so replace denominator by $1 - 2t_x - 2t_y + t_x q_x^2 + t_y q_y^2$

Now write $q_x = \frac{\tilde{q}_x}{\sqrt{t_x}}$, $q_y = \frac{\tilde{q}_y}{\sqrt{t_y}}$, $W = \frac{1}{(2\pi)^2} \int \frac{d\tilde{q}_x d\tilde{q}_y}{\sqrt{t_x t_y}} \frac{e^{-i(\tilde{q}_x \frac{x}{\sqrt{t_x}} + \tilde{q}_y \frac{y}{\sqrt{t_y}})}}{1 - \frac{\bar{t}}{t_c} + \tilde{q}^2}$

By rotational symmetry in $\vec{\tilde{q}}$, W is a function of $\frac{x^2}{t_x} + \frac{y^2}{t_y}$

ie curves of constant $W(x,y)$ are ellipses.

c) From the first line in (a) we see that $W_L(\overset{q_x=}{0}, \overset{q_y=}{0})$ is the total weight of all RWs of length L regardless of end point.

So $\langle L \rangle = \frac{\sum L W_L(0,0)}{\sum W_L(0,0)} = \frac{\sum L (4\bar{t})^L}{\sum (4\bar{t})^L} = \frac{4\bar{t}}{1 - 4\bar{t}} = \frac{\bar{t}}{t_c - \bar{t}}$

so $\langle L \rangle$ diverges like $\frac{1}{4\delta t}$ as $\bar{t} \to t_c$.

3c continued

the correlation length in the planes will be long (many lattice spacings)
and between planes will not be long (few lattice spacings)
and then we will see $d=2$, $n=3$ type behaviour.

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3 a) The L-G Hamiltonian must have different strengths of coupling terms in different directions but still have rotational invariance in the n dimensional $\vec{m}$ space. The leading terms will be

$$-\beta H = \int d^d x \left[\frac{1}{2} \sum_{i=1}^{d} K_i \frac{\partial \vec{m}}{\partial x_i} \cdot \frac{\partial \vec{m}}{\partial x_i} + \frac{1}{2} t \vec{m}(x) \vec{m}(x) + u \left\{ \vec{m}(x) \cdot \vec{m}(x) \right\}^2 \right]$$

b) As a first step we can rescale the different x_i to make all the K_i the same ie let $x_i = x_i' \sqrt{\frac{K_i}{K_0}}$. The L-G Hamiltonian is then isotropic in the d dimensional x space and we will get same behaviour as isotropic case ie anisotropy is irrelevant. Of course if we started with a short distance cut off $\sum q_i^2 < \Lambda^2$ we will now have an ellipsoidal cut off but the RG equations for the low q effective βH will not be changed.

c) This combination of antiferromagnetic and ferromagnetic couplings is equivalent to all ferromagnetic (change the sign of every other column of spins)

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

This works for a cubic lattice but not, e.g., for a triangular lattice. The diagram shows the alignment of each column of spins at zero temperature. The phase transition will be a 3-dimensional space, 3 dimensional $\vec{m}$ type (by argument in (b)).

However there will be a range of temperatures where

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8.334. Problem Set 11, Solutions Apr 92

2) $-\beta H = \sum_{x,y}\left\{K_x \sigma_{x,y}\sigma_{x+1,y} + K_y \sigma_{x,y}\sigma_{x,y+1}\right\}$

High temp. series is $Z = \left[\sum t_x^{l_x} t_y^{l_y}\right] \times \left(2\cosh K_x \cosh K_y\right)^N$

$(t = \tanh K)$

where the sum is over all closed graphs on the lattice ie all choices of a subset of bonds with 0, 2 or 4 bonds at each site and l_x, l_y are the numbers of bonds in the x and y directions.

As in the isotropic case, the sum is the exponential of the sum over all single closed RWs with no U-turns weighted by (-1) for each crossing. We count directed RWs from $(0,0)$ to $(0,0)$. Let $\langle 0|W^*(l_x,l_y)|0\rangle =$

$\frac{1}{2}$ the sum of $(-1)^{\#\text{ of crossings}}$ over all directed, $\cancel{U}$ RWs from $(0,0)$ to $(0,0)$ in (l_x, l_y) steps.

Then $f = \frac{\ln Z}{N} = \ln\left(2\cosh K_x \cosh K_y\right) + \sum_{l_x,l_y} \frac{t_x^{l_x} t_y^{l_y}}{l_x+l_y} \langle 0|W^*(l_x l_y)|$

As in the isotropic case we generate all the terms corresponding to RWs of length $l = l_x + l_y$ by taking the trace of the l th power of the $4N \times 4N$ matrix

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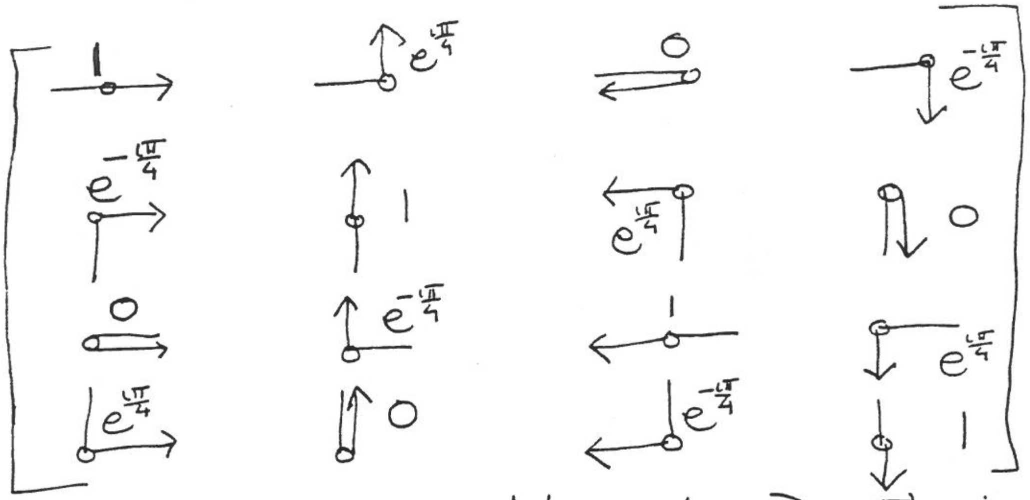
$T^* =$



(the indicated factors come from Whitney's theorem). There is one overall factor of -1 in every term of the sum, from Whitney's theorem.

After we go to Fourier transform of (x, y), the matrix elements $\langle \mu' | T^*(q_x\, q_y) | \mu \rangle$ will be as above with the columns multiplied by $t_x e^{-iq_x}$, $t_y e^{-iq_y}$, $t_x e^{iq_x}$, $t_y e^{iq_y}$ respectively. The appearance of the different t_x, t_y here, instead of multiplying the whole matrix by t, is the only change due to anisotropy. We get

$$f = \ln 2 c_x c_y + \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \operatorname{tr} \ln \left[1 - T^*(q) \right]$$

$\operatorname{tr} \ln A = \ln \det A$. Working out the determinant gives

$$f = \ln(2 \cosh K_x \cosh K_y) + \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \ln \left[(1+t_x^2)(1+t_y^2) - 2\cos q_x\, t_x (1-t_y^2) - 2\cos q_y\, t_y (1-t_x^2) \right]$$

$$= \ln 2 + \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \ln \left[\cosh 2K_x \cosh 2K_y - \cos q_x \sinh 2K_x - \cos q_y \sinh 2K_y \right]$$

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i) Minimum value of argument of ln is when $q_x = q_y = 0$, when it equals $\cosh 2K_x \cosh 2K_y - \sinh 2K_x - \sinh 2K_y$

$$= \frac{1}{2} e^{2K_x}(\cosh 2K_y - 1) - \sinh 2K_y + \frac{1}{2} e^{-2K_x}(\cosh 2K_y + 1)$$

$$= \frac{1}{2}\left\{ e^{K_x}\sqrt{\cosh 2K_y - 1} - e^{-K_x}\sqrt{\cosh 2K_y + 1} \right\}^2$$

Critical surface is therefore $e^{2K_x} = \sqrt{\dfrac{\cosh 2K_y + 1}{\cosh 2K_y - 1}}$

which is same as
$\sinh 2K_x = \frac{1}{2}\left(\sqrt{\dfrac{\cosh 2K_y + 1}{\cosh 2K_y - 1}} - \sqrt{\dfrac{\cosh 2K_y - 1}{\cosh 2K_y + 1}} \right) = \dfrac{1}{\sinh 2K_y}$

Duality relation is $\sinh 2K \sinh 2\tilde{K} = 1$ so critical surface is $K_x = \tilde{K}_y$

j) Near critical surface singular part of $\dfrac{\ln Z}{N} = f_s$

$$= \frac{1}{2}\int^{\Lambda} \frac{d^2q}{(2\pi)^2} \ln\left[\left\{ e^{K_x}\sinh K_y - e^{-K_x}\cosh K_y \right\}^2 + \sinh 2K_x \frac{q_x^2}{2} + \sinh 2K_y \frac{q_y^2}{2} \right]$$

Let $q_{x,y} = \sqrt{\dfrac{2}{\sinh 2K_{x,y}}}\, q'_{x,y}$, $\delta t = e^{K_x}\sinh K_y - e^{-K_x}\cosh K_y$ follows a curve which will go linearly through zero as (K_x, K_y) intersecting the critical line $K_x = \tilde{K}_y$.

Then $f_s = \dfrac{1}{\sqrt{\sinh 2K_x \sinh 2K_y}} \displaystyle\int^{\Lambda} \frac{d^2q'}{(2\pi)^2} \ln\left[\delta t^2 + q'^2\right]$ which is same
$\uparrow$ $=1$ on critical line

as isotropic case and gives $f_s = -\dfrac{1}{2\pi}(\delta t)^2 \ln \delta t$

Thus exponents and amplitude ratios are unchanged by anisotropy. However the amplitudes themselves (i.e the coefficients of $|\ln \delta t|$ in the specific heat) depend on where and in which direction we cross the critical line.

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The Roughening Transition

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1. *Renormalization:* In problem set 3, we examined a continuum interface problem which in $d = 3$ is described by

$$-\beta\mathcal{H}_0 = -\frac{K}{2}\int d^2\mathbf{x}(\nabla h)^2,$$

where $h(\mathbf{x})$ is the interface height at $\mathbf{x}$. For a crystalline facet the allowed values of h are multiples of the lattice spacing. In the continuum, this tendency for integer h can be mimicked by adding a term

$$-\beta U = y_0 \int d^2\mathbf{x}\cos(2\pi h)$$

to the Hamiltonian. Treat $-\beta U$ as a perturbation, and proceed to construct a renormalization group as follows:

(a) Show that

$$\langle \exp[i\sum_\alpha q_\alpha h(\mathbf{x}_\alpha)]\rangle_0 = \exp[\frac{1}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C(\mathbf{x}_\alpha - \mathbf{x}_\beta)]$$

for $\sum_\alpha q_\alpha = 0$, and zero otherwise. ($C(\mathbf{x}) = \ln|\mathbf{x}|/2\pi$ is the Coulomb interaction in two dimensions.)

(b) Prove that

$$\langle|h(\mathbf{x}) - h(\mathbf{y})|^2\rangle = -\frac{d^2}{dk^2}G_k(\mathbf{x}-\mathbf{y})\,|_{k=0},$$

where $G_k(\mathbf{x}-\mathbf{y}) = \langle\exp[ik\big(h(\mathbf{x}) - h(\mathbf{y})\big)]\rangle$.

(c) Use the results in (a) to calculate $G_k(\mathbf{x}-\mathbf{y})$ in perturbation theory to order of y_0^2.

(d) Write the perturbation result in terms of an effective interaction K, and show that perturbation theory fails for K larger than a critical K_c.

(e) Recast the perturbation result in (d) into R.G. equations for K and y_0 by changing the "lattice spacing" from a to ae^ℓ.

(f) Using the recursion relations discuss the phase diagram and phases of this model.

(g) For large separations $|\mathbf{x}-\mathbf{y}|$, find the magnitude of the discontinuous jump in $\langle|h(\mathbf{x}) - h(\mathbf{y})|^2\rangle$ at the transition.

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2. *Duality:* Consider a discretized version of the Hamiltonian in problem one: for each site i of a square lattice define an integer valued height h_i. The Hamiltonian is

$$\beta\mathcal{H} = -\frac{K}{2}\sum_{<ij>} |h_i - h_j|^{\infty};$$

where the "∞" power means that there is no energy cost for $\Delta h = 0$; an energy cost of $K/2$ for $\Delta h = \pm 1$; and $\Delta h = \pm 2$ or higher *are not allowed* for neighboring sites. (This is known as the restricted solid on solid (RSOS) model.)

(a) Construct the dual model either diagrammatically, or by following these steps:

(i) Change from the N site variables h_i to the $2N$ bond variables $n_{ij} = h_i - h_j$. Show that the sum of n_{ij} around any plaquette is constrained to be zero.

(ii) Impose the constraints using the identity $\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta n} = \delta_{n,0}$ for integer n.

(iii) After imposing the constraints you can sum freely over the bond variables n_{ij} to obtain a dual interaction $\tilde{v}(\theta_i - \theta_j)$ between dual variables θ_i on neighboring plaquettes.

(b) Show that for K large the dual problem is just the XY model. Is this conclusion consistent with the renormalization group results of problem 1?

(c) Does the one dimensional version of this Hamiltonian; i.e. a 2d interface with

$$\beta\mathcal{H} = -\frac{K}{2}\sum_{i} |h_i - h_{i+1}|^{\infty};$$

have a roughening transition?

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Suggested Reading: The XY and roughening transitions have not made their way into standard textbooks yet. See H. van Beijeren and I. Nolden, in *Structure and Dynamics of Surfaces II*, edited by Schommers and P. von Blackenhagen (Springer -Verlag, Berlin, 1987); and D.R. Nelson's contribution in, *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebovitz (Academic, New York, 1983).

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8.334 Problem Set 12, Solutions May 92

1) $$Z = \int \mathcal{D}h(\vec{x}) \exp\left[-\frac{K}{2}\int d^2x (\nabla h)^2 + y_0 \int d^2x \cos 2\pi h\right]$$

Zero temperature: h = constant = integer

a) H_0 does nothing to tie down a constant $h(x) = h_0$, so in calculating

$\left\langle e^{i\sum q_\alpha h(x_\alpha)}\right\rangle_0$ there is a factor $\int dh\, e^{ih\sum q_\alpha} = 0$ if $\sum q_\alpha \neq 0$

If $\sum q_\alpha = 0$, $$\left\langle e^{i\sum q_\alpha h(x_\alpha)}\right\rangle_0 = e^{-\frac{1}{2}\sum_{\alpha,\beta} q_\alpha q_\beta \langle h(x_\alpha) h(x_\beta)\rangle_0}$$

$$\sum_{\alpha,\beta} q_\alpha q_\beta \langle h(x_\alpha) h(x_\beta)\rangle_0 = \int \frac{d^2q}{Kq^2}\frac{1}{(2\pi)^2}\sum_{\alpha\beta} q_\alpha q_\beta \left\{e^{i\vec{q}\cdot(\vec{x}_\alpha - \vec{x}_\beta)} - 1\right\}$$

$$= \frac{2}{K}\sum_{\alpha<\beta}\int \frac{d^2q}{(2\pi)^2} q_\alpha q_\beta \left\{\frac{e^{iq\cdot x_\alpha - x_\beta} - 1}{q^2}\right\} = -\frac{2}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C(x_\alpha - x_\beta)$$

$\nabla^2 C(x) = +\delta^2(x)$, $C(x) = \frac{1}{2\pi}\ln\frac{|x|}{a}$ so $\left\langle e^{i\sum q_\alpha h(x_\alpha)}\right\rangle_0 = e^{\frac{1}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C(x_\alpha - x_\beta)}$

b) $$G_k = \left\langle e^{ik(h(x)-h(y))}\right\rangle = 1 + ik\langle h(x) - h(y)\rangle - \frac{k^2}{2}\langle (h(x) - h(y))^2\rangle \cdots$$

so $$\langle |h(x) - h(y)|^2\rangle = -\frac{d^2}{dk^2} G_k(x-y)\Big|_{k=0}$$

c) $$\left\langle e^{ik[h(x)-h(y)]}\right\rangle = \frac{\frac{1}{Z_0}\int \mathcal{D}h\, e^{ik[h(x)-h(y)]}\, e^{-\beta H_0}\left\{1 - \beta U + \frac{1}{2}(\beta U)^2 \cdots\right\}}{\frac{1}{Z_0}\int \mathcal{D}h\, e^{-\beta H_0}\left\{1 - \beta U + \frac{1}{2}(\beta U)^2 \cdots\right\}}$$

$$-\beta U = \frac{y_0}{2}\int d^2z \left\{e^{2\pi i h(z)} + e^{-2\pi i h(z)}\right\}$$

We see that $\langle \beta U\rangle_0$ and $\left\langle e^{ik\{h(x)-h(y)\}}\beta U\right\rangle_0$ are of the form in (a) with $\sum q_\alpha = \pm 2\pi$ so give zero. Then to order y_0^2

$$G_k = \left\langle e^{ik\{h(x)-h(y)\}}\right\rangle_0 + \frac{1}{2}\left\langle e^{ik\{h(x)-h(y)\}}(\beta U)^2\right\rangle - \frac{1}{2}\left\langle e^{ik\{h(x)-h(y)\}}\right\rangle_0 \left\langle (\beta U)^2\right\rangle_0$$

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$$= e^{-\frac{k^2}{K}C(x-y)} + \frac{y_0^2}{8}\int d^2z\, d^2z' \left\langle e^{ik\{h(x)-h(y)\}}\left[e^{2\pi i h(z)} + e^{-2\pi i h(z)}\right]\left[e^{2\pi i h(z')} + e^{-2\pi i h(z')}\right]\right\rangle_{0,\text{connected}}$$

$$= e^{-\frac{k^2}{K}C(x-y)} + \frac{y_0^2}{4}\int d^2z\, d^2z' \left\langle e^{ikh(x) - ikh(y) + 2\pi i h(z) - 2\pi i h(z')}\right\rangle_{0,\text{connected}}$$

[other terms have $\sum q_\alpha \neq 0$]

$$= e^{-\frac{k^2}{K}C(x-y)}\left[1 + \frac{y_0^2}{4}\int d^2z\, d^2z'\, e^{-\frac{4\pi^2}{K}C(z-z')}\left\{e^{\frac{2\pi k}{K}\left[C(x-z) - C(y-z) - C(x-z') + C(y-z')\right]} - 1\right\}\right]$$

d) We now see a formal analogy to the Coulomb gas of vortices in the X-Y model, with y_0 as a fugacity and 'charges' $\pm k$ at x, y with interaction modified by 'charges' $\pm 2\pi$ created at z, z'. However here the interaction $\propto \frac{1}{K}$, as against K in the X-Y vortex calculation ie we have a suggestion of duality to X-Y, see problem 2

We calculate the effect of a pair $\pm 2\pi$ with $|z-z'|$ small.

Let $\vec{r} = \vec{z} - \vec{z}'$, $\vec{R} = \frac{\vec{z}+\vec{z}'}{2}$, expand in r

$$\left[C(x-z) - C(x-z') - C(y-z) + C(y-z')\right] = \vec{r}\cdot\vec{\nabla}\left\{C(y-R) - C(x-R) + O(r^3)\right\}$$

$$e^{\frac{2\pi k}{K}\left[\;\right]} - 1 = +\frac{2\pi k}{K}\, r\cdot\nabla_R\left\{C(x-R) - C(y-R)\right\} + \frac{2\pi^2 k^2}{K^2}\left[r\cdot\nabla_R\{\;\}\right]^2 + O(r^3)$$

Now we need $\displaystyle\int d^2r\, d^2R\, e^{-\frac{4\pi^2}{K}C(r)}\left[\;\right]$

The term linear in k gives zero when we integrate over R since $C(x-R) - C(y-R) \to 0$ as $R \to \infty$.

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In the k^2 term, first integrate over the angle of $\vec{r}$ to replace $(r\cdot\nabla_R f)^2$ by $\frac{1}{2}r^2(\nabla_R f)^2$, then use

$$\int d^2R\,(\nabla_R f)^2 = -\int d^2R\, f\,\nabla_R^2 f \quad \text{to get}$$

$$\frac{2\pi^2k^2}{K^2}\int d^2r\,\frac{1}{2}r^2 e^{-\frac{4\pi^2}{K}C(r)}\left[-\int d^2R\left\{C(x-R)-C(y-R)\right\}\left\{+\delta^{(2)}(x-R)-\delta^{(2)}(y-\right.\right.$$

$$= \frac{2\pi^2k^2}{K^2}\int 2\pi r^3 dr\, e^{-\frac{4\pi^2}{K}C(r)}\left[C(r)-C(0)\right]$$

But $e^{-\frac{4\pi^2}{K}C(r)} = r^{-\frac{2\pi}{K}}$ so we get

$$G_k(x-y) = e^{-\frac{k^2}{K}C(r)}\left[1+\frac{\pi^3k^2y_0^2}{K^2}C(r)\int_a^\infty r^{3-\frac{2\pi}{K}}dr\right]$$

$$= e^{-\frac{k^2}{K_{eff}}C(r)} \quad \text{with} \quad \frac{1}{K_{eff}} = \frac{1}{K} - \frac{\pi^3y_0^2}{K^2}\int_a^\infty r^{3-\frac{2\pi}{K}}dr$$

Integral is well behaved at large r for $\frac{2\pi}{K} > 4$, $K < \frac{\pi}{2}$

ie at high T effect of y_0 is to increase K_{eff}, surface is still rough.

e) R.G., $a \to ae^l \approx a(1+l)$. First coarse grain

$$K_{eff} = K + \pi^3y_0^2\int_{ae^l}^\infty r^{3-\frac{2\pi}{K}}dr + \pi^3y_0^2a^{4-\frac{2\pi}{K}}l$$

so $\tilde{K} = K + l\,\pi^3y_0^2a^{4-\frac{2\pi}{K}}$. Now rescale, $r' = re^{-l}$

so that r' runs from a to ∞.

$$K_{eff} = \tilde{K} + \pi^3y_0^2e^{l(4-\frac{2\pi}{K})}\int_a^\infty r^{3-\frac{2\pi}{K}}dr$$

of same form as original equation if we take $\tilde{y}_0 = y_0e^{l(2-\frac{\pi}{K})}$

[In the integral replace K by $\tilde{K}$, correct up to order y_0^2]

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R.G. equations for flow $(K, y_0) \to (\tilde{K}, \tilde{y}_0)$ are

$$\tilde{K} = K + l\pi^3 y_0^2 a^{4-\frac{2\pi}{K}}, \quad \tilde{y}_0 = y_0 + y_0 l\left(2 - \frac{\pi}{K}\right)$$

ie $$\frac{dK}{dl} = \pi^3 y_0^2 a^{4-\frac{2\pi}{K}}, \quad \frac{dy_0}{dl} = y_0\left(2 - \frac{\pi}{K}\right)$$

Just like the X-Y model with K changed to K^{-1}!

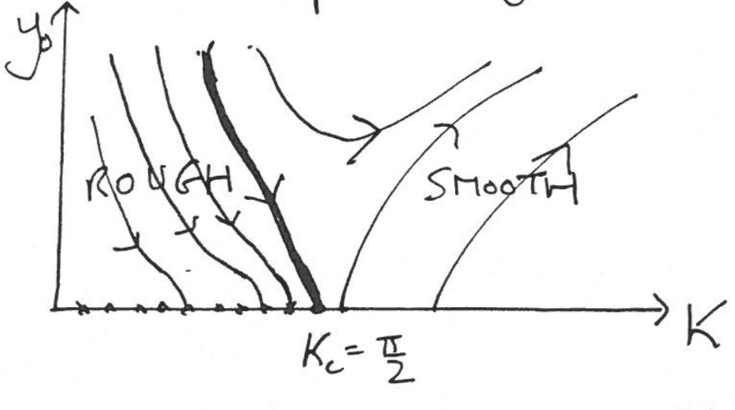
F) For $K < K_c = \frac{\pi}{2}$, $\frac{dy_0}{dl} < 0$ and there is a line of fixed points at $y_0 = 0$, $0 < K < K_c$.

Term $\cos 2\pi h$ in energy is irrelevant and we have capillary waves on interface, correlations decaying like power, surface is rough.

For $K > K_c$, flow increases y_0 and K so system at large scales is driven towards flatness $[K(\nabla h)^2$ with K large$]$ and integer values of h $[y_0 \cos 2\pi h$ with y_0 large$]$.

Excitations are islands of integer h.

Just as in X-Y model, phase diagram is



Transition line is $K = K_c - \frac{\pi^2}{2} y_0$ near $y_0 = 0$.

g) $$\langle |h(x) - h(y)|^2 \rangle = \frac{2C(x-y)}{K_{eff}} \quad \text{(from (b) and (d))} = \frac{\ln|x-y|}{\pi K_{eff}}$$

At transition line, large scale K_{eff} jumps from ∞ to $K_c = \frac{\pi}{2}$

so $\langle (\Delta h)^2 \rangle$ jumps by $\frac{2}{\pi^2} \ln|x-y|$

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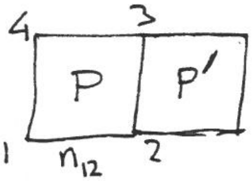
2) a) i)



$$n_{12} = h_1 - h_2$$

$$\sum_P n_{ij} = n_{12} + n_{23} + n_{34} + n_{41} = 0$$

ii) $$Z = \sum_{\{h_i\}} e^{-\frac{K}{2}\sum_{\langle ij\rangle} |h_i - h_j|^\infty} = \sum_{\{n_{ij}\}} e^{-\frac{K}{2}\sum |h_i - h_j|^\infty} \prod_P \delta_{\sum_P n_{ij},0}$$

$$= \sum_{\{n_{ij}\}} \int \prod_P \frac{d\theta_P}{2\pi} \prod_{\langle ij\rangle} e^{-\frac{K}{2}|n_{ij}|^\infty + i n_{ij}(\theta_P - \theta_{P'})}$$

↑ plaquettes on either side of $\langle ij\rangle$

iii) $$= \int \prod_P \frac{d\theta_P}{2\pi} e^{\sum_{\langle ij\rangle} \tilde{v}(\theta_P - \theta_{P'})}$$

$$e^{\tilde{v}(\theta)} = 1 + e^{-\frac{K}{2}}(e^{i\theta} + e^{-i\theta}), \quad \tilde{v}(\theta) = \ln\left\{1 + 2e^{-\frac{K}{2}}\cos\theta\right\}$$

On dual lattice we now have XY type model with interaction $\tilde{v}(\theta)$ between nearest neighbors.

b) $K \gg 1$, $e^{-\frac{K}{2}} \ll 1$, $\tilde{v}(\theta) \approx \tilde{K}\cos\theta$, $\tilde{K} = 2e^{-\frac{K}{2}} \ll 1$, $\tilde{K}^{-1} = \frac{1}{2}e^{K/2} \gg 1$. So smooth phase corresponds to high temperature phase of XY model, but with finite correlation lengths. This is consistent with (1) where we found similar phase diagrams with K corresponding to $\tilde{K}^{-1}$

c) In one-dimensional chain, $n_i = h_i - h_{i+1}$ are independent variables.

$$\left\langle e^{i\theta(h_m - h_0)}\right\rangle = \left\langle e^{i\theta n}\right\rangle^m = \left[\frac{1 + 2e^{-K/2}\cos\theta}{1 + 2e^{-K/2}}\right]^m$$

$\approx \exp\{-m\theta^2 e^{-K/2}\}$ so $\langle (h_m - h_0)^2\rangle \approx 2e^{-K/2} m$ ie $\Delta h \propto \sqrt{m}$, same result as in $(\nabla h)^2$ theory ie rough except at $T=0$.

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Review Problems

The endterm quiz will take place on Friday May 8, in room 13–3101 from 3:00 to 4:30pm (with an additional half hour if necessary). All topics presented in the course will be discussed, with emphasis on the second half (sections V to VII). It will be a closed book exam, but you may bring a two–sided sheet of formulas if you wish. The following problems, taken form previous exams, are provided to help you review the material. *This is not a problem set* and you are not expected to turn in any solutions.

1. Consider the Hamiltonian

$$\beta\mathcal{H}=\int d^d\mathbf{x}[\frac{K}{2}(\nabla m)^2+\frac{t}{2}m^2+v_n m^n] \quad ,$$

for a one component field $m(\mathbf{x})$.

(a) Using the saddle point approximation, calculate the exponents β and α for the critical behaviors of magnetization and heat capacity as $t\to 0$.

(b) Using power counting, or otherwise, find the upper critical dimension $d_u(n)$, for breakdown of saddle point results due to fluctuations.

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2. Consider the nonlinear σ model in $d=2$ and for $n>2$. The recursion relations for temperature T and magnetic field h are

$$\begin{aligned}\frac{dT}{d\ell}=&\frac{(n-2)}{2\pi}T^2 \quad ,\\ \frac{dh}{d\ell}=&2h \quad .\end{aligned}$$

(a) How does the correlation length diverge as $T\to 0$?

(b) Write down the singular form of the free energy as $T,h\to 0$.

(c) How does the susceptibility χ diverge as $T\to 0$ for $h=0$?

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3. Consider the Ising Model on a *hexagonal lattice* with a Hamiltonian

$$-\beta\mathcal{H}=K\sum_{<ij>}\sigma_i\sigma_j \quad .$$

(a) Calculate the first two (nontrivial) terms in the *high temperature* expansion of the free energy.

(b) Calculate the first two (nontrivial) terms in the *low temperature* expansion of the free energy.

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4. *The spin 1 model:* Consider a linear chain where the spin s_i at each site takes on three values $s_i = -1, 0, +1$. The spins interact via a Hamiltonian

$$-\beta\mathcal{H} = K\sum_i s_i s_{i+1} \quad .$$

(a) Write down the transfer matrix $\langle s|T|s'\rangle = \exp(Kss')$ explicitly.

(b) How are the free energy per site ($\ln Z/N$) and the correlation length ξ related to the properties of a transfer matrix in general?

(c) Use symmetry properties to find the largest eigenvalue of T and hence obtain the expression for the free energy per site ($\ln Z/N$).

(d) Obtain the expression for the correlation length ξ and note its behavior as $K \to \infty$.

(e) If we try to perform a renormalization group by decimation on the above chain we find that additional interactions are generated. Write down the simplest generalization of $\beta\mathcal{H}$ whose parameter space is closed under such decimation RG.

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