Chapter 16

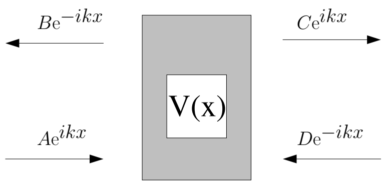
Problem sets

Before starting these problems, you might want to revise some of the examples from the Part IB Quantum Physics course. The examples marked with a † are typically more challenging and are the ones to omit if your time is very short, or if you are finding the course difficult. Some of the questions involve a routine piece of bookwork. This is the kind of thing you will have to do in the exam. You are strongly encouraged to do these parts and get feedback in supervisions.

16.1 Problem Set I

1. Quantum mechanics in one-dimension: The following question introduces the concept of a scattering matrix (or S-matrix) in relation to scattering from a potential in one dimension. These concepts will prepare the ground for the study of the three-dimensional scattering problem addressed later in the course.

   Consider a *localized* potential in one-dimension (i.e. a potential that is non-zero only over a finite region in space) subject to a beam of quantum particles incident from the left and from the right (see figure). Outside the region of the potential, we know that the wavefunction of the particles is described by a plane wave of wavevector $k = \sqrt{2mE}/\hbar$. The relation between the incoming and outgoing components of the plane wave are specified by a **scattering matrix** (commonly referred to as the (**S-matrix**)), i.e. referring to the figure,



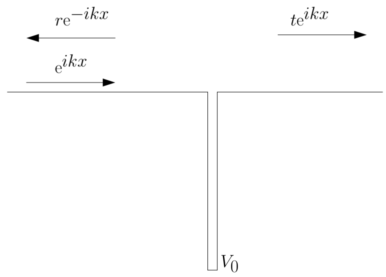
$$|\Psi_{\rm out}\rangle = S|\Psi_{\rm in}\rangle \Rightarrow \left(\begin{array}{c} C \\ B \end{array}\right) = \left(\begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S_{22} \end{array}\right)\left(\begin{array}{c} A \\ D \end{array}\right)$$

   (a) Consider the action of the probability current operator on a plane wave and hence show that conservation of probability implies that $|A|^2 + |D|^2 = |B|^2 + |C|^2$. Show that this condition is equivalent to demanding that the S-matrix is unitary, i.e. $S^\dagger S = \mathbb{I}$. For matrices which are unitary, the eigenvalues have unit magnitude,[1] $e^{i\theta}$ – i.e. two **scattering phase shifts**, in general functions of k, completely describe the scattering in one dimension.

   For the case of a symmetric potential, $V(x) = V(-x)$, the S-matrix assumes the simple form

$$S = \left(\begin{array}{cc} t & r \\ r & t \end{array}\right),$$

[1] The proof is as follows: for an eigenvector $|v\rangle$, such that $S|v\rangle = \lambda|v\rangle$, we have the norm $\langle v|S^\dagger S|v\rangle = |\lambda|^2\langle v|v\rangle = \langle v|v\rangle$, i.e. $|\lambda|^2 = 1$.



where r and t are the reflection and transmission amplitudes.

(b) Show that unitarity demands that $rt^* + r^*t = 0$ and $|r|^2 + |t|^2 = 1$, and hence that $|r \pm t|^2 = 1$. Find θ_1 and θ_2 in terms of r and t. What is the difference in phase between r and t?

(c) By matching the boundary conditions, show the elements of the S-matrix for the scattering of particles of mass m and energy $E = \frac{\hbar^2 k^2}{2m}$ from a δ-function potential, $aV_0\delta(x)$, are given by $r = -\frac{\gamma}{\gamma+ik}$ and $t = \frac{ik}{\gamma+ik}$, where $\gamma = \frac{maV_0}{\hbar^2}$. Obtain the corresponding scattering phase shifts.

2. Operator methods: This problem addresses simple relations that follow from the orthogonality of eigenfunctions and the time-development of wavefunctions.

The Hamiltonian $\hat{H}$ has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ which correspond to different eigenvalues E_1 and E_2.

(a) Show that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

(b) For an observable $\hat{A}$ where $\hat{A}|\psi_1\rangle = |\psi_2\rangle$ and $\hat{A}|\psi_2\rangle = |\psi_1\rangle$, calculate the eigenvalues and eigenvectors (which are combinations of $|\psi_1\rangle$ and $|\psi_2\rangle$).

(c) Assuming that at $t = 0$ the system is in the state $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}[|\psi_1\rangle - |\psi_2\rangle]$, find the state of the system $|\psi(t)\rangle$ at time t and show that the probability of the system returning to its initial state is given by $P = \cos^2[(E_1 - E_2)t/2\hbar]$.

3. Operator methods: This question relates to the probem of **coherent** or **Glauber** states. It is included in the problem set as it presents a useful arena in which to practice operator methods. The aim of the problem is to explore properties of coherent states and establish their connection to classical dynamics of the harmonic oscillator.

(a) By using the commutation relation $[a, a^\dagger] = 1$, show that

$$e^{-\beta a^\dagger} a e^{\beta a^\dagger} = \beta + a\,.$$

Hint: To prove this result most straightforwardly, consider the β derivative of this expression.

Using this result, show that $|\beta\rangle = Ne^{\beta a^\dagger}|0\rangle$ is a coherent state, i.e. $a|\beta\rangle = \beta|\beta\rangle$. Finally, show that the normalization, $N = e^{-|\beta|^2/2}$.

(b) Calculate the expectation values, $x_0 = \langle\hat{x}\rangle$ and $p_0 = \langle\hat{p}\rangle$, with respect to $|\beta\rangle$ and, by considering $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$, show that

$$(\Delta p)^2(\Delta x)^2 = \frac{\hbar^2}{4}\,,$$

where $\Delta p = \hat{p} - \langle\hat{p}\rangle$ (similarly x).

Hint: Remember how creation and annihilation operators are related to the phase space operators $\hat{x}$ and $\hat{p}$. Also, note that the Hermitian conjuate of the eigenvalue equation $a|\beta\rangle = \beta|\beta\rangle$ leads to the relation $\langle\beta|a^\dagger = \langle\beta|\beta^*$.

(c) To determine the coordinate representation to the coherent state, $\psi(x) = \langle x|\beta\rangle$, it is helpful to revert back to the expression for a as a differential operator. Show that the eigenvalue equation $a|\beta\rangle = \beta|\beta\rangle$ translates to the equation,

$$\left(x + \frac{\hbar}{m\omega}\partial_x\right)\psi(x) = \beta\psi(x)\,.$$

Show that this equation has the solution

$$\psi(x) = N \exp\left[-\frac{(x-x_0)^2}{4(\Delta x)^2} + i\frac{p_0 x}{\hbar}\right],$$

where x_0 and p_0 are defined in part (b) above.

(d) By expressing $|\beta\rangle$ in the number basis, show that

$$|\beta(t)\rangle = e^{-i\frac{\omega t}{2}}|\beta e^{-i\omega t}\rangle .$$

As a result, deduce expressions for $x_0(t)$ and $p_0(t)$ and show they represent solutions to the classical equations of motion. How does the width of the coherent state wavepacket evolve with time?

4. Charged particles in a magnetic field: In lectures, we studied the motion of an electron in a uniform magnetic field working in the Landau gauge, $\mathbf{A} = (-By, 0, 0)$. With this gauge choice, the Hamiltonian may be straightforwardly brought to a quantum harmonic oscillator form. In this question, we will address the problem by working the "symmetric" gauge $\mathbf{A} = (-By/2, Bx/2, 0)$. The advantage of this gauge choice is that it facilitates the development of the many-particle wavefunction in an aesthetic and useful form.

(a) A spinless electron of charge $q = -e$ is confined to the xy-plane and subject to a perpendicular magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$. Working in the symmetric gauge $\mathbf{A} = (-y, x, 0)B/2$, show that the electron Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x - \frac{1}{2}m\omega y\right)^2 + \frac{1}{2m}\left(\hat{p}_y + \frac{1}{2}m\omega x\right)^2,$$

where $\omega = \frac{eB}{m}$ denotes the cyclotron frequency.

(b) If units are chosen such that $\omega = m = \hbar = 1$, show that the Hamiltonian can be recast in the dimensionless form

$$\hat{H} = \frac{1}{2}\left(-i\partial_x - \frac{y}{2}\right)^2 + \frac{1}{2}\left(-i\partial_y + \frac{x}{2}\right)^2 .$$

Physicists often choose to work in dimensionless units setting $\hbar = 1$, etc. Think about what this choice entails and, at each stage of the problem, try to infer how dimensionful parameters can be restored.

(c) Introducting the complex coordinate, $\bar{z} = x + iy$ and the complex conjugate, $z = x - iy$, the corresponding derivatives are given by $\partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y)$, and $\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$ where $\partial_z \equiv \frac{\partial}{\partial z}$. From this definition, confirm that

$$[z, \partial_z] = [\bar{z}, \partial_{\bar{z}}] = -1, \qquad [z, \partial_{\bar{z}}] = [\bar{z}, \partial_z] = 0 .$$

The choice $z = x - iy$ (as opposed to $z = x + iy$) is made on purely aesthetic grounds – see the wavefunction below. Also, following convention in this subject, we have chosen to write the complex conjugate as $\bar{z}$ and not z^*.)

Show that the operators,

$$a = \sqrt{2}\left(\partial_{\bar{z}} + \frac{z}{4}\right), \qquad a^\dagger = \sqrt{2}\left(-\partial_z + \frac{\bar{z}}{4}\right),$$
$$b = \sqrt{2}\left(\partial_z + \frac{\bar{z}}{4}\right), \qquad b^\dagger = \sqrt{2}\left(-\partial_{\bar{z}} + \frac{z}{4}\right),$$

fulfil the commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$ characteristic of creation and annihilation operators. Applied to the Hamiltonian, show that

$$\hat{H} = a^\dagger a + \frac{1}{2},$$

independent of b, i.e. a quantum harmonic oscillator. As a result, we can identify a Landau level Hamiltonian with the level set by the eigenvalue, n of the number operator, $\hat{n} = a^\dagger a$.

(d) Show that the angular momentum operator takes the form,

$$\hat{L}_z = -(b^\dagger b - a^\dagger a)\hbar \,.$$

Therefore, if we define the eigenvalues of $\hat{L}_z$ as $-m\hbar$ (with the sign following an accepted convention), the quantum numbers m can take values $m = -n, -n+1, \cdots, 0, 1, \cdots$. The corresponding (normalized) states of Landau level, n, and angular momentum, m, are given by

$$|n, m\rangle = \frac{(b^\dagger)^{m+n}}{\sqrt{(m+n)!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0, 0\rangle \,,$$

where $|0, 0\rangle$ denotes the zero angular momentum ground state of the lowest Landau level.

(e) Going back to the definition of the annihilation operator, show that the ground state in the real space representation is given by,[2]

Note that, for the ground state, we must have $a|0,0\rangle = b|0,0\rangle = 0$.

$$\langle \mathbf{r}|0, 0\rangle = \frac{1}{\sqrt{2\pi}} e^{-\bar{z}z/4} \,.$$

Finally, show that in the lowest Landau level,

$$\langle \mathbf{r}|0, m\rangle = \frac{1}{\sqrt{2\pi 2^m m!}} z^m e^{-\bar{z}z/4} \,,$$

i.e. aside from the Gaussian factor, the states of the lowest Landau level are given by a polynomial in z – they are said to be *analytic* functions of z.

(f) The last part of this problem anticipates our studies of the many-particle fermionic system in chapter 8 and should not be attempted until this ground is covered. In this chapter, we will show that the properly antisymmetrized many-particle wavefunction of a non-interacting system of N identical spinless electrons (fermions) is given by the Slater determinant, $\psi(\mathbf{r}_1, \mathbf{r}_2, \cdots) = \det_{ij} \phi_i(\mathbf{r}_j)$ where $\phi_i(\mathbf{r})$ denote eigenstates of the single-particle problem.

Using the identity

$$\begin{vmatrix} 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = \prod_{j<k} (z_j - z_k) \,,$$

known as a **Vandemonde determinant**, show that the ground state wavefunction of the filled lowest Landau level is given by

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \cdots) = \prod_{j<k} (z_j - z_k) \exp\left[-\frac{1}{4} \sum_i |z_i|^2 \right] \,.$$

[2] In the symmetric gauge, we therefore find that the Landau level states are localized in both x and y directions. This contrasts with what was found from the Landau gauge condition where states were localized along only one direction. Of course, there is no contradiction between these two representations: since the Landau levels have a huge degeneracy, we are at liberty to reconstruct states within the basis.

Note that, as required, the wavefunction is antisymmetric under the exchange of any two fermions. This expression also emphasizes the exclusion character of the fermionic system with the wavefunction vanishing as any two particles move together.

5. Spin: This question develops the concept of a spinor wavefunction.

Using the Pauli matrices, σ_x, σ_y and σ_z, write down the operator corresponding to a component of spin along the axis (θ, ϕ) in spherical polar coordinates. Show that the eigenvalues of spin in this direction are $\pm\hbar/2$ (as expected), and deduce the corresponding wavefunctions. Hence, infer the wavefunctions for particles whose spins are aligned along the $+x$, $-x$, $+y$ and $-y$ directions.

Hint: You can find the relevant spin operator by taking the dot product of the vector $\boldsymbol{\sigma}$ with a unit vector in the desired direction.

6. Spin: This problem addresses the addition of spin angular momenta.

Consider two identical spin 1/2 fermions, and let $\chi_+(i)$ represent the state of particle i with spin up, and $\chi_-(i)$ the state with spin down. Write down the four possible states of the system which have definite exchange symmetry.

Show that $\hat{\mathbf{S}}^2 = \hat{S}_+\hat{S}_- + \hat{S}_z^2 - \hbar\hat{S}_z$, where the spin raising and lowering operators are given by $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$. Using this result, or otherwise, show that the four states of definite exchange symmetry are eigenstates of $\hat{\mathbf{S}}^2$ and find the corresponding eigenvalues and hence the total spin quantum number for each state.

At a given moment, the system is in a state,

$$\psi = \sqrt{\frac{2}{3}}\chi_+(1)\chi_-(2) + \sqrt{\frac{1}{3}}\chi_-(1)\chi_+(2)\,.$$

What is the probability of a measurement of the total spin giving the result $S = 1$?

7. †Spin: This is a challenging problem which addresses several important aspects of the coursework: quantum spin algebra, time-evolution, and spin precession in a field.

A Stern-Gerlach apparatus is used as a filter which rejects para-H_2 and passes molecules of ortho-H_2 (which has resultant nuclear spin one) with spin component $+\hbar$ in the x direction travelling in the y direction. A magnetic field B in the z direction acts over 20mm of path between two such filters in series, and it is found that no molecules of kinetic energy 0.025eV emerge when $B = 1.8(n + 1/2) \times 10^{-3}$T, where n is an integer. Explain this phenomenon and deduce a value for the magnetic moment of the proton. (Part II 1966)

Hint: Write the wavefunction corresponding to spin $+\hbar$ in the x direction using as a basis the eigenstates of $\hat{S}_z$. Then write the time dependence of this wavefunction in the presence of the uniform field B, and find the fraction of the $S_x = +\hbar$ state in this wavefunction at a later time. You can also obtain the same result from a classical precession argument.
You may note that $J_\pm|j, m\rangle = \hbar[j(j+1) - m(m+1)]^{1/2}|j, m\pm 1\rangle$.

8. Spin: We have seen that the "ladder operator" formalism provides a framework in which to define and classify the states of the quantum harmonic oscillator. In the following, we will see that the same ladder operator formalism provides a representation of the quantum spin algebra. This representation, known as the Holstein-Primakoff transformation, can be used to

develop a controlled perturbation theory of spin Hamiltonians. For present purposes, it also gives us an opportunity to practice the operator formalism and spin algebra.

According to the quantum spin algebra, the spin operators are defined by the commutation relations, $[\hat{S}^i, \hat{S}^j] = i\hbar\epsilon_{ijk}\hat{S}^k$, where ϵ^{ijk} denotes the antisymmetric tensor. According to the Holstein-Primakoff transformation,[3] it is stated that the quantum mechanical spin S operators can be represented by

$$\hat{S}^- = \hbar\sqrt{2S}\,a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2}, \qquad \hat{S}^+ = (\hat{S}^-)^\dagger, \qquad \hat{S}^z = \hbar(S - a^\dagger a),$$

where the ladder operators a and $a^\dagger$ obey the usual commutation relations, $[a, a^\dagger] \equiv aa^\dagger - a^\dagger a = 1$.[4] Making use of these relations, show that this definition is indeed consistent with the quantum spin algebra, i.e. $[\hat{S}^+, \hat{S}^-] = 2\hbar\hat{S}^z$.

Hint: If you find yourself expanding the square root, you should stop and consider whether there is a simpler method...

Comments: Physically, the ladder operators simply "count" the number of "spin deviations" away from $\hat{e}_z$. For large spin, S – the analogue the semi-classical limit for quantum mechanical spins – the Holstein-Primakoff transformation affords the expansion, $S^- = \hbar\sqrt{2S}\,a^\dagger + O(S^{-1/2})$ and $S^+ = \hbar\sqrt{2S}\,a + O(S^{-1/2})$. In this limit, quantum spin models typically become bilinear (i.e. quadratic) in ladder operators and can be "diagonalized" (i.e. solved) in the same manner as the quantum harmonic oscillator Hamiltonian.

9. Addition of angular momenta: The addition of two or more angular momenta is a common problem which will arise this year in atomic, molecular, nuclear and particle physics. It is therefore very important to understand the basic principles.

   Consider the addition of two angular momenta, $\ell_1 = 1$ and $\ell_2 = 2$. According to quantum mechanics, the possible values for the total angular momentum quantum number L range from $\ell_1 + \ell_2$ to $|\ell_1 - \ell_2|$, i.e. 3, 2, 1 in this case. Tabulate the possible values of the corresponding quantum numbers m_1, m_2 and $M = m_1 + m_2$ (i.e. those relating to $\hat{L}_z$), and show that the values of M correspond to the expected values of L. Repeat for the case $\ell_1 = 3$, $\ell_2 = 1$.

   For the case $\ell_1 = 2$ and $\ell_2 = 1$, the state $|L = 3, M = 3\rangle$ can be written down straightforwardly as $|\ell_1 = 2, m_1 = 2\rangle \otimes |\ell_2 = 1, m_2 = 1\rangle$. Use ladder operators to construct explicitly the state $|L = 3, M = 2\rangle$ and then orthogonality to construct the state $|L = 2, M = 2\rangle$ in terms of the $|\ell_1, m_1\rangle \otimes |\ell_2, m_2\rangle$ states. This method can obviously be extended to construct all such states.

10. Addition of angular momenta:

    Write down the commutation relations between the angular momentum operators J_x, J_y, and J_z (hats not shown!).

[3] T. Holstein and H. Primakoff, *Field dependence of the intrinsic domain magnetization of a ferromagnet*, Phys. Rev. **58**, 1098 (1940).

[4] If you feel disturbed by the concept of the "square root of an operator", you should think of it as defined by a Taylor expansion in the argument, $\frac{a^\dagger a}{2s}$.

(a) Show that the operators $J_\pm = J_x \pm iJ_y$ act as raising and lowering operators for the z-component of angular momentum, by first calculating the commutator $[J_z, J_\pm]$.

(b) State the allowed values of the total spin angular momentum for a system of three electrons.

(c) The 'coupled basis' state $|S = 3/2, m_S = 3/2\rangle$ (an eigenstate of total spin) is also a state of the 'uncoupled basis', which may be denoted by $|\uparrow\uparrow\uparrow\rangle \equiv |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle$. By an application of total spin lowering operator, show that

You may note that $J_\pm|j,m\rangle = \hbar[j(j+1)-m(m+1)]^{1/2}|j,m\pm1\rangle$.

$$|S = 3/2, m_S = 1/2\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right) .$$