Synopsis: Lectures 5-10

5. **Charged particle in an electromagnetic field:**
   Classical and quantum mechanics of particle in a field; normal Zeeman effect; gauge invariance and the Aharonov-Bohm effect; Landau levels.

6. **Spin:**
   Stern-Gerlach experiment; spinors, spin operators and Pauli matrices; spin precession in a magnetic field; parametric resonance; addition of angular momenta.

7. **Time-independent perturbation theory:**
   Perturbation series; first and second order expansion; degenerate perturbation theory; Stark effect; nearly free electron model.

8. **Variational and WKB method:**
   Variational method: ground state energy and eigenfunctions; application to helium; Semiclassics and the WKB method.

Lecture 8

Approximation methods for stationary states

Approximation methods: outline

- We have succeeded in developing formal analytical solutions for stationary states of Schrödinger operator, $\hat{H}$ in variety of settings.

- But majority of "real-life" applications are formally intractable!

  e.g. the "three-body problem" already non-integrable – rules out exact solution for non-hydrogenic atoms!

- It is therefore essential to develop approximation schemes: In the following, we wil develop three complementary approaches:

  1. Perturbation series expansion (degenerate and non-degenerate)
  2. Variational method
  3. WKB approximation

Time-independent perturbation theory

- In the perturbative series expansion, states of $\hat{H}$ obtained through sequence of corrections to some reference, $\hat{H}_0$, for which states are known.

- Although perturbative scheme is effective, there are – typically very interesting – problems which cannot be solved using this approach.

  e.g. in 1d, arbitrarily weak attractive potential causes $k = 0$ free particle state to drop below continuum and become bound.

- **Adiabatic continuity:** In general, perturbation theory useful (valid) when states of unperturbed system, $\hat{H}_0$, flow smoothly into states of $\hat{H}$ – viz. symmetries, node structures, etc.

Perturbation series expansion

- Consider unperturbed Hamiltonian, $\hat{H}^{(0)}$, having known eigenstates $|n^{(0)}\rangle$ and eigenvalues $E_n^{(0)}$,

$$\hat{H}^{(0)}|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle$$

- How are the eigenstates and eigenenergies modified by small perturbation, $\hat{H}^{(1)}$?

$$(\hat{H}^{(0)} + \hat{H}^{(1)})|n\rangle = E_n|n\rangle$$

e.g. external electric or magnetic field applied to charged particle

- If perturbation small, $\langle n^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle \ll E_n^{(0)}$, on "turning on" $\hat{H}^{(1)}$, expect eigenfunctions and eigenvalues to change adiabatically:

$$|n^{(0)}\rangle \longmapsto |n\rangle, \qquad E_n^{(0)} \longmapsto E_n$$

Perturbation series expansion

- Basic assumption: for $\hat{H}^{(1)}$ small, leading corrections are $O(\hat{H}^{(1)})$
- Perturbed eigenfunctions and eigenvalues obtained by successive series of corrections, each $O(\langle \hat{H}^{(1)} \rangle / \langle \hat{H}^{(0)} \rangle)$ compared with previous.
- To identify terms of same order in $\langle \hat{H}^{(1)} \rangle / \langle \hat{H}^{(0)} \rangle$, convenient to extract from $\hat{H}^{(1)}$ a dimensionless parameter λ characterising relative magnitude of perturbation and then expand:

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + \cdots = \sum_{m=0}^{\infty} \lambda^m |n^{(m)}\rangle$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots = \sum_{m=0}^{\infty} \lambda^m E_n^{(m)}$$

Perturbation series expansion

$$|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots = \sum_{m=0}^{\infty} \lambda^m |n^{(m)}\rangle$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots = \sum_{m=0}^{\infty} \lambda^m E_n^{(m)}$$

- Applied to Schrödinger equation, $(\hat{H}^{(0)} + \lambda\hat{H}^{(1)})|n\rangle = E_n|n\rangle$

$$\begin{aligned}&(\hat{H}^{(0)} + \lambda\hat{H}^{(1)})(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots) \\ &= (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots)(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots)\end{aligned}$$

and then relate terms of equal order in λ, e.g. to $O(\lambda)$

$$\hat{H}^{(0)}|n^{(1)}\rangle + \hat{H}^{(1)}|n^{(0)}\rangle = E_n^{(0)}|n^{(1)}\rangle + E_n^{(1)}|n^{(0)}\rangle$$

First order perturbation theory

- To $O(\lambda)$, taking inner product with $\langle n^{(0)}|$

$$\hat{H}^{(0)}|n^{(1)}\rangle + \hat{H}^{(1)}|n^{(0)}\rangle = E_n^{(0)}|n^{(1)}\rangle + E_n^{(1)}|n^{(0)}\rangle$$

$$\underbrace{\langle n^{(0)}|\hat{H}^{(0)}}_{=\langle n^{(0)}|E_n^{(0)}}|n^{(1)}\rangle + \langle n^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle = E_n^{(0)}\langle n^{(0)}|n^{(1)}\rangle + E_n^{(1)}\underbrace{\langle n^{(0)}|n^{(0)}\rangle}_{=1}$$

$$E_n^{(1)} = \langle n^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle$$

- If instead take the inner product with $\langle m^{(0)}|$ $(m \neq n)$

$$\underbrace{\langle m^{(0)}|\hat{H}^{(0)}}_{\langle m^{(0)}|E_m^{(0)}}|n^{(1)}\rangle + \langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle = E_n^{(0)}\langle m^{(0)}|n^{(1)}\rangle + E_n^{(1)}\underbrace{\langle m^{(0)}|n^{(0)}\rangle}_{=0}$$

$$\langle m^{(0)}|n^{(1)}\rangle = \frac{\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle}{}$$

First order perturbation theory

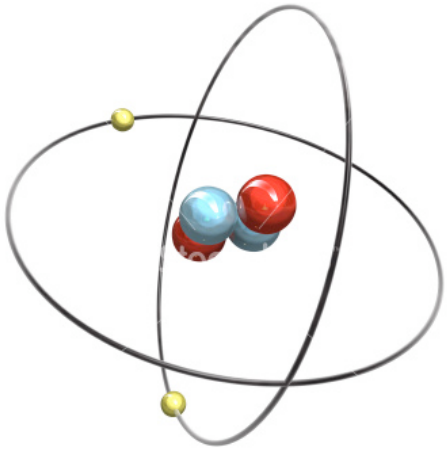
For $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$,

$$E_n \simeq E_n^{(0)} + \langle n^{(0)} | \hat{H}^{(1)} | n^{(0)} \rangle$$

$$|n\rangle \simeq |n^{(0)}\rangle + \sum_{m \neq n} |m^{(0)}\rangle \frac{\langle m^{(0)} | \hat{H}^{(1)} | n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}}$$

Example: Ground state energy of Helium atom

- Helium atom, two electrons bound to nucleus of two protons and two neutrons ($Z = 2$)



- To leading order in electron-electron interaction, what is ground state energy?

  - Without Coulomb interaction between electrons, ground state:

$$|\mathrm{g.s.}^{(0)}\rangle = \frac{1}{\sqrt{2}}\left(|100,\uparrow\rangle \otimes |100,\downarrow\rangle - |100,\downarrow\rangle \otimes |100,\uparrow\rangle\right)$$

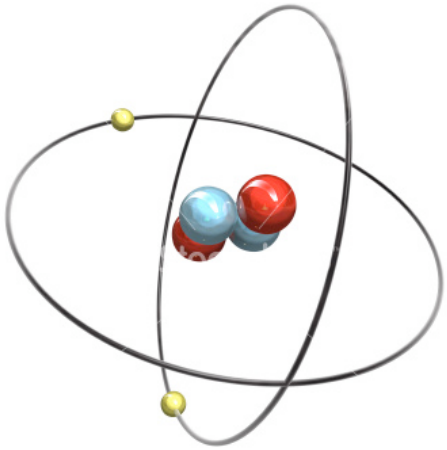
where $E^{(0)} = -2 \times Z^2 \times \mathrm{Ry}$, with $\mathrm{Ry} = \dfrac{e^2}{4\pi\epsilon_0}\dfrac{1}{2a_0}$, $a_0 = \dfrac{4\pi\epsilon_0}{e^2}\dfrac{\hbar^2}{m_e}$ and

$$\psi_{100}(\mathbf{r}) \equiv \langle \mathbf{r}|n = 1, \ell = 0, m = 0\rangle = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}$$

Example: Ground state energy of Helium atom

- Treating electron-electron interaction as perturbation,

$$\hat{H}^{(1)} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$



  - To first order, energy shift: $E^{(1)} = \langle \mathrm{g.s.}^{(0)}|\hat{H}^{(1)}|\mathrm{g.s.}^{(0)}\rangle$

$$E^{(1)} = \frac{e^2}{4\pi\epsilon_0}\left(\frac{Z^3}{\pi a_0^3}\right)^2 \int d^3r_1 d^3r_2 \frac{e^{-2Z(r_1+r_2)/a_0}}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{e^2}{4\pi\epsilon_0}\frac{Z}{2a_0} \times C_0$$

$$\text{with } C_0 = \frac{1}{(4\pi)^2}\int d^3z_1 d^3z_2 \frac{e^{-(z_1+z_2)}}{|\mathbf{z}_1 - \mathbf{z}_2|} = \frac{5}{4}\ ,\ E^{(1)} = \frac{5}{4}Z\,\mathrm{Ry}$$

- $E \simeq (2Z^2 - \frac{5}{4}Z)\,\mathrm{Ry} = -5.5\,\mathrm{Ry}$ (cf. experiment: $-5.807\mathrm{Ry}$)

Second order perturbation theory

- Recall expansion: $(\hat{H}^{(0)} + \lambda\hat{H}^{(1)})|n\rangle = E_n|n\rangle$

$$(\hat{H}^{(0)} + \lambda\hat{H}^{(1)})(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots)$$
$$= (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots)(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots)$$

- To $O(\lambda^2)$, taking inner product with $\langle n^{(0)}|$

$$\hat{H}^{(0)}|n^{(2)}\rangle + \hat{H}^{(1)}|n^{(1)}\rangle$$
$$= E_n^{(0)}|n^{(2)}\rangle + E_n^{(1)}|n^{(1)}\rangle + E_n^{(2)}|n^{(0)}\rangle$$

$$\underbrace{\langle n^{(0)}|\hat{H}^{(0)}}_{=\langle n^{(0)}|E^{(0)}}|n^{(2)}\rangle + \langle n^{(0)}|\hat{H}^{(1)}|n^{(1)}\rangle$$
$$= E_n^{(0)}\langle n^{(0)}|n^{(2)}\rangle + E_n^{(1)}\langle n^{(0)}|n^{(1)}\rangle + E_n^{(2)}\underbrace{\langle n^{(0)}|n^{(0)}\rangle}_{=1}$$

- Collecting terms:

$$E_n^{(2)} = \langle n^{(0)}|\hat{H}^{(1)}|n^{(1)}\rangle - E_n^{(1)}\langle n^{(0)}|n^{(1)}\rangle$$

Second order perturbation theory

$$E_n^{(2)} = \langle n^{(0)}|\hat{H}^{(1)}|n^{(1)}\rangle - E_n^{(1)}\langle n^{(0)}|n^{(1)}\rangle$$

- Although we have assumed normalization of $|n^{(0)}\rangle$, we have said nothing about $|n\rangle$.
- While we would want to fix normalization of $|n\rangle$ eventually to unity, it is convenient to impose "normalization":

$$\langle n^{(0)}|n\rangle = 1 = \langle n^{(0)}|n^{(0)}\rangle + \lambda\langle n^{(0)}|n^{(1)}\rangle + \lambda^2\langle n^{(0)}|n^{(2)}\rangle + \cdots$$

i.e. $\langle n^{(0)}|n^{(1)}\rangle = \langle n^{(0)}|n^{(2)}\rangle = \cdots = 0$, and

$$E_n^{(2)} = \langle n^{(0)}|\hat{H}^{(1)}|n^{(1)}\rangle = \langle n^{(0)}|\hat{H}^{(1)} \sum_{m\neq n} |m^{(0)}\rangle \frac{\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}$$

i.e. $$\boxed{E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}}$$

Second order perturbation theory

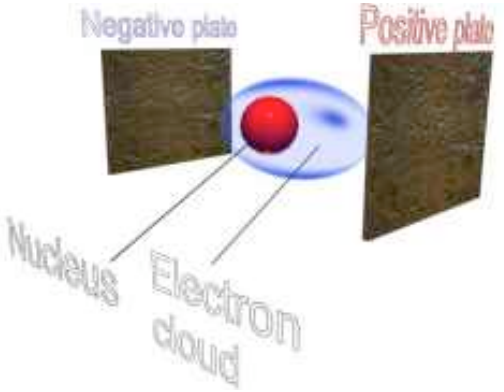
$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Remarks:

- For ground state, second order energy shift always negative.
- If matrix elements of $\hat{H}^{(1)}$ are of comparable magnitude, neighbouring levels make larger contribution in second order perturbation theory than distant levels.
- If portion of states belong to continuum, sum replaced by intergral.
- Levels that lie in close proximity tend to be pushed apart – degeneracies are never accidental!

Example: Quadratic Stark Effect

- What is influence of external electric field on ground state of hydrogen atom?



- Electron and proton drawn in different directions by field
  $\rightsquigarrow$ creation of dipole, $\hat{\mathbf{d}} = q\mathbf{r}$, which lowers energy.
- To explore effect, we can treat field, $\mathbf{E} = \mathcal{E}\hat{\mathbf{e}}_z$, as a perturbation:

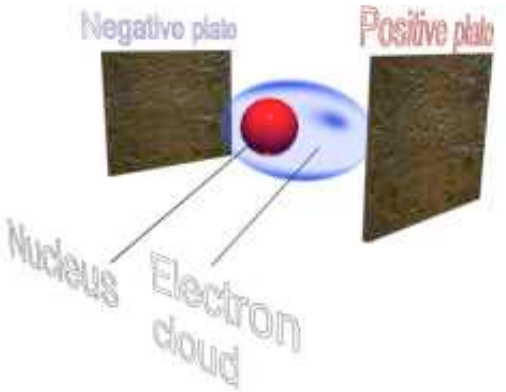
$$\hat{H}^{(1)} = -\mathbf{E}\cdot\mathbf{d} = -q\mathcal{E}z = -q\mathcal{E}r\cos\theta, \qquad q = -e$$

- Unperturbed energy spectrum: $E^{(0)}_{n\ell m} \equiv E^{(0)}_n = -\frac{\mathrm{Ry}}{n^2}$,
  ground state energy $E^{(0)} \equiv E^{(0)}_{100} = -\mathrm{Ry}$.
- At first order in field, $E^{(1)}_1 = \langle 100|e\mathcal{E}z|100\rangle = 0$ by symmetry.

Example: Quadratic Stark Effect

- At second order of perturbation theory, (neglecting continuum of unbound states),

$$E_1^{(2)} = \sum_{n\neq 1,\ell,m} \frac{|\langle n\ell m|e\mathcal{E}z|100\rangle|^2}{E_1^{(0)} - E_n^{(0)}}$$



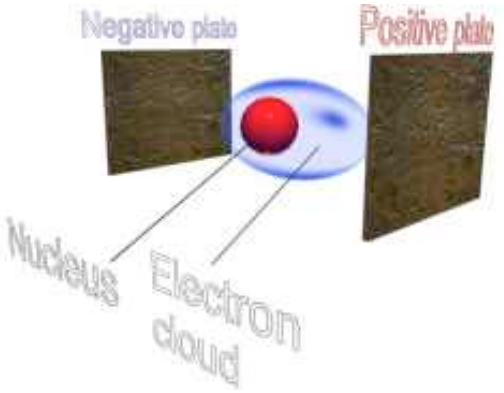
where $|n\ell m\rangle$ denote hydrogen bound states.

- Although $E_1^{(2)}$ can be evaluated exactly (using various tricks), we can place a strong bound by a simpler argument.
- Since, for $n > 2$, $|E_1^{(0)} - E_n^{(0)}| > |E_1^{(0)} - E_2^{(0)}|$,

$$|E_1^{(2)}| < \frac{1}{E_2^{(0)} - E_1^{(0)}} \sum_{n\neq 1,\ell,m} \langle 100|e\mathcal{E}z|n\ell m\rangle\langle n\ell m|e\mathcal{E}z|100\rangle$$

Example: Quadratic Stark Effect

$$|E_1^{(2)}| < \frac{1}{E_2^{(0)} - E_1^{(0)}} \times \sum_{n\neq 1,\ell,m} \langle 100|e\mathcal{E}z|n\ell m\rangle\langle n\ell m|e\mathcal{E}z|100\rangle$$

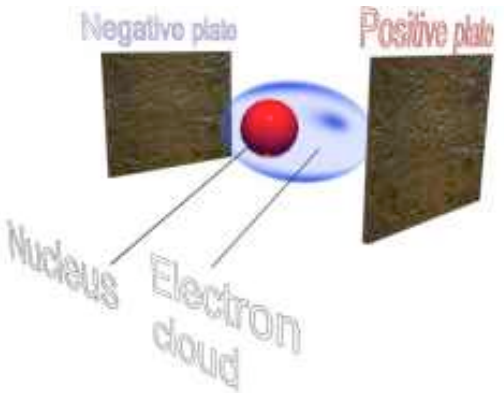


- Using $\sum_{n\ell m} |n\ell m\rangle\langle n\ell m| = \mathbb{I} = \sum_{n\neq 1,\ell,m} |n\ell m\rangle\langle n\ell m| + |100\rangle\langle 100|$,

$$\begin{aligned}
|E_1^{(2)}| &< \frac{1}{E_2^{(0)} - E_1^{(0)}} \langle 100|e\mathcal{E}z\left(\mathbb{I} - |100\rangle\langle 100|\right) e\mathcal{E}z|100\rangle \\
&< \frac{1}{E_2^{(0)} - E_1^{(0)}} \left[\langle 100|(e\mathcal{E}z)^2|100\rangle - \left(\langle 100|e\mathcal{E}z|100\rangle\right)^2\right] \\
&< \frac{1}{E_2^{(0)} - E_1^{(0)}} \langle 100|(e\mathcal{E}z)^2|100\rangle
\end{aligned}$$

Example: Quadratic Stark Effect

$$|E_1^{(2)}| < \frac{1}{E_2^{(0)} - E_1^{(0)}} \langle 100|(e\mathcal{E}z)^2|100\rangle$$



$$E_1^{(2)} = \sum_{n\neq 1,\ell,m} \frac{|\langle n\ell m|e\mathcal{E}z|100\rangle|^2}{E_1^{(0)} - E_n^{(0)}}$$

- With $\langle 100|z^2|100\rangle = a_0^2$, $E_1^{(0)} = -\dfrac{e^2}{4\pi\epsilon_0}\dfrac{1}{2a_0} = -\mathrm{Ry}$, $E_2^{(0)} = \dfrac{E_1^{(0)}}{4}$,

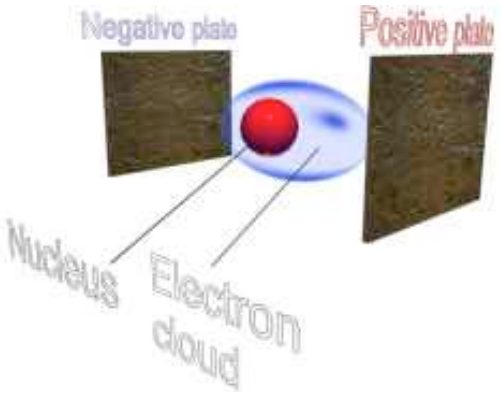
$$|E_1^{(2)}| < \frac{1}{\frac{3}{4}e^2/8\pi\epsilon_0 a_0}(e\mathcal{E})^2 a_0^2 = \frac{8}{3}4\pi\epsilon_0\mathcal{E}^2 a_0^3$$

- Furthermore, since all terms in perturbation series for $E^{(2)}$ are negative, first term sets lower bound,

$$|E_1^{(2)}| > \frac{|\langle 210|e\mathcal{E}z|100\rangle|^2}{E_2^{(0)} - E_1^{(0)}}$$

Example: Polarizability

$$|E_1^{(2)}| < \frac{8}{3} 4\pi\epsilon_0 \mathcal{E}^2 a_0^3$$



- In general, the induced electric dipole moment in an electric field is given by $\mathbf{d} = \alpha\epsilon_0\mathbf{E}$ where α is the polarizability (generally a tensor).
- The energy shift created by an electric field on a dipole is given by

$$\Delta E = -\frac{1}{2}\mathbf{d}\cdot\mathbf{E} = -\frac{1}{2}\alpha\epsilon_0\mathcal{E}^2$$

- Perturbation series expansion shows that

$$\alpha \simeq \frac{64}{3}\pi a_0^3$$

Summary: perturbation series

How are states of a quantum system modified by the action of a small perturbation $\hat{H}^{(1)}$?

$$(\hat{H}^{(0)} + \hat{H}^{(1)})|n\rangle = E_n|n\rangle$$

- If perturbed states are "adiabatically" connected to unperturbed, corrections can be treated through series expansion in $\langle \hat{H}^{(1)} \rangle / \langle \hat{H}^{(0)} \rangle$.
- At second order,

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Degenerate Perturbation Theory

- Validity of perturbation series expansion relies upon matrix elements being smaller than corresponding energy level differences, e.g.

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

- If unperturbed states $|m^{(0)}\rangle$ and $|n^{(0)}\rangle$ are degenerate, and $\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle \neq 0$, perturbation theory fails!

- However, problem is easily fixed...

Degenerate Perturbation Theory

- ... to understand how, consider two-dimensional oscillator,

$$\hat{H}^{(0)} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$$

  wavefunctions are separable into two one-dimensional oscillators.

- Setting $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ and $\eta = \sqrt{\frac{m\omega}{\hbar}}y$, ground state given by $|0,0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-(\xi^2+\eta^2)/2}$, and two *degenerate* first excited states,

$$\left\{ \begin{array}{l} |1,0\rangle \\ |0,1\rangle \end{array} \right. = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-(\xi^2+\eta^2)/2} \left\{ \begin{array}{l} \xi \\ \eta \end{array} \right.$$

- Consider effect of perturbation,

$$\hat{H}^{(1)} = \alpha m\omega^2 xy$$

  controlled by a small parameter α.

Degenerate Perturbation Theory

$$\hat{H}^{(0)} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2), \qquad \hat{H}^{(1)} = \alpha m\omega^2 xy$$

- By symmetry, $\langle 0,0|\hat{H}^{(1)}|0,0\rangle = \langle 1,0|\hat{H}^{(1)}|1,0\rangle = \langle 0,1|\hat{H}^{(1)}|0,1\rangle = 0$
  $\therefore$ according to a naïve perturbation theory, no first-order correction.

$$E_n^{(1)} = \langle n^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle = 0$$

- However, second-order correction appears to diverge!

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

since $\langle 1,0|\hat{H}^{(1)}|0,1\rangle \neq 0$, but $|0,1\rangle$ and $|1,0\rangle$ are degenerate

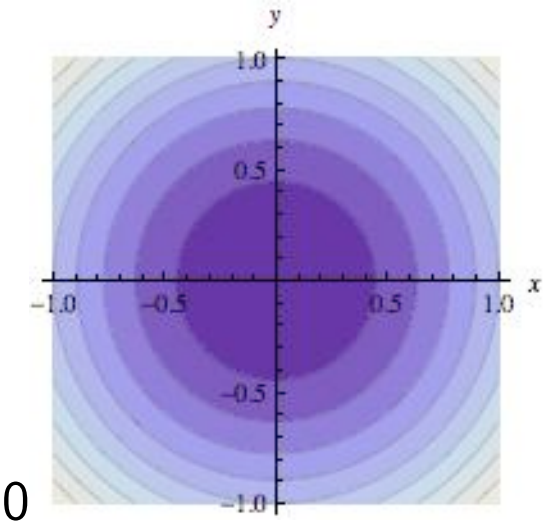
- Yet we know that perturbation will not spoil two-dimensional harmonic oscillator – so what's gone wrong with approach?

Degenerate Perturbation Theory

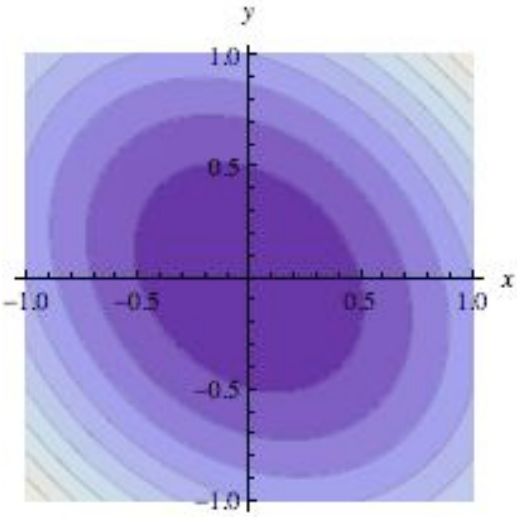
$$\hat{H}^{(0)} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2), \qquad \hat{H}^{(1)} = \alpha m\omega^2 xy$$

- Consider contours of constant potential energy:

$\alpha = 0$



$\alpha \neq 0$



- For $\alpha \neq 0$, circles of constant potential become ellipses, with axes aligned along $x = \pm y$.
- As soon as perturbation is introduced, eigenstates lie in direction of the new elliptic axes – switch not proportional to α.
- But original unperturbed problem had circular symmetry, and there was no particular reason to choose axes along x and y.

Degenerate Perturbation Theory

$$\hat{H}^{(0)} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2), \qquad \hat{H}^{(1)} = \alpha m\omega^2 xy$$

- By rearranging the coordinates along the principle axes,

$$\frac{1}{2}m\omega^2(x^2 + y^2) + \alpha m\omega^2 xy$$
$$= \frac{1}{2}m\omega^2\left[(1+\alpha)\left(\frac{x+y}{\sqrt{2}}\right)^2 + (1-\alpha)\left(\frac{x-y}{\sqrt{2}}\right)^2\right]$$

despite the results of naïve first order perturbation theory, there is indeed a first order energy shift:

$$\hbar\omega \to \hbar\omega\sqrt{1 \pm \alpha} \approx \hbar\omega(1 \pm \alpha/2)$$

Degenerate Perturbation Theory: formal

- So, generally, suppose we have a Hamiltonian, $\hat{H}^{(0)}$ in which the following states $|n_a^{(0)}\rangle, |n_b^{(0)}\rangle, \cdots |n_k^{(0)}\rangle$ are degenerate,

$$\hat{H}^{(0)}|n_i^{(0)}\rangle = \epsilon|n_i^{(0)}\rangle$$

- Since perturbation theory is an expansion in

$$\frac{\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}$$

the theory will fail. Instead we can proceed by transforming basis states of the degenerate subspace, $|n_i^{(0)}\rangle$, to eigenstates of $\hat{H}^{(1)}$, $|n_\alpha^{(0)}\rangle$

$$\langle n_\alpha^{(0)}|\hat{H}^{(1)}|n_\beta^{(0)}\rangle = H_\alpha^{(1)}\delta_{\alpha\beta}$$

Degenerate Perturbation Theory: formal

$$\langle n_\alpha^{(0)} | \hat{H}^{(1)} | n_\beta^{(0)} \rangle = H_\alpha^{(1)} \delta_{\alpha\beta}$$

- The eigenvalues $H_\alpha^{(1)}$ are determined by the **secular equation**,

$$\det(H_{ij}^{(1)} - H_\alpha^{(1)}) = 0$$

  where the matrix elements $H_{ij}^{(1)} = \langle n_i^{(0)} | \hat{H}^{(1)} | n_j^{(0)} \rangle$ involve *only the degenerate states in the original basis.*

- The new states $|n_\alpha^{(0)}\rangle = \sum_i c_{i\alpha} |n_i^{(0)}\rangle$, defined by the eigenstates $c_{i\alpha}$ of $H_{ij}^{(1)}$ now define a non-degenerate basis in which one can develop a perturbative series expansion involving all states.
- In practice, this change of basis is often sufficient.

Example I: Two-dimensional oscillator

$$\hat{H}^{(0)} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2), \qquad \hat{H}^{(1)} = \alpha m\omega^2 xy$$

- Working in the degenerate subspace of the first excited states, $|1,0\rangle$ and $|0,1\rangle$, the diagonal matrix elements of $\hat{H}^{(1)}$ vanish, $\langle 1,0|\hat{H}^{(1)}|1,0\rangle = \langle 0,1|\hat{H}^{(1)}|0,1\rangle = 0$.
- However, off-diagonal matrix elements are non-zero,

$$\langle 1,0|\hat{H}^{(1)}|0,1\rangle = \langle 0,1|\hat{H}^{(1)}|1,0\rangle = \hbar\omega\frac{\alpha}{2}$$

- In *two*-dimensional degenerate subspace spanned by $|1,0\rangle$ and $|0,1\rangle$,

$$H_{ij}^{(1)} = \hbar\omega\frac{\alpha}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Eigenstates $(|1,0\rangle \pm |0,1\rangle)/\sqrt{2}$ have eigenvalue $E_1^{(1)} = \pm\hbar\omega\frac{\alpha}{2}$, i.e. $E_1 = E_1^{(0)} + E_1^{(1)} = \hbar\omega(1 \pm \alpha/2)$ as expected from exact solution.

Example II: Linear Stark Effect

- Previously, we used second order perturbation theory to explore influence of a static electric field on the polarizability of the hydrogen atom ground state. There we showed that the leading correction to the energy scaled quadratically with field, $\mathcal{E}^2$.
- But how does the electric field influence the excited states?
- For atomic hydrogen, the four unperturbed $n = 2$ states are all degenerate with $E_2^{(0)} = -\frac{1}{4}\mathrm{Ry}$.
- How are these states influenced by a weak electric field?

$$\hat{H}^{(1)} = e\mathcal{E}\, r \cos\theta$$

- To answer this question, we must turn to degenerate perturbation theory.

Example II: Linear Stark Effect

$$\hat{H}^{(1)} = e\mathcal{E} r \cos\theta$$

- To implement degenerate perturbation theory, we must find matrix elements $\langle n_i^{(0)}|\hat{H}^{(1)}|n_j^{(0)}\rangle$ on the degenerate subspace.
- From the structure of the four $n = 2$ states,

$$\begin{cases} \psi_{200}(\mathbf{r}) \\ \psi_{210}(\mathbf{r}) \\ \psi_{21,\pm1}(\mathbf{r}) \end{cases} = \left(\frac{1}{32\pi a_0^3}\right)^{1/2} e^{-r/2a_0} \begin{cases} \left(2 - \frac{r}{a_0}\right) \\ \frac{r}{a_0}\cos\theta \\ \frac{r}{a_0} e^{\pm i\phi}\sin\theta \end{cases} .$$

  and the symmetry of $\hat{H}^{(1)}$, it is clear that only the matrix element $\Delta = \langle 200|\hat{H}^{(1)}|210\rangle$ is non-vanishing (and given by $\Delta = 3e\mathcal{E}a_0$).
- In *two*-dimensional degenerate subspace spanned by $|200\rangle$ and $|210\rangle$,

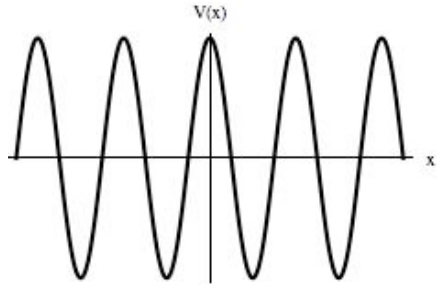
$$H_{ij}^{(1)} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

- Eigenstates: $(|200\rangle \pm |210\rangle)/\sqrt{2}$ with energy $\pm\Delta$ linear in $\mathcal{E}$.

Example III: Nearly free electron model

- How is free quantum particle influenced by weak periodic potential, $V(x) = 2V\cos(2\pi x/a)$?

  cf. conduction electrons in simple crystalline solid



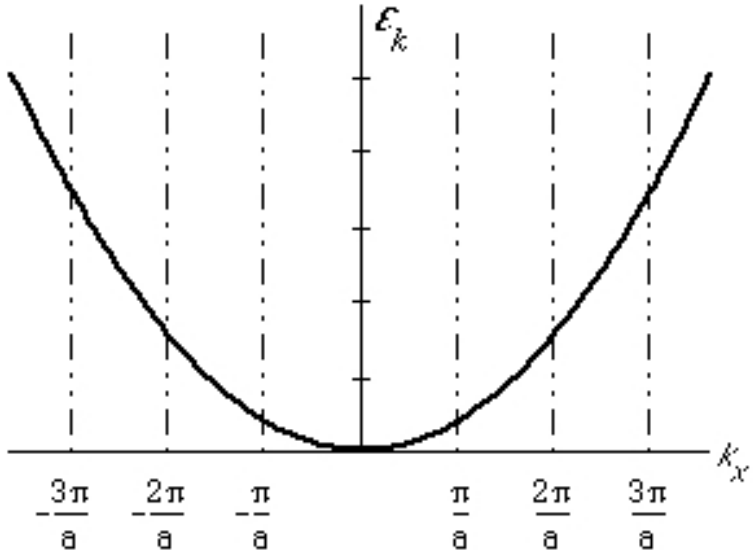
- If $V \ll E$, $V(x)$ may be treated as a perturbation of $\hat{H}_0 = \dfrac{\hat{\mathbf{p}}^2}{2m}$
- With periodic b.c., $L = Na$, free particle states are plane waves $\psi_k(x) = \langle x|k\rangle = \dfrac{1}{\sqrt{L}}e^{ikx}$, with $k = 2\pi n/L$, n integer, $E_k^{(0)} = \dfrac{\hbar^2 k^2}{2m}$.
- Matrix elements of $V(x)$:

$$\boxed{\langle k|V|k'\rangle = V\delta_{k'-k,\pm 2\pi/a}}$$

  i.e. only states separated by $G \equiv 2\pi/a$ coupled by perturbation.

Example III: Nearly free electron model

$$\langle k|V|k'\rangle = V\delta_{k'-k,\pm G}, \qquad G = 2\pi/a$$

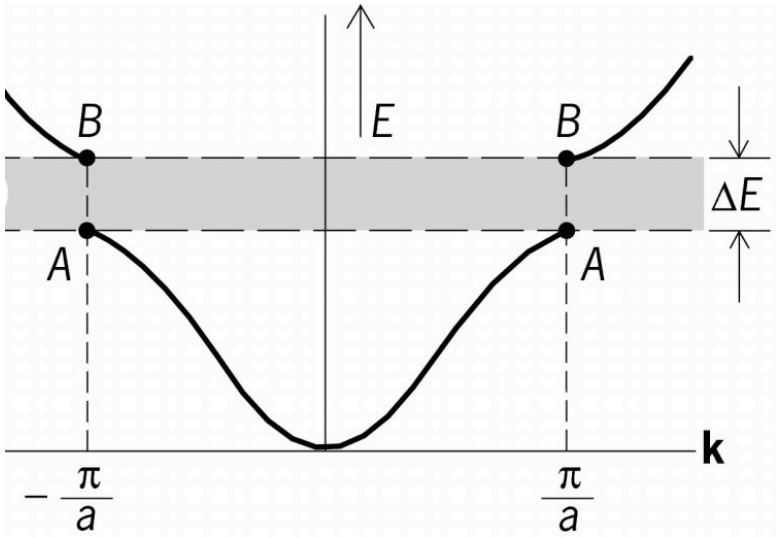


- In particular, the degenerate states with $k = \pm\pi/a$ are coupled by $V(x)$ – demands application of degenerate perturbation theory.
- Taking all pairs of coupled states, $|k = G/2 + q\rangle$, $|k = -G/2 + q\rangle$, in neighbourhood of $k = \pm G/2$ (i.e. q small), the matrix elements of the total Hamiltonian $\hat{H} = \hat{H}^{(0)} + V(x)$ are given by

$$H_q = \begin{pmatrix} E^{(0)}_{G/2+q} & V \\ V & E^{(0)}_{-G/2+q} \end{pmatrix}$$

Example III: Nearly free electron model

$$H_q = \begin{pmatrix} E^{(0)}_{G/2+q} & V \\ V & E^{(0)}_{-G/2+q} \end{pmatrix}$$



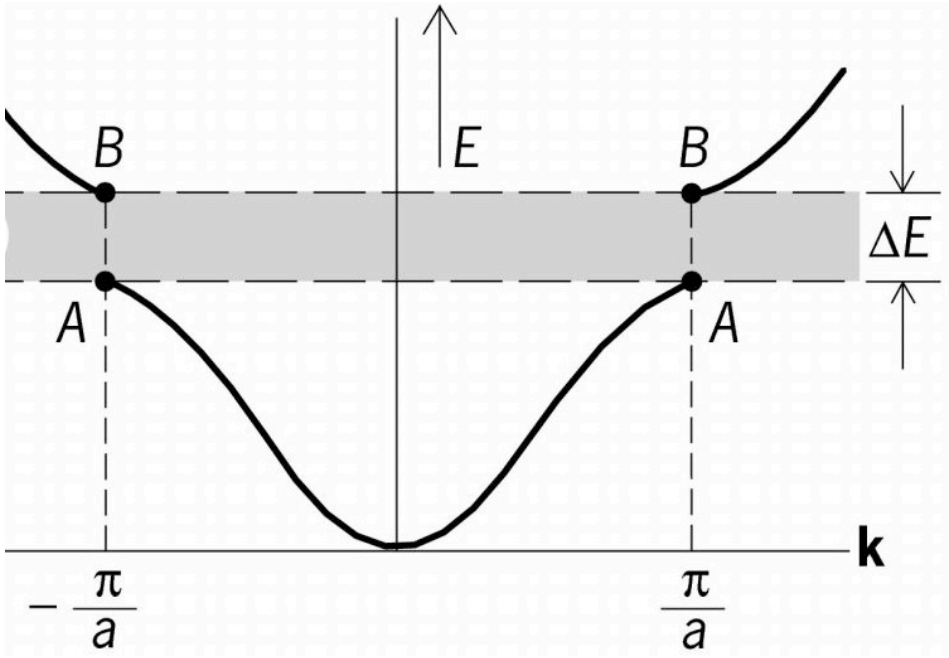
- Solving secular equation $\det(H_q - E_q \mathbb{I}) = 0$, obtain

$$E^{\pm}_q = \frac{E^{(0)}_{G/2+q} + E^{(0)}_{-G/2+q}}{2} \pm \left[\left(\frac{E^{(0)}_{G/2+q} - E^{(0)}_{-G/2+q}}{2} \right)^2 + V^2 \right]^{1/2}$$

- i.e. for $k = \pm G/2$ ($q = 0$), degeneracy lifted by potential. For $|q| \ll G$, spectrum has a gap of size $\Delta E = 2V$, cf. Kronig-Penney.

Example III: Nearly free electron model

- Appearance of gap has important consequences in theory of solids.



- Electrons are fermions and have to obey Pauli's exclusion principle.
- **Metal:** electrons occupy states up to "Fermi" energy; low-energy excitations allows current flow in electric field.
- **Band insulator:** When Fermi energy lies in gap, electric field is unable to create excitations and induce current flow.

Summary: perturbation series

How are states of a quantum system modified by the action of a small perturbation $\hat{H}^{(1)}$?

$$(\hat{H}^{(0)} + \hat{H}^{(1)})|n\rangle = E_n|n\rangle$$

- If perturbed states are "adiabatically" connected to unperturbed, corrections can be treated through series expansion in $\langle \hat{H}^{(1)} \rangle / \langle \hat{H}^{(0)} \rangle$.
- At second order,

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

- When the perturbation acts on system with degeneracies, perturbation theory must be developed using basis of degenerate subspace in which perturbation is diagonal.

Lecture 9

Approximation methods: Variational method

Variational method

- So far, we have focused on perturbative scheme in which states of non-perturbed system provid platform, i.e. unperturbed states mirror those of new Hamiltonian – adiabatic contunity.

  e.g. anharmonic oscillator

- However, often new states may not be adiabatically connected.

  e.g. nucleation of bound states, or strongly interacting quantum systems where many-particle correlations can effect transitions to new states such as superfluid or fractional quantum Hall fluid – typically associated with breaking of fundamental symmetry.

- To address such systems it is often extremely effective to "guess" and then optimize a trial ground state wavefunction
  – the variational method.

Variational method

- Method involves optimization of some trial wavefunction, $|\psi_{\text{trial}}\rangle$, on the basis of one or more adjustable "variational" parameters.
- Achieved by minimizing expectation value of the energy,

$$E = \langle \psi_{\text{trial}} | \hat{H} | \psi_{\text{trial}} \rangle$$

  of the trial wavefunction.
- Seemingly crude approach can provide a (surprisingly) good approximation to the ground state energy (but not quite so good for wavefunction).

- Crucially, the method can be extended to many-particle quantum, and to problems for which a perturbative expansion is invalid.

Variational method

- Consider Hamiltonian $\hat{H}$ with (unknown) eigenstates, $|n\rangle$ and energy E_n. A normalized trial state $|\psi(\alpha)\rangle$ (a function of some set of parameters α) can be expanded as

$$|\psi(\alpha)\rangle = \sum_n a_n(\alpha)|n\rangle$$

with the normalization $\sum_n |a_n|^2 = 1$.

- Therefore, for *any* $|\psi(\alpha)\rangle$,

$$E(\alpha) \equiv \langle\psi(\alpha)|\hat{H}|\psi(\alpha)\rangle = \sum_n |a_n|^2 E_n$$

- Then, since the ground state energy $E_0 \leq E_n$, we have

$$E(\alpha) \geq E_0 \sum_n |a_n|^2 = E_0$$

showing that $E(\alpha)$ places an upper bound on E_0.

Variational method

$$E(\alpha) \equiv \langle \psi(\alpha) | \hat{H} | \psi(\alpha) \rangle \geq E_0$$

- The variational method entails minimizing the function $E(\alpha)$ with respect to α from which follows an upper bound on ground state energy, E_0.
- Generally, the approach provides a good approximation to the energy, while the estimate of the wavefunction is less effective.
- e.g. if optimum state includes ca. 20% admixtue of first excited state $|\alpha_{\min}\rangle = \frac{1}{\sqrt{1+0.2^2}}(|0\rangle + 0.2|1\rangle)$, energy estimate will be too high by only,

$$\frac{1}{1+0.2^2}(E_0 + 0.2^2 E_1) - E_0 = \frac{0.2^2}{1+0.2^2}(E_1 - E_0)$$

typically a much smaller error.

Example I: Ground state of hydrogen

- Previously, we have shown that the radial Schrödinger equation for atomic hydrogen is given by

$$\left[-\frac{\hbar^2}{2m}\left(\partial_r^2+\frac{2}{r}\partial_r\right)+\frac{\hbar^2}{2mr^2}\ell(\ell+1)-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right]R(r)=ER(r)$$

- Setting $R(r)=u(r)/r$,

$$\left[-\frac{\hbar^2\partial_r^2}{2m}+\frac{\hbar^2}{2mr^2}\ell(\ell+1)-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\right]u(r)=uR(r)$$

- Finally,

$$\left[-\partial_r^2+\frac{1}{r^2}\ell(\ell+1)-\frac{e^2}{4\pi\epsilon_0}\frac{2m}{\hbar^2}\frac{1}{r}\right]u(r)=\frac{2mE}{\hbar^2}u(r)$$

  introducing coordinate, $\rho=r/a_0$, where $a_0=\frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{m}$ is Bohr radius,

$$\boxed{\left[-\partial_\rho^2+\frac{1}{\rho^2}\ell(\ell+1)-\frac{2}{\rho}\right]u(\rho)=\frac{2mEa_0^2}{\hbar^2}u(\rho)}$$

Example I: Ground state of hydrogen

$$\left[-\partial_\rho^2 + \frac{1}{\rho^2}\ell(\ell+1) - \frac{2}{\rho}\right] u(\rho) = \frac{2mEa_0^2}{\hbar^2} u(\rho)$$

- So, in dimensionless coordinates, effective Hamiltonian given by

$$\hat{H}_{\text{eff}} = -\partial_\rho^2 + \frac{1}{\rho^2}\ell(\ell+1) - \frac{2}{\rho}, \qquad \epsilon = \frac{2mEa_0^2}{\hbar^2} = \frac{E}{\text{Ry}}$$

  with $\hat{H}_{\text{eff}}\, u(\rho) = \epsilon u(\rho)$ and normalization $\int_0^\infty d\rho |u(\rho)|^2 = 1$.

- In dimensionless variables, we will use variational method to estimate ground state energy of hydrogen atom for three families of trial functions,

$$u_1(\rho) = \rho e^{-\alpha\rho}, \qquad u_2(\rho) = \frac{\rho}{\alpha^2 + \rho^2}, \qquad u_3(\rho) = \rho^2 e^{-\alpha\rho}$$

Example I: Ground state of hydrogen

$$\hat{H}_{\rm eff} = -\partial_\rho^2 + \frac{1}{\rho^2}\ell(\ell+1) - \frac{2}{\rho}, \qquad \epsilon = \frac{2mEa_0^2}{\hbar^2} = \frac{E}{\rm Ry}$$

- For trial (real) radial function $u(\rho)$, $\rho = r/a_0$, variational energy:

$$\frac{E(u)}{\rm Ry} = \frac{\int_0^\infty d\rho\, u(\rho)\left(-\partial_\rho^2 - \frac{2}{\rho}\right) u(\rho)}{\int_0^\infty d\rho\, u^2(\rho)}$$

- For three families of trial functions,

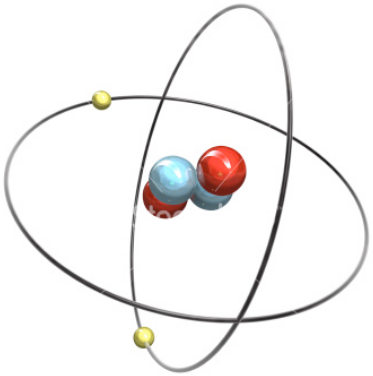
$$u_1(\rho) = \rho e^{-\alpha\rho}, \qquad u_2(\rho) = \frac{\rho}{\alpha^2+\rho^2}, \qquad u_3(\rho) = \rho^2 e^{-\alpha\rho}$$

find $\alpha_{\rm min} = 1$, $\pi/4$, and $3/2$.

- Predicted energy of optimal state out by ca. 0, 25%, and 21% respectively. (N.B. $u_1(\rho)$ includes ground state.)
- Error in wavefunction, $1 - |\langle u_0|u_{\rm var}\rangle|^2 = 0$, 0.21, and 0.05 respectively. (N.B. singular behaviour of potential at origin!)

Example II: Helium atom by variational method

- For atomic hydrogen, ground state energy is $-1\,\mathrm{Ry}$
- He^+ ion (with just a single electron) has nuclear charge $Z = 2$, so $E_{\mathrm{g.s.}} = -2^2\,\mathrm{Ry}$.
- For Helium atom, if we negect Coulomb interaction between electrons, $E_{\mathrm{g.s.}} = -2 \times 2^2\,\mathrm{Ry}$



  - To get better estimate, retain form of ionic wavefunction, $(\frac{Z^3}{\pi a_0^3})^{1/2} e^{-Zr/a_0}$, but treat Z as variational parameter.

    i.e. effect of electron-electron repulsion pushes wavefunction to larger radii $\rightsquigarrow$ effective reduction in Z.

Example II: Helium atom by variational method

- To find **potential energy** from interaction with nucleus, must use the actual nuclear charge $Z = 2$, but impose a variable Z for wavefunction,

$$\text{p.e.} = -2 \times \frac{2e^2}{4\pi\epsilon_0} \int_0^\infty 4\pi r^2 dr \frac{Z^3}{\pi a_0^3} \frac{e^{-2Zr/a_0}}{r} = -8Z\,\text{Ry}$$

- **Kinetic energy** determined solely by trial function and translates to Z^2 Ry per electron, i.e. total $\text{k.e.} = 2Z^2\,\text{Ry}$.

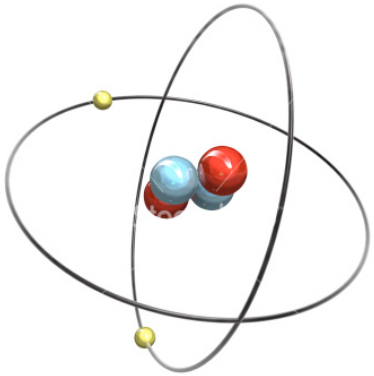
- Contribution from **electron-electron interaction**,

$$\frac{e^2}{4\pi\epsilon_0} \frac{Z^3}{(\pi a_0^3)^2} \int d^3r_1 d^3r_2 \frac{e^{-2Z(r_1+r_2)/a_0}}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{5}{4} \frac{e^2}{4\pi\epsilon_0} \frac{Z}{2a_0} = \frac{5}{4} Z\,\text{Ry}$$

Example II: Helium atom by variational method

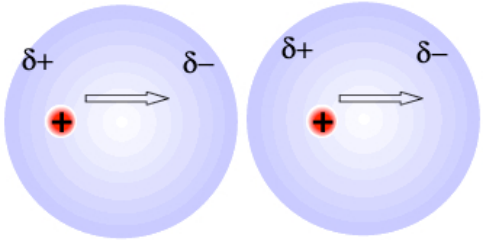
- Altogether, variational state energy:

$$E(Z) = -2\left(4Z - Z^2 - \frac{5}{8}Z\right)\,\mathrm{Ry}$$

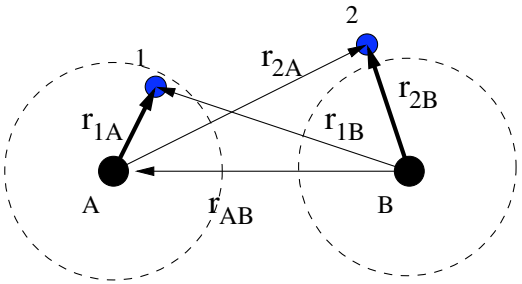


  - Minimization of $E(Z)$ w.r.t. $Z \rightsquigarrow Z = 2 - \frac{5}{16}$ which translates to energy 77.5 eV ca. 1 eV smaller than true ground state energy.
  - So electron-electron interaction leads effectively to a shielding of nuclear charge by an amount of ca. $(5/16)e$.

Example III: Van der Waals interaction



How do electric dipole fluctuations influence interaction between neutral atoms?



- For two hydrogen atoms, if we ignore dynamics of the nuclei A and B, the total Hamiltonian for the two electrons 1 and 2 is given by $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$ where

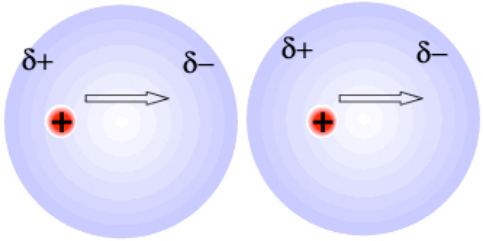
$$\hat{H}^{(0)} = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r_{1A}} + \frac{1}{r_{2B}}\right)$$

$$\hat{H}^{(1)} = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r_{AB}} + \frac{1}{r_{12}} - \frac{1}{r_{1B}} - \frac{1}{r_{2A}}\right)$$

- Since $\langle r_{1A}\rangle \sim \langle r_{2B}\rangle \sim a_0$, if the atomic separation is large, $|r_{AB}| \gg a_0$, we can treat $\hat{H}^{(1)}$ as a small perturbation which can be expanded in r

Example III: Van der Waals interaction

$$\hat{H}^{(1)} = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r_{AB}} + \frac{1}{r_{12}} - \frac{1}{r_{1B}} - \frac{1}{r_{2A}}\right)$$



- If we define the z-axis to lie along the direction $\mathbf{r}_{AB}$, an expansion obtains (exercise – cf. multipole expansion)

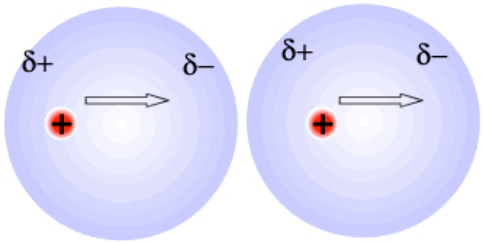
$$\hat{H}^{(1)} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^3}\left[\mathbf{r}_{1A}\cdot\mathbf{r}_{2B} - 3(\mathbf{r}_{1A}\cdot\hat{\mathbf{e}}_z)(\mathbf{r}_{2B}\cdot\hat{\mathbf{e}}_z)\right]$$

- Physically, fluctuations of the electron charge cloud on each hydrogen atom result in an electric dipole moment leading to an instantaneous **dipole-dipole interaction**.

- The unperturbed ground state involves both electrons in the $1s$ orbital, $|0^{(0)}\rangle = |100\rangle \otimes |100\rangle$. Since the matrix element $\langle 0^{(0)}|\hat{H}^{(1)}|0^{(0)}\rangle$ vanishes (by symmetry), we have to turn to higher orders to obtain a non-zero contribution to the energy shift.

Example III: Van der Waals interaction

$$\hat{H}^{(1)} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^3}\left[\mathbf{r}_{1A}\cdot\mathbf{r}_{2B} - 3(\mathbf{r}_{1A}\cdot\hat{\mathbf{e}}_z)(\mathbf{r}_{2B}\cdot\hat{\mathbf{e}}_z)\right]$$



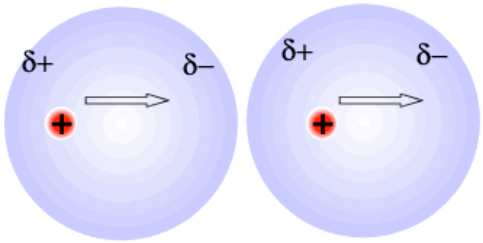
- At second order of **perturbation theory**, energy shift is given by

$$E^{(2)} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^6}\sum_{n\neq 0}\frac{|\langle 0^{(0)}|\mathbf{r}_{1A}\cdot\mathbf{r}_{2B} - 3z_{1A}z_{2B})|n^{(0)}\rangle|^2}{E_0^{(0)} - E_n^{(0)}} < 0$$

- The instantaneous dipole-dipole interaction always leads to an **attractive** interaction which scales as $1/r_{AB}^6$ – **Van der Waals**.
- Result is valid for any pair of atoms in spherically symmetric states.
- To estimate the scale of the perturbation theory, we can make use of a trick to provide an upper bound (cf. quadratic Stark effect).

Example III: Van der Waals interaction

$$E^{(2)} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^6}\sum_{n\neq 0}\frac{|\langle 0^{(0)}|\mathbf{r}_{1A}\cdot\mathbf{r}_{2B} - 3z_{1A}z_{2B})|n^{(0)}\rangle|^2}{E_0^{(0)} - E_n^{(0)}}$$



- If $|1^{(0)}\rangle = |21m\rangle \otimes |21m\rangle$ denotes a lowest excited state with non-vanishing dipole matrix element with $|0^{(0)}\rangle$, we may set

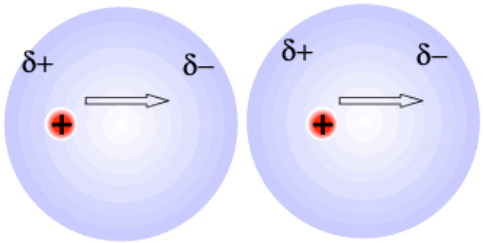
$$E^{(2)} \geq \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^6}\frac{1}{E_0^{(0)} - E_1^{(0)}}\sum_{n\neq 0}|\langle 0^{(0)}|\mathbf{r}_{1A}\cdot\mathbf{r}_{2B} - 3z_{1A}z_{2B})|n^{(0)}\rangle|^2$$

- Then, since the $n = 0$ matrix element in the sum vanishes, we may extend the sum and make use of the identity $\sum_n |n^{(0)}\rangle\langle n^{(0)}| = \mathbb{I}$,

$$E^{(2)}(r_{AB}) \geq \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^6}\frac{1}{E_0^{(0)} - E_1^{(0)}}\langle 0^{(0)}|\left(\mathbf{r}_{1A}\cdot\mathbf{r}_{2B} - 3z_{1A}z_{2B}\right)^2|0^{(0)}\rangle$$

Example III: Van der Waals interaction

$$E^{(2)} \geq \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^6}\frac{\langle 0^{(0)}|\left(\mathbf{r}_{1A}\cdot\mathbf{r}_{2B}-3z_{1A}z_{2B}\right)^2|0^{(0)}\rangle}{E_0^{(0)}-E_1^{(0)}}$$



- Noting that the matrix elements of the cross-terms (such as $\langle x_{1A}x_{2B}z_{1A}z_{2B}\rangle$) in the expansion of $(\mathbf{r}_{1A}\cdot\mathbf{r}_{2B}-3z_{1A}z_{2B})^2$ vanish, and using $E_0^{(0)}=-2\times\mathrm{Ry}$ and $E_1^{(0)}=-2\times\frac{\mathrm{Ry}}{4}$, we find

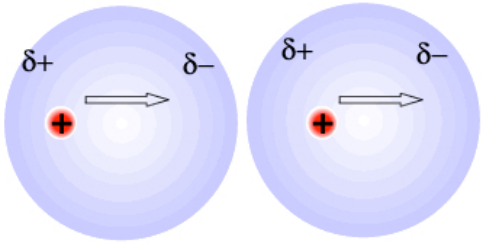
$$E^{(2)} \geq \frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{AB}^6}\frac{6\langle 0^{(0)}|z_{1A}^2z_{2B}^2|0^{(0)}\rangle}{-2(1-\frac{1}{4})\mathrm{Ry}}$$

- Evaluating the matrix element, we finally obtain (exercise)

$$E^{(2)} \geq -8\frac{e^2}{4\pi\epsilon_0}\frac{a_0^5}{r_{AB}^6}$$

- However, we can also place an upper limit by making use of the variational method...

Example III: Van der Waals interaction



- For reasons that will become clear, consider the (non-normalized) variational state wavefunction,

$$|\psi_{\text{trial}}\rangle = (1 + A\hat{H}^{(1)})|0^{(0)}\rangle$$

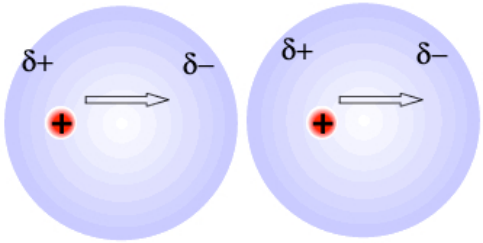
where A is a variational parameter.

- According to the variational principle, the ground state energy is bound by the inequality,

$$E_0 \leq \frac{\langle\psi_{\text{trial}}|\hat{H}|\psi_{\text{trial}}\rangle}{\langle\psi_{\text{trial}}|\psi_{\text{trial}}\rangle} = \frac{\langle 0^{(0)}|(1 + A\hat{H}^{(1)})\hat{H}(1 + A\hat{H}^{(1)})|0^{(0)}\rangle}{\langle 0^{(0)}|(1 + A\hat{H}^{(1)})^2|0^{(0)}\rangle}$$

Example III: Van der Waals interaction

$$E_0 \leq \frac{\langle 0^{(0)}|(1+A\hat{H}^{(1)})\hat{H}(1+A\hat{H}^{(1)})|0^{(0)}\rangle}{\langle 0^{(0)}|(1+A\hat{H}^{(1)})^2|0^{(0)}\rangle}$$



- Noting that $\langle 0^{(0)}|\hat{H}^{(1)}|0^{(0)}\rangle = 0$, dropping terms $O(\hat{H}^{(1)})^3 \sim \frac{1}{r_{AB}^9}$, and taking A small, we find

$$E_0 \leq E_0^{(0)} + \sum_{n\neq 0} |\langle 0^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2 \left[2A + A^2(E_n^{(0)} - E_0^{(0)})\right]$$

  [N.B. We are more concerned with principle than practice!]

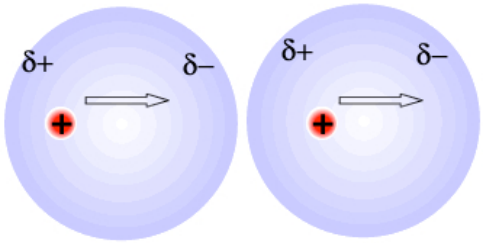
- Optimizing on the parameter A,

$$A = \frac{\sum_{n\neq 0} |\langle 0^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2}{\sum_{n\neq 0} |\langle 0^{(0)}|\hat{H}^{(1)}|n^{(0)}\rangle|^2 (E_n^{(0)} - E_0^{(0)})}$$

Example III: Van der Waals interaction

- Substituting for optimal A,

$$E_0 \leq E_0^{(0)} + A \sum_{n \neq 0} |\langle 0^{(0)} | \hat{H}^{(1)} | n^{(0)} \rangle|^2$$



- Evaluating matrix elements, we obtain

$$E_0 \leq E_0^{(0)} - 6 \frac{e^2}{4\pi\epsilon_0} \frac{a_0^5}{r_{AB}^6}$$

- Combined with first perturbative result, we therefore find that Van der Waals interaction energy has the following bound:

$$E_0^{(0)} - 8 \frac{e^2}{4\pi\epsilon_0} \frac{a_0^5}{r_{AB}^6} \leq E_0(r_{AB}) \leq E_0^{(0)} - 6 \frac{e^2}{4\pi\epsilon_0} \frac{a_0^5}{r_{AB}^6}$$

Perturbation theory: summary

- When a general quantum system is subject to a small perturbation,

$$\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$$

we have two approximation schemes at our disposal:

1. When the perturbed states are, by symmetry, "adiabatically" connected to the unperturbed system, we can adopt a **series expansion** in the small parameter $\langle \hat{H}^{(1)} \rangle / \langle \hat{H}^{(0)} \rangle$.

2. When states are disconnected – often due to some symmetry breaking (e.g. development of a bound state) – we can implement the **variational approach**.

Variational method: summary

- By introducing a trial wavefunction $|\psi_{\text{trial}}(\alpha)\rangle$, a function of variational parameters α, a minimization of the expectation value,

$$E(\alpha) = \langle \psi_{\text{trial}}(\alpha) | \hat{H} | \psi_{\text{trial}} \rangle$$

  provides an upper bound on the ground state energy.

- The variational method can be very effective – but it demands some insight into the underlying physics of the ground state wavefunction.

- This is rarely a problem when investigating a single-particle Hamiltonian.

- However, in strongly interacting many-particle quantum systems – the arena in which the variational approach has provided ground-breaking discoveries (e.g. fractional quantum Hall effect and superconductivity) – the approach can demand great inspiration.

Looking ahead

- Although the perturbation series expansion and the variational method provide a general approximation scheme, neither exploit the connection between quantum and classical mechanics.
- However, we know that when "$\hbar \to 0$" or, more intuitively, when the energy scales of interest are large as compared to the characteristic energy scales in the problem, we know that the properties become increasing classical.
- Can we develop an approach which exploits this correspondence? In the next lecture, we will introduce such a semiclassical scheme – the **WKB method**.
- Although the method demands some technical ingenuity, its insight and utility mean that it should not be reserved exclusively for TP2!

Lecture 10

Approximation methods: WKB method

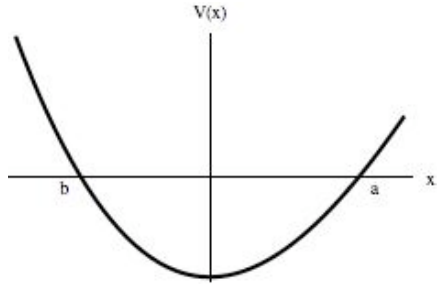
Wentzel, Kramers and Brillouin (WKB) method

- The **WKB method** provides a "semi-classical" approach for solving the one-dimensional time-independent Schrödinger equation.
- History predates Wentzel, Krammers & Brillouin (1926): developed independently by Jeffreys in 1923 and aspects utilized by Liouville and Green a century earlier!
- Provides an effective approach to treating general wave-like systems, e.g. fluids, electromagnetic waves, etc.

  In optics, WKB is known as **eikonal method**, and in general referred to by mathematicians as **short wavelength asymptotics**.
- In quantum mechanics, it provides conceptual framework for understanding the classical limit ($\hbar \to 0$) – hence "semi-classics"

Wentzel, Kramers and Brillouin (WKB) method

- Consider propagation of a quantum particle in a slowly-varying one-dimensional potential, $V(x)$



$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x)$$

  - For a uniform potential V, solutions are plane waves, $\psi = e^{ikx}$.
  - For a smooth potential, let us parameterize wavefunction as $\psi(x) = e^{i\sigma(x)/\hbar}$, complex $\sigma(x)$ encompasses amplitude and phase.
  - Making use of the identity, $\hbar^2\partial_x^2 e^{i\sigma(x)/\hbar} = e^{i\sigma(x)/\hbar}[i\hbar\,\partial_x^2\sigma - (\partial_x\sigma)^2]$, substitution in the Schrödinger equation leads to nonlinear equation,

$$-i\hbar\,\partial_x^2\sigma(x) + (\partial_x\sigma)^2 = p^2(x), \qquad p(x) \equiv \frac{h}{\lambda(x)} = \sqrt{2m(E - V(x))}$$

Wentzel, Kramers and Brillouin (WKB) method

$$-i\hbar\,\partial_x^2\sigma(x) + (\partial_x\sigma)^2 = p^2(x) \qquad (*)$$

- Since we're looking for semi-classical approximation, its makes sense to expand $\sigma(x)$ as power series in $\hbar$,

$$\sigma = \sigma_0 + (\hbar/i)\sigma_1 + (\hbar/i)^2\sigma_2 + \cdots$$

- At the leading (zeroth) order of the expansion, can drop the first term in (*),

$$(\partial_x\sigma_0)^2 = p^2(x)$$

- Fixing sign of $p(x) = +\sqrt{2m(E - V(x))}$,

$$\sigma_0(x) = \pm\int p(x)\,dx$$

N.B. for free particle, this is equivalent to classical action.

Wentzel, Kramers and Brillouin (WKB) method

$$-i\hbar\,\partial_x^2\sigma(x) + (\partial_x\sigma)^2 = p^2(x) \qquad (*)$$

- Solution $\sigma_0(x) = \pm\int p(x)\,dx$ valid when first term can be neglected:

$$\left|\frac{\hbar\,\partial_x^2\sigma(x)}{(\partial_x\sigma(x))^2}\right| \equiv \left|\partial_x\left(\frac{\hbar}{\partial_x\sigma}\right)\right| \ll 1$$

- In leading approximation, $\partial_x\sigma \simeq p(x)$ and $p(x) = 2\pi\hbar/\lambda(x)$, so

$$\frac{1}{2\pi}\,|\partial_x\lambda(x)| \ll 1$$

  i.e. change in wavelength over distance of one wavelength must be small.

- Approximation must fail at boundary of classically allowed region, the classical turning points: when $E = V(x)$, $p(x) = 0$ and wavelength infinite – see later!

Wentzel, Kramers and Brillouin (WKB) method

$$-i\hbar\partial_x^2\sigma(x) + (\partial_x\sigma)^2 = p^2(x) \qquad (*)$$

- Retaining terms of order $\hbar$, with $\sigma = \sigma_0 + (\hbar/i)\sigma_1 + \cdots$,

$$-i\hbar\,\partial_x^2\sigma_0 + 2\partial_x\sigma_0(\hbar/i)\partial_x\sigma_1 = 0$$

- Rearranging, recalling $\partial_x\sigma_0 = p$, and integrating,

$$\partial_x\sigma_1 = -\frac{\partial_x^2\sigma_0}{2\partial_x\sigma_0} = -\frac{\partial_x p}{2p}, \quad \sigma_1(x) = -\frac{1}{2}\ln p(x), \quad e^{\sigma_1(x)} = \frac{1}{\sqrt{p(x)}}$$

- So, to this order, $$\psi(x) = \frac{C_1}{\sqrt{p(x)}}e^{(i/\hbar)\int p\,dx} + \frac{C_2}{\sqrt{p(x)}}e^{-(i/\hbar)\int p\,dx}$$

  where C_1 and C_2 denote constants of integration.

- Physically: probability of finding the particle, $|\psi(x)|^2 dx \simeq dx/p(x)$, proportional to the time it spends there.

Wentzel, Kramers and Brillouin (WKB) method

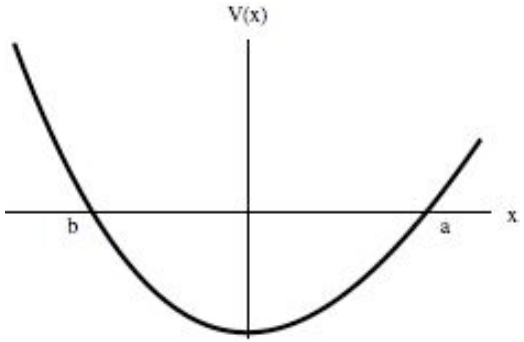
$$\psi(x) = \frac{C_1}{\sqrt{p(x)}} e^{(i/\hbar)\int p\, dx} + \frac{C_2}{\sqrt{p(x)}} e^{-(i/\hbar)\int p\, dx}$$

- In classically forbidden region, where $\frac{p^2(x)}{2m} = E - V(x) < 0$, $p(x)$ is pure imaginary, but same formal solution applies

$$\psi(x) = \frac{C_1'}{\sqrt{|p(x)|}} e^{-(1/\hbar)\int |p|\, dx} + \frac{C_2'}{\sqrt{|p(x)|}} e^{(1/\hbar)\int |p|\, dx}$$

- This completes formulation of semi-classical approximation – but to apply it, we have to understand how to deal with regions close to classical turning points $\leftrightarrow$ **energy quantization condition**.

WKB: Connection formulae, and quantization rules



- Consider 1d confining potential where classically allowed region $b \leq x \leq a$.
- How to connect three regions together?
  - Close to turning point at, say, $x = a$, $E - V(x) \simeq F_0(x - a)$, formal solution to Schrödinger equation is an Airy function with

$$\lim_{x \gg a} \psi(x) \simeq \frac{C}{2\sqrt{|p(x)|}} e^{-(1/\hbar)\int_a^x |p|\, dx}$$

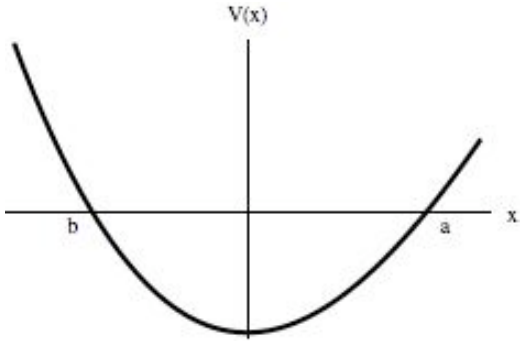
translating to decay into classically forbidden region while, to left,

$$\lim_{b \ll x < a} \psi(x) = \frac{C}{\sqrt{|p(x)|}} \cos\left[\frac{\pi}{4} - \frac{1}{\hbar}\int_x^a p\, dx\right]$$

  - Similarly, at second classical turning point at $x = b$,

$$\lim_{b < x \ll a} \psi(x) = \frac{C'}{\sqrt{|p(x)|}} \cos\left[\frac{1}{\hbar}\int_b^x p\, dx - \frac{\pi}{4}\right]$$

WKB: Connection formulae, and quantization rules



- For expressions to be consistent, we must have $|C'| = |C|$ and

$$\left(\frac{1}{\hbar}\int_b^x p\,dx - \frac{\pi}{4}\right) - \left(\frac{\pi}{4} - \frac{1}{\hbar}\int_x^a p\,dx\right) = n\pi$$

  where, for n even, $C' = C$ and for n odd, $C' = -C$.

- Therefore, $\frac{1}{\hbar}\int_b^a p\,dx = (n + 1/2)\pi$, or when cast in terms of a complete periodic cycle of classical motion,

$$\boxed{\oint p\,dx = 2\pi\hbar(n + 1/2)}$$

n counts nodes, cf. **Bohr-Sommerfeld quantization condition**

A few words about "semi-classics"

- What is meant by semi-classics being an $\hbar \to 0$ limit?
  $\hbar$ is a fundamental constant – not easily adjusted!

- Validity of WKB approximation relies upon condition $\lambda/L \ll 1$.
- From de Broglie relation, we may write inequality as $h/pL \ll 1$,
  where p denotes particle momentum.
- Both p and L can be considered as "classical" scales.
- So, formally, we can think of think of accessing the semi-classical limit by adjusting $\hbar$ so that it is small enough to fulfil inequality.
- Alternatively, at fixed $\hbar$, we can access the semi-classical regime by reaching to higher and higher energy scales (larger and larger p).

Example I: Quantum harmonic oscillator

- For the harmonic oscillator, $H = p^2/2m + m\omega^2 x^2/2$, classical momentum given by

$$p(x) = \sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)}$$

- The classical turning points: set by $E = \frac{1}{2}m\omega^2 x_0^2$, i.e. $x_0 = \pm\frac{2E}{m\omega^2}$
- Over periodic cycle, the classical action is given by

$$\oint p(x)dx = 2\int_{-x_0}^{x_0} dx\sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)} = 2\pi\frac{E}{\omega} \stackrel{!}{=} 2\pi\hbar(n + 1/2)$$

i.e. $E_n = (n + 1/2)\hbar\omega$.

Example I: Quantum harmonic oscillator

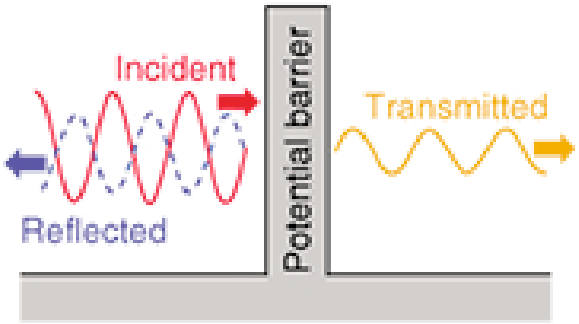
- In WKB approximation, wavefuntions given by

$$\psi(x) = \begin{cases} \frac{C}{\sqrt{p(x)}} \cos\left(\frac{\pi}{4} - \frac{1}{\hbar}\int_x^{x_0} p\,dx\right) & x < x_0 \\ \frac{C}{2\sqrt{p(x)}} \exp\left(-\frac{1}{\hbar}\int_{x_0}^{x} |p|\,dx\right) & x > x_0 \end{cases}$$

$$\psi(x) = \begin{cases} \frac{C}{\sqrt{p(x)}} \cos\left(\frac{n\pi}{2} + \frac{E}{\hbar\omega}\left[\arcsin\left(\frac{x}{x_0}\right) + \frac{x}{x_0}\sqrt{1-\frac{x^2}{x_0^2}}\right]\right) & 0 < x < x_0 \\ \frac{C}{2\sqrt{p(x)}} \exp\left(-\frac{E}{\hbar\omega}\left[\frac{x}{x_0}\sqrt{\frac{x^2}{x_0^2}-1} - \operatorname{arccosh}\left(\frac{x}{x_0}\right)\right]\right) & x > x_0 \end{cases}$$

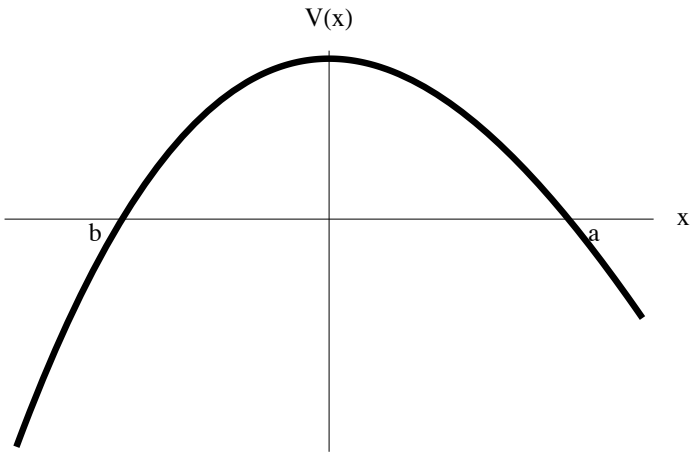
Example II: Quantum tunneling

- Consider beam of particles incident upon a localized potential barrier, $V(x)$.



  - Suppose that, over single continuous region, from b to a, potential rises above incident energy of incoming particles so that, classically, all particles reflected.
  - In quantum system, some particles incident from left may tunnel through the barrier and continue propagating to the right.
  - What is transmission probability?

Example II: Quantum tunneling



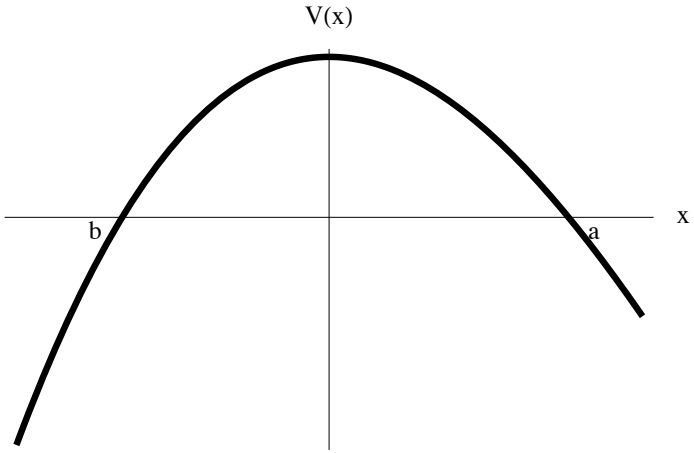
- From WKB, to left of barrier (region 1),

$$\psi_1(x) = \frac{1}{\sqrt{p}} \exp\left[\frac{i}{\hbar}\int_b^x p\,dx\right] + r(E)\frac{1}{\sqrt{p}} \exp\left[-\frac{i}{\hbar}\int_b^x p\,dx\right]$$

with $p(E) = \sqrt{2m(E - V(x))}$, while, to right (region 3),

$$\psi_3(x) = t(E)\frac{1}{\sqrt{p}} \exp\left[\frac{i}{\hbar}\int_a^x p\,dx\right]$$

Example II: Quantum tunneling



- In barrier region (2),

$$\psi_2(x) = \frac{C_1}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_a^x |p|\,dx\right] + \frac{C_2}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_a^x |p|\,dx\right]$$

- Applying the continuity condition on the wavefunction, one obtains the transmissivity,

$$T(E) = |t(E)|^2 \simeq \exp\left[-\frac{2}{\hbar}\int_a^b |p|\,dx\right]$$

Summary

- Most problems in quantum mechanics are formally intractable. Fortunately, we can draw upon several approximation schemes.
- In cases where a small perturbation conserves the character of the states, we can adopt a **perturbative series expansion**.
- Where a series expansion is invalid, the **variational method** can be deployed – but typically it's application requires some intuition (or prejudice!) about the nature of perturbed states.
- Finally, in systems which are either one-dimensional, or rendered such by symmetry, we can engage the power of the **semi-classical WKB approach**.

Synopsis: Lectures 5-10

5. **Charged particle in an electromagnetic field:**

   Classical and quantum mechanics of particle in a field; normal Zeeman effect; gauge invariance and the Aharonov-Bohm effect; Landau levels.

6. **Spin:**

   Stern-Gerlach experiment; spinors, spin operators and Pauli matrices; spin precession in a magnetic field; parametric resonance; addition of angular momenta.

7. **Time-independent perturbation theory:**

   Perturbation series; first and second order expansion; degenerate perturbation theory; Stark effect; nearly free electron model.

8. **Variational and WKB method:**

   Variational method: ground state energy and eigenfunctions; application to helium; Semiclassics and the WKB method.

Synopsis: Lectures 11-15

9. **Identical particles:**

   Particle indistinguishability and quantum statistics; space and spin wavefunctions; consequences of particle statistics; ideal quantum gases; degeneracy pressure in neutron stars; Bose-Einstein condensation in ultracold atomic gases.

10. **Atomic structure:**

    Relativistic corrections – spin-orbit coupling; Darwin structure; Lamb shift; hyperfine structure. Multi-electron atoms; Helium; Hartree approximation and beyond; Hund's rule; periodic table; coupling schemes LS and jj; atomic spectra; Zeeman effect.

11. **Molecular structure:**

    Born-Oppenheimer approximation; H_2^+ ion; H_2 molecule; ionic and covalent bonding; solids; molecular spectra; rotation and vibrational transitions.

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