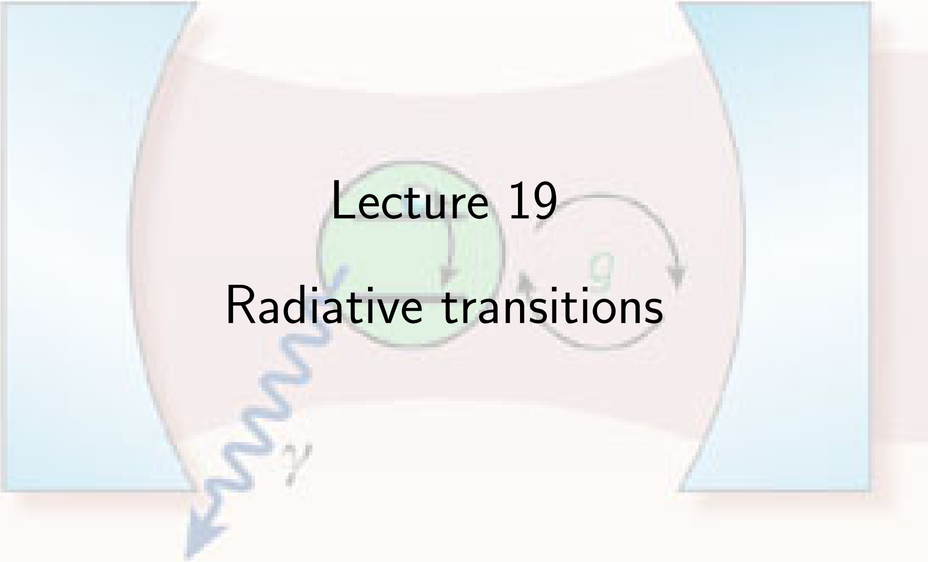


Lecture 19

Radiative transitions

Radiative transitions: background

- Previously, we have formulated a quantum theory of atoms (matter) coupling to a classical time-independent electromagnetic field, cf. Zeeman and Stark effects.
- To develop a fully quantum theory of light-matter systems, we have to address both the quantum theory of the electromagnetic field and formulate a theory of the coupling of light to matter.
- In the following we will address each of these components in turn, starting with light-matter coupling.
- Our motivation for developing such a consistent theory is that it:
  - **(a)** provides a platform to study radiative tranistions in atoms (which will address)
  - **(b)** forms the basis of **quantum optics** (which will not address – but which is well-represented in subsequent courses).

Radiative transitions: outline

- Coupling of matter to electromagnetic field
- Spontaneous emission, absorption and stimulated emission
- Einstein's A and B coefficients
- Selection rules
- Theory of the laser and coherent states

Coupling of matter to the electromagnetic field

- For a single-electron atom in a time-dependent external EM field, the Hamiltonian takes the form,

$$\boxed{\hat{H}_{\rm atom} = \frac{1}{2m}\left(\hat{\mathbf{p}} + e\mathbf{A}(\mathbf{r}, t)\right)^2 - e\phi(\mathbf{r}, t) + V(\mathbf{r})}$$

  (with a straightforward generalization to multi-electron atoms).

- Previously, we have seen that it is profitable to expand Hamiltonian in $\hat{A}$, $\hat{H}_{\rm atom} = \hat{H}_0 + \hat{H}_{\rm para} + \hat{H}_{\rm dia.}$, where $\hat{H}_0 = \dfrac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}) - e\phi(t)$

$$\boxed{\hat{H}_{\rm para}(t) = \frac{e}{m}\mathbf{A}(t)\cdot\hat{\mathbf{p}}}$$

  the paramagnetic term describes coupling of the atom to the EM field, and $\hat{H}_{\rm dia} = (e\mathbf{A})^2/2m$ represents diamagnetic term.

- Since we will be interested in absorption and emission of single photons, influence of diamagnetic term is (as usual) negligible.

Coupling of matter to the electromagnetic field

- When quantized, EM field is described by the photon Hamiltonian,

$$\boxed{\hat{H}_{\rm rad} = \sum_{\mathbf{k},\lambda=1,2} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \frac{1}{2} \right), \qquad \omega_{\mathbf{k}} = c|\mathbf{k}|}$$

where $a^\dagger_{\mathbf{k}\lambda}/a_{\mathbf{k}\lambda}$ create/annihilate photons with polarization λ.

- These operators obey (bosonic) commutation relations,

$$[a_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}, \qquad [a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}] = [a^\dagger_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = 0$$

and act on photon number states, $|n_{\mathbf{k}\lambda}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}\lambda}!}}(a^\dagger_{\mathbf{k}\lambda})^{n_{\mathbf{k}\lambda}}|\Omega\rangle$, as

$$a_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}}|n_{\mathbf{k}\lambda} - 1\rangle, \qquad a^\dagger_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}+1}|n_{\mathbf{k}\lambda} + 1\rangle$$

- With these definitions, the vector potential is given by

$$\boxed{\hat{\mathbf{A}}(\mathbf{r}) = \sum_{\mathbf{k}\lambda=1,2} \sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}} V}} \left[\hat{\mathbf{e}}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{\mathbf{e}}^*_{\mathbf{k}\lambda} a^\dagger_{\mathbf{k}\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}} \right]}$$

Heisenberg representation

$$\boxed{\hat{H}_{\mathrm{rad}} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \frac{1}{2} \right)}$$

- To determine radiative transition rates, we will exploit **Fermi's Golden rule**. To prepare for this, it is convenient to transfer time-dependence to operators (Heisenberg representation).
- As with any operator, the field operators obey equations of motion,

$$\dot{a}_{\mathbf{k}\lambda} = \frac{i}{\hbar}[\hat{H}, a_{\mathbf{k}\lambda}] = i\omega_{\mathbf{k}}[a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}, a_{\mathbf{k}\lambda}] = -i\omega_{\mathbf{k}} a_{\mathbf{k}\lambda}$$

i.e. $a_{\mathbf{k}\lambda}(t) = a_{\mathbf{k}\lambda}(0)e^{-i\omega_{\mathbf{k}}t}$, $a^\dagger_{\mathbf{k}\lambda}(t) = a^\dagger_{\mathbf{k}\lambda}(0)e^{i\omega_{\mathbf{k}}t}$ and

$$\boxed{\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}} V}} \left[\hat{\mathbf{e}}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \hat{\mathbf{e}}^*_{\mathbf{k}\lambda} a^\dagger_{\mathbf{k}\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} \right]}$$

Coupling of matter to the electromagnetic field

- Putting together all of these components, the total Hamiltonian is then given by $\hat{H} = \hat{H}_{\rm atom} + \hat{H}_{\rm para} + \hat{H}_{\rm rad}$ where (with $\phi = 0$)

$$\hat{H}_{\rm atom} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}), \quad \hat{H}_{\rm rad} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \frac{1}{2} \right) \quad \text{and}$$

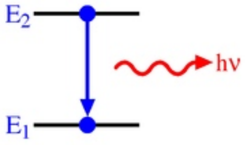
$$\hat{H}_{\rm para}(t) = \frac{e}{m} \hat{\mathbf{A}}(\mathbf{r}, t) \cdot \hat{\mathbf{p}} \quad \text{with}$$

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_{\mathbf{k}} V}} \left[\hat{\mathbf{e}}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} + \hat{\mathbf{e}}^*_{\mathbf{k}\lambda} a^\dagger_{\mathbf{k}\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} \right]$$

- In the following, we will apply this Hamiltonian to the problem of radiative transitions in single electron atoms.

Spontaneous emission

- Consider probability for an atom, initially in state $|\mathrm{i}\rangle$ to make transition to $|\mathrm{f}\rangle$ with emission of a photon of wavevector $\mathbf{k}$ and polarization λ – spontaneous emission.



- If radiation field initially prepared in vacuum state, $|\Omega\rangle$, then final state involves one photon, $a^\dagger_{\mathbf{k}\lambda}|\Omega\rangle$.
- Therefore, making use of Fermi's Golden rule, with the perturbation

$$\hat{H}_{\mathrm{para}}(t) = \frac{e}{m}\hat{\mathbf{A}}(\mathbf{r},t)\cdot\hat{\mathbf{p}} = \sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\left[\hat{\mathbf{e}}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \hat{\mathbf{e}}^*_{\mathbf{k}\lambda}a^\dagger_{\mathbf{k}\lambda}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}\right]\cdot\frac{e}{m}\hat{\mathbf{p}}$$

transition probability given by,

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f}}(t) = \frac{2\pi}{\hbar^2}|\langle\mathrm{f}|\otimes\langle\mathbf{k}\lambda|\hat{H}_{\mathrm{para}}|\mathrm{i}\rangle\otimes|\Omega\rangle|^2\delta(\omega_{\mathrm{if}}-\omega)}$$

Spontaneous emission

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} = \frac{2\pi}{\hbar}\left|\langle \mathrm{f}|\frac{e}{m}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle\right|^2 \delta(E_{\mathrm{i}} - E_{\mathrm{f}} - \hbar\omega_{\mathbf{k}})}$$

- To determine transition rate, we must analyse matrix elements of the form $\langle \mathrm{f}|e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle$. For typical state, $\langle\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}\rangle \sim p \sim Zmc\alpha$.
- But what about exponential factor? With $r \sim \hbar/p \simeq \hbar/mZc\alpha$, and $\omega_{\mathbf{k}} = c|\mathbf{k}| \sim \frac{p^2}{2m}$ (for electronic transitions), we have

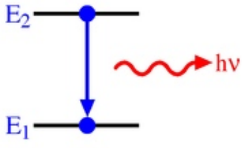
$$\mathbf{k}\cdot\mathbf{r} \simeq \frac{\omega_{\mathbf{k}}}{c}\frac{\hbar}{p} \simeq \frac{\hbar p}{mc} \simeq Z\alpha$$

  i.e. for $Z\alpha \ll 1$, we can expand exponential as power series in $\mathbf{k}\cdot\mathbf{r}$ with lowest terms dominant.
- Taking zeroth order term, and using $\hat{\mathbf{p}} = \frac{im}{\hbar}[\hat{H}_0, \mathbf{r}]$ (cf. Ehrenfest)

$$\langle \mathrm{f}|\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle = m\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\langle \mathrm{f}|\frac{i}{\hbar}[\hat{H}_0, \mathbf{r}]|\mathrm{i}\rangle = im\frac{E_{\mathrm{f}} - E_{\mathrm{i}}}{\hbar}\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\langle \mathrm{f}|\mathbf{r}|\mathrm{i}\rangle$$

Spontaneous emission: electric dipole approximation

$$\boxed{\langle \mathrm{f}|\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle = -im\omega_{\mathbf{k}}\langle \mathrm{f}|\hat{\mathbf{e}}^*_{\mathbf{k}\lambda}\cdot\mathbf{r}|\mathrm{i}\rangle}$$



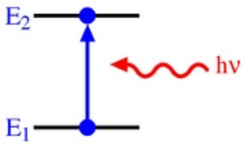
- This result, which emerges from leading approximation in $Z\alpha$, is known as **electric dipole approximation**: Effectively, we have set

$$\boxed{\hat{H}_{\mathrm{para}} = \frac{e}{m}\hat{\mathbf{A}}(\mathbf{r},t)\cdot\hat{\mathbf{p}} \simeq e\hat{\mathbf{E}}(\mathbf{r},t)\cdot\mathbf{r} = -\hat{\mathbf{E}}(\mathbf{r},t)\cdot\hat{\mathbf{d}}}$$

  translating to the potential energy of a dipole, with moment $\hat{\mathbf{d}} = -e\mathbf{r}$, in an oscillating electric field.

Stimulated absorption and emission

- Consider now absorption of a photon. If we assume that, in the initial state, there are $n_{\mathbf{k}\lambda}$ photons in mode $(\mathbf{k}\lambda)$ then, after the transition, there will be $n_{\mathbf{k}\lambda} - 1$ photons.



- Then, if initial state of the atom is $|\mathrm{i}\rangle$ and final state is $|\mathrm{f}\rangle$,

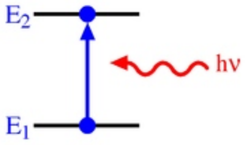
$$\langle \mathrm{f}| \otimes \langle (n_{\mathbf{k},\lambda} - 1)|\hat{H}_{\mathrm{para}}|\mathrm{i}\rangle \otimes |n_{\mathbf{k}\lambda}\rangle$$
$$= \langle \mathrm{f}| \otimes \langle (n_{\mathbf{k},\lambda} - 1)|\frac{e}{m}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} \cdot \hat{\mathbf{p}}|\mathrm{i}\rangle \otimes |n_{\mathbf{k}\lambda}\rangle$$

- Then, using the relation $a_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}}|(n_{\mathbf{k}\lambda} - 1)\rangle$,

$$\langle \mathrm{f}| \otimes \langle (n_{\mathbf{k},\lambda} - 1)|\hat{H}_{\mathrm{para}}|\mathrm{i}\rangle \otimes |n_{\mathbf{k}\lambda}\rangle = \langle \mathrm{f}|\frac{e}{m}\sqrt{\frac{\hbar n_{\mathbf{k}\lambda}}{2\epsilon_0\omega_{\mathbf{k}}V}}e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \hat{\mathbf{p}}|\mathrm{i}\rangle$$

Stimulated absorption and emission

- Consider now absorption of a photon. If we assume that, in the initial state, there are $n_{\mathbf{k}\lambda}$ photons in mode $(\mathbf{k}\lambda)$ then, after the transition, there will be $n_{\mathbf{k}\lambda} - 1$ photons.



- As a result, using Fermi's Golden rule,

$$\Gamma_{\mathrm{i}\to\mathrm{f}}(t) = \frac{2\pi}{\hbar^2}|\langle \mathrm{f}| \otimes \langle (n_{\mathbf{k}\lambda} - 1)|\hat{H}_{\mathrm{para}}|\mathrm{i}\rangle \otimes |n_{\mathbf{k}\lambda}\rangle|^2 \delta(\omega_{\mathrm{fi}} - \omega)$$

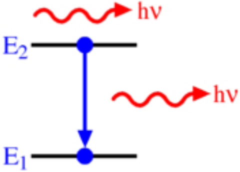
  we obtain the transition amplitude,

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} = \frac{2\pi}{\hbar}\left|\langle \mathrm{f}|\frac{e}{m}\sqrt{\frac{\hbar n_{\mathbf{k}\lambda}}{2\epsilon_0 \omega_{\mathbf{k}} V}} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \hat{\mathbf{p}}|\mathrm{i}\rangle\right|^2 \delta(E_{\mathrm{f}} - E_{\mathrm{i}} - \hbar\omega_{\mathbf{k}})}$$

- In particular, we find that the absorption rate **increases** linearly with photon number, $n_{\mathbf{k}\lambda}$.

Stimulated absorption and emission

- Similarly, if we consider emission process in which there are already $n_{\mathbf{k}\lambda}$ photons in initial state,

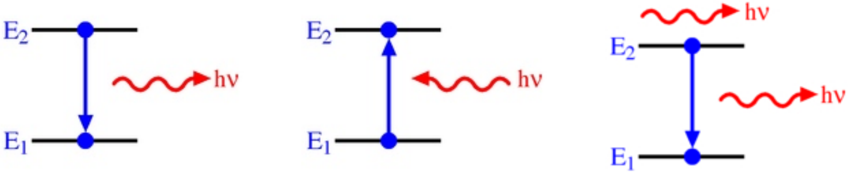


- using the relation $a^{\dagger}_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}+1}|(n_{\mathbf{k}\lambda}+1)\rangle$, we have revised transition rate,

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} = \frac{2\pi}{\hbar}\left|\langle\mathrm{f}|\frac{e}{m}\sqrt{\frac{\hbar(n_{\mathbf{k}\lambda}+1)}{2\epsilon_0\omega_{\mathbf{k}}V}}e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle\right|^2\delta(E_{\mathrm{f}}-E_{\mathrm{i}}-\hbar\omega_{\mathbf{k}})}$$

- Enhancement of transition rate by photon occupancy known as **stimulated emission**.

Radiative transitions: summary



- Altogether, in dipole approximation $\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle \simeq -im\omega_{\mathbf{k}}\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\mathbf{r}|\mathrm{i}\rangle$

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} \simeq \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda}\ \delta(E_{\mathrm{f}}-E_{\mathrm{i}}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k}\lambda}+1)\ \delta(E_{\mathrm{i}}-E_{\mathrm{f}}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{cases}}$$

where $\hat{\mathbf{d}} = -e\mathbf{r}$ is electric dipole operator.

- If there are no photons present initially, $\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda}$ reduces to result for spontaneous emission.
- The coincidence of $n_{\mathbf{k}\lambda}$-independent coefficients for absorption and emission coincide is known as **detailed balance**.

Absorption and stimulated emission

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} \simeq \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \left\{\begin{array}{rl} n_{\mathbf{k}\lambda}\ \delta(E_{\mathrm{f}}-E_{\mathrm{i}}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k}\lambda}+1)\ \delta(E_{\mathrm{i}}-E_{\mathrm{f}}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{array}\right.}$$

- Integrated transition rate associated with a small solid angle $d\Omega$ in the direction $\mathbf{k}$ given by

$$dR_{\mathrm{i}\to\mathrm{f},\lambda} = \sum_{\mathbf{k}\in d\Omega}\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} = d\Omega\, V\int\frac{k^2 dk}{(2\pi)^3}\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda}$$

- If we assume that the photon number, $n_{\mathbf{k}\lambda}$ is isotropic, independent of angle Ω, using the dispersion relation $\omega_{\mathbf{k}} = ck$, we obtain

$$\frac{dR_{\mathrm{i}\to\mathrm{f},\lambda}}{d\Omega} = \frac{V}{c^3}\int\frac{\omega^2 d\omega}{(2\pi)^3}\frac{\pi\omega}{\epsilon_0 V}\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \left\{\begin{array}{l} n_\lambda(\omega)\ \delta(E_{\mathrm{f}}-E_{\mathrm{i}}-\hbar\omega) \\ (n_\lambda(\omega)+1)\ \delta(E_{\mathrm{i}}-E_{\mathrm{f}}-\hbar\omega) \end{array}\right.$$

$$\boxed{\frac{dR_{\mathrm{i}\to\mathrm{f},\lambda}}{d\Omega} = \frac{\omega^3}{8\pi^2\epsilon_0\hbar c^3}|\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\mathbf{d}|\mathrm{i}\rangle|^2 \left\{\begin{array}{l} n_\lambda(\omega) \\ n_\lambda(\omega)+1 \end{array}\right.}$$

where $\hbar\omega = |E_{\mathrm{f}} - E_{\mathrm{i}}|$.

- From this expression, we can obtain the power loss as $P_\lambda = \hbar\omega R_\lambda$

Einstein's A and B coefficients

In fact, frequency dependence of spontaneous emission rate can be inferred using ingenious argument due to Einstein who showed that stimulated and spontaneous transitions must be related.

- Consider ensemble of atoms exposed to black-body radiation at temperature T. Let us consider transitions between states $|\psi_j\rangle$ and $|\psi_k\rangle$, with $E_k - E_j = \hbar\omega$.
- If number of atoms in two states given by n_j and n_k, transition rates per atom given by:

$$\begin{array}{rcc} \text{absorption} & j \to k & B_{j\to k}u(\omega) \\ \text{stimulated emission} & k \to j & B_{k\to j}u(\omega) \\ \text{spontaneous emission} & k \to j & A_{k\to j}(\omega) \end{array}$$

  where $u(\omega)$ denotes energy density of radiation.
- A and B are known as **Einstein's A and B coefficients**, and, as we have seen, are properties of atomic states.

Einstein's A and B coefficients

$$\begin{array}{rcl} \text{absorption} & j \to k & B_{j\to k}u(\omega) \\ \text{stimulated emission} & k \to j & B_{k\to j}u(\omega) \\ \text{spontaneous emission} & k \to j & A_{k\to j}(\omega) \end{array}$$

- In thermodynamic equilibrium the rates must balance, so that

$$n_k \left[A_{k\to j}(\omega) + B_{k\to j}u(\omega)\right] = n_j B_{j\to k}u(\omega)$$

- At the same time, relative populations of two states given by Boltzmann factor,

$$\frac{n_j}{n_k} = \frac{e^{-E_j/k_\mathrm{B}T}}{e^{-E_k/k_\mathrm{B}T}} = e^{\hbar\omega/k_\mathrm{B}T}$$

Thus we have:

$$\boxed{A_{k\to j}(\omega) = \left[B_{j\to k}e^{\hbar\omega/k_\mathrm{B}T} - B_{k\to j}\right]u(\omega)}$$

Einstein's A and B coefficients

$$\boxed{A_{k\to j}(\omega) = \left[B_{j\to k}e^{\hbar\omega/k_{\mathrm{B}}T} - B_{k\to j}\right] u(\omega)}$$

- For black-body, energy density $u(\omega)$ set by Planck formula,

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\bar{n}(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T} - 1}$$

$$A_{k\to j}(\omega) = \left[B_{j\to k}e^{\hbar\omega/k_{\mathrm{B}}T} - B_{k\to j}\right]\frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T} - 1}$$

- Since $A_{k\to j}$ is intrinsic (independent of temperature), T must cancel on right hand side, i.e.

$$\boxed{B_{k\to j} = B_{j\to k} \qquad \text{and} \qquad A_{k\to j}(\omega) = B_{k\to j}\frac{\hbar\omega^3}{\pi^2 c^3}}$$

- So, A and B coefficients are related, and if we can calculate B coefficient for stimulated emission from Fermi's Golden rule, we can infer A, and *vice versa*.

Selection rules: parity

$$\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} \simeq \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda}\ \delta(E_\mathrm{f}-E_\mathrm{i}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k}\lambda}+1)\ \delta(E_\mathrm{i}-E_\mathrm{f}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{cases}$$

- Formulae for rates $\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda}$ show that radiative transitions will not occur between states $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$ unless at least one component of the dipole matrix element $\langle \mathrm{f}|\hat{\mathbf{d}}|\mathrm{i}\rangle$ is non-zero.
- If matrix elements are zero for certain pairs, they are disallowed (at least in the electric dipole approximation) leading to **selection rules**.
- Since dipole operator $\hat{\mathbf{d}} = -e\mathbf{r}$ changes sign under parity ($\mathbf{r}\to-\mathbf{r}$), matrix element $\langle \mathrm{f}|\hat{\mathbf{d}}|\mathrm{i}\rangle$ will vanish if $|\mathrm{f}\rangle$ and $|\mathrm{i}\rangle$ have same parity.

1. The parity of the wavefunction must change in an electric dipole transition.

Selection rules: spin

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} \simeq \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle\mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda}\;\delta(E_\mathrm{f}-E_\mathrm{i}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k}\lambda}+1)\;\delta(E_\mathrm{i}-E_\mathrm{f}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{cases}}$$

- Separating wavefunction into spatial and spin components, $|\mathrm{f}\rangle = |\phi_\mathrm{f}\rangle \otimes |\chi_\mathrm{f}\rangle$, since dipole operator acts only on spatial part,

$$\langle\mathrm{f}|\hat{\mathbf{d}}|\mathrm{i}\rangle = -\langle\chi_\mathrm{f}|\chi_\mathrm{i}\rangle \int d^3r\,\phi_\mathrm{f}^*(\mathbf{r})\,e\mathbf{r}\phi_\mathrm{i}(\mathbf{r})$$

i.e. spin term, $\langle\chi_\mathrm{f}|\chi_\mathrm{i}\rangle$, vanishes unless $|\chi_\mathrm{i}\rangle$ and $|\chi_\mathrm{f}\rangle$ are identical,

$$\boxed{\Delta s = 0, \qquad \Delta m_s = 0}$$

2. The spin state is not altered in an electric dipole transition.

Selection rules: orbital angular momentum

$$\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} \simeq \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda}\ \delta(E_\mathrm{f}-E_\mathrm{i}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k}\lambda}+1)\ \delta(E_\mathrm{i}-E_\mathrm{f}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{cases}$$

- From the operator identity, $[\hat{L}_i, r_j] = i\hbar\epsilon_{ijk}r_k$, it follows that

$$[\hat{L}_z, z] = 0, \qquad [\hat{L}_z, x \pm iy] = \pm(x \pm iy)\hbar$$

- We therefore obtain,

$$\langle \ell', m'|[\hat{L}_z, z]|\ell, m\rangle = (m' - m)\hbar\langle \ell', m'|z|\ell, m\rangle = 0$$

- Similarly, since $\langle \ell', m'|[\hat{L}_z, x \pm iy]|\ell, m\rangle = \pm\hbar\langle \ell', m'|x \pm iy|\ell, m\rangle$,

$$(m' - m \mp 1)\langle \ell', m'|x \pm iy|\ell, m\rangle = 0$$

3. Therefore, to get non-zero component of dipole matrix element, require $\Delta m_\ell = 0, \pm 1$.

Selection rules: orbital angular momentum

- Using operator identity $[\hat{\mathbf{L}}^2, [\hat{\mathbf{L}}^2, \mathbf{r}]] = 2\hbar^2(\mathbf{r}\hat{\mathbf{L}}^2 + \hat{\mathbf{L}}^2\mathbf{r})$, we have

$$\begin{aligned}\langle \ell', m'|[\hat{\mathbf{L}}^2, [\hat{\mathbf{L}}^2, \mathbf{r}]]|\ell, m\rangle &= [\ell'(\ell'+1) - \ell(\ell+1)]^2 \langle \ell', m'|\mathbf{r}|\ell, m\rangle \\ &= 2[\ell'(\ell'+1) + \ell(\ell+1)]\langle \ell', m'|\mathbf{r}|\ell, m\rangle\end{aligned}$$

  i.e. $(\ell+\ell')(\ell+\ell'+2)[(\ell'-\ell)^2 - 1]\langle \ell', m'|\mathbf{r}|\ell, m\rangle = 0$. Since $\ell, \ell' \geq 0$, dipole matrix element non-vanishing only if $\ell' = \ell \pm 1$.

4. To effect an electric dipole transition, we must have $\Delta\ell = \pm 1$.

- One may summarize the selection rules for ℓ and m_ℓ is by saying that the photon carries off (or brings in, in an absorption transition) one unit of angular momentum.
- N.B. it is possible, though much less likely in the case of an atom, for EM field to interact with magnetic dipole or electric quadrupole moment with different selection rules.

Selection rules: polarization

$$\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k},\lambda} = \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle\mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\mathbf{d}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k},\lambda}\ \delta(E_\mathrm{f}-E_\mathrm{i}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k},\lambda}+1)\ \delta(E_\mathrm{i}-E_\mathrm{f}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{cases}$$

- For transitions with $\Delta m_\ell = 0$, the dipole matrix element $\langle\mathrm{f}|\mathbf{d}|\mathrm{i}\rangle \sim \hat{\mathbf{e}}_z$ – and there is no component of polarization along z-direction.

- Similarly, for electric dipole transitions with $m' = m \pm 1$, $\langle\ell', m'|x \mp iy|\ell, m\rangle = 0 = \langle\ell', m'|z|\ell, m\rangle$, and $\langle\mathrm{f}|\mathbf{d}|\mathrm{i}\rangle \sim (1, \mp i, 0)$.

  (a) If the wavevector of photon lies along z, the emitted light is circularly polarized with a polarization which depends on helicity.

  (b) If the wavevector lies in xy place, the emitted light is linearly polarized, while in general it is elliptically polarized.

Selection rules: LS coupling

- In the presence of spin-orbit coupling, stationary states labelled by quantum numbers J, m_J, ℓ, s where $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$.
- The selection rules in this case can be inferred by looking for the conditions for non-zero matrix elements $\langle J', m_{J'}, \ell', s' | \mathbf{r} | J, m_J, \ell, s \rangle$.
- By expanding states $|J, m_J, \ell, s\rangle$ in basis states $|\ell, m_\ell\rangle \otimes |s, m_s\rangle$, one may uncover the following set of selection rules:

5. For dipole transitions to take place, we require that

$$\Delta m_j = 0, \pm 1$$
$$\Delta j = 0, \pm 1 \quad \text{not} \quad 0 \to 0$$

- N.B. These conclusions are consistent with photon carrying on unit of angular momentum.

Radiative transitions: recap

- When coupled to a quantized electromagnetic field, the total Hamiltonian for atomic system given by $\hat{H} = \hat{H}_{\rm atom} + \hat{H}_{\rm para} + \hat{H}_{\rm rad}$ where

$$\boxed{\hat{H}_{\rm atom} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}), \qquad \hat{H}_{\rm rad} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \frac{1}{2} \right)}$$

denotes the Hamiltonian of the isolated atomic and radiation field, and

$$\boxed{\hat{H}_{\rm para}(t) = \frac{e}{m} \hat{\mathbf{A}}(\mathbf{r}, t) \cdot \hat{\mathbf{p}}}$$

denotes the coupling with

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_{\mathbf{k}} V}} \left[\hat{\mathbf{e}}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} + \hat{\mathbf{e}}^*_{\mathbf{k}\lambda} a^\dagger_{\mathbf{k}\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} \right]$$

Radiative transitions: recap

- The transition rate between an initial and final state of the atom and electromagnetic field can be estimated using Fermi's Golden rule

$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f}} = \frac{2\pi}{\hbar^2}|\langle \mathrm{f}|\hat{H}_{\mathrm{para}}|\mathrm{i}\rangle|^2\delta(\omega_{\mathrm{if}}-\omega)}$$

  where $\hbar\omega_{\mathrm{if}} = E_{\mathrm{i}} - E_{\mathrm{f}}$.

- Crucially, since the photon creation/annihilation operators obey the relations, $a^\dagger_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}+1}|(n_{\mathbf{k}\lambda}+1)\rangle$ and $a_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}}|(n_{\mathbf{k}\lambda}-1)\rangle$ the transition rate depends on the photon number, $n_{\mathbf{k}\lambda}$.

- When $Z\alpha \ll 1$, the effective range of the interaction of the atom with the field is small (i.e. $kr \sim Z\alpha$) and we can effect the dipole approximation,

$$\boxed{\langle \mathrm{f}|e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{p}}|\mathrm{i}\rangle \simeq \frac{im\omega_{\mathbf{k}}}{e}\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\mathbf{d}|\mathrm{i}\rangle, \qquad \mathbf{d} = -e\mathbf{r}}$$

Radiative transitions: recap

- In the electric dipole approximation, the transition rate is given by

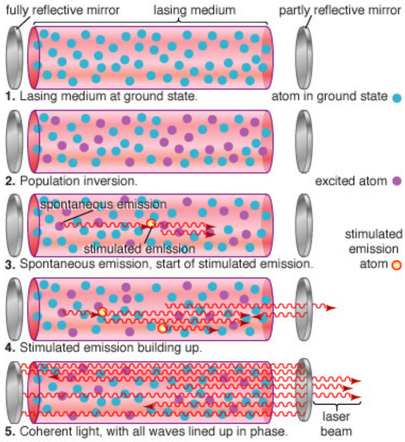
$$\boxed{\Gamma_{\mathrm{i}\to\mathrm{f},\mathbf{k}\lambda} \simeq \frac{\pi\omega_{\mathbf{k}}}{\epsilon_0 V}|\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda}\ \delta(E_{\mathrm{f}}-E_{\mathrm{i}}-\hbar\omega_{\mathbf{k}}) & \text{absorption} \\ (n_{\mathbf{k}\lambda}+1)\ \delta(E_{\mathrm{i}}-E_{\mathrm{f}}-\hbar\omega_{\mathbf{k}}) & \text{emission} \end{cases}}$$

where $\hat{\mathbf{d}} = -e\mathbf{r}$ is electric dipole operator.

- The coincidence of $n_{\mathbf{k}\lambda}$-independent coefficients for absorption and emission coincide is known as **detailed balance**.
- From these results, we turn now to consider the principle of the operation of an atomic laser.

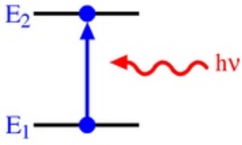
Theory of laser

- Principle of stimulated emission provides basis of laser operation: light amplification by stimulated emission of radiation.
- However, laser not only amplifies light, but provides source of **monochromatic** (single mode), **coherent** (spatial/temporal), **directional** and **intense** radiation.
- In atomic laser, the gain medium provided by a gas of atoms confined to a cavity and bound by highly reflective mirrors.



Theory of laser: rate equations

$$\boxed{\Gamma_{\mathbf{k}\lambda} \sim |\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda} & \text{abs.} \\ (n_{\mathbf{k}\lambda}+1) & \text{emiss.} \end{cases}}$$



- Consider gas of atoms in a cavity subject to an EM field of intensity $I \propto n(\omega)$ and angular frequency ω tuned to energy difference between two discrete energy levels of the atoms, i.e. $\hbar\omega = E_2 - E_1$.
- Taking into account stimulated absorption, atoms are transferred from level 1 to level 2 at a rate

$$\Gamma_{12} = WN_1 n(\omega)$$

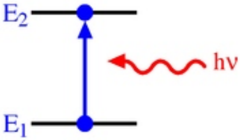
where N_1 atoms in level 1 and W includes matrix elements.

- From spontaneous and stimulated emission processes, the rate of transfer of atoms from level 2 to level 1 is given by

$$\Gamma_{21} = WN_2(n(\omega) + 1)$$

Theory of laser: photon equations

$$\boxed{\Gamma_{\mathbf{k}\lambda} \sim |\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \left\{ \begin{array}{rl} n_{\mathbf{k}\lambda} & \text{abs.} \\ (n_{\mathbf{k}\lambda}+1) & \text{emiss.} \end{array} \right.}$$



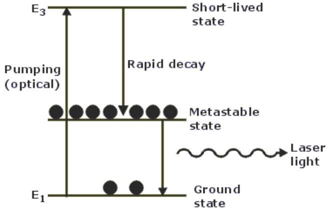
- Since transfer of particles from level 2 to 1 leads to creation of photons in cavity while from 1 to 2 they involve absorption, the rate of change of photon number is given by $\dot{n} = W(N_2(n+1) - N_1 n)$.
- However, to make use of cavity as a photon source, we have to allow photons to leak from the cavity through imperfect mirrors. Taking into account this and other loss processes, we have

$$\boxed{\dot{n} = DWn + N_2 W - \frac{n}{\tau_{\mathrm{ph}}}}$$

where $D = N_2 - N_1$ denotes population imbalance and $1/\tau_{\mathrm{ph}}$ is the total loss rate.

Theory of laser: matter equations

$$\boxed{\Gamma_{\mathbf{k}\lambda} \sim |\langle \mathrm{f}|\hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \hat{\mathbf{d}}|\mathrm{i}\rangle|^2 \begin{cases} n_{\mathbf{k}\lambda} & \text{abs.} \\ (n_{\mathbf{k}\lambda} + 1) & \text{emiss.} \end{cases}}$$



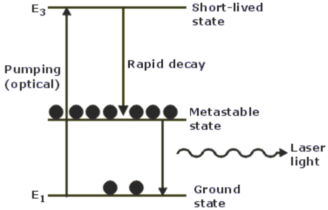
- Without further external processes, photons would escape from cavity and the system would relax into ground state – To create a steady-state photon population, energy must be pumped into the system in the form of excitations.
- Achieved by transferring atoms between 1 and 2 via level 3 by non-resonant optical pump. If lifetime of 3 is short, occupancy is effectively zero, rate of transfer of particles from 2 to 1,

$$\dot{N}_2 \simeq -w_{21} N_2 + w_{12} N_1 - (N_2 - N_1) W n \simeq -\dot{N}_1$$

where we have dropped small contribution from spontaneous emission, and w_{12}, w_{21} denote net non-resonant transition rates.

Theory of laser: stationary equations

$$\dot{N}_2 \simeq -w_{21}N_2 + w_{12}N_1 - (N_2 - N_1)Wn \simeq -\dot{N}_1$$



- Without cavity photons ($n = 0$), since $N_1 + N_2 \simeq N$, in steady state,

$$D^{(0)} = N_2^{(0)} - N_1^{(0)} = N\frac{w_{12} - w_{21}}{w_{12} + w_{21}}$$

denotes **unsaturated inversion**.

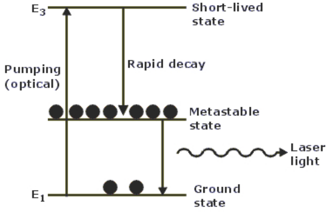
- Restoring the cavity photons, we have

$$\boxed{\dot{D} = \dot{N}_2 - \dot{N}_1 = \frac{D^{(0)} - D}{T} - 2DWn}$$

where $1/T = w_{12} + w_{21}$ represents typical relaxation rate.

Theory of laser: stationary equations

$$\boxed{\dot{n} = DWn - \frac{n}{\tau_{\rm ph}}, \quad \dot{D} = \frac{D^{(0)} - D}{T} - 2DWn}$$



- In steady-state operation, $\dot{n} = \dot{D} = 0$, population imbalance

$$\boxed{D \equiv N_2 - N_1 = \frac{D^{(0)}}{1 + 2TWn}}$$

- From this result, we find the steady state photon number

$$\boxed{n = \frac{D^{(0)}W - 1/\tau_{\rm ph}}{2TW/\tau_{\rm ph}}}$$

- This result shows that the system will only start lasing when the unsaturated inversion exceeds a threshold, $D^{(0)} > 1/\tau_{\rm ph}W$.

Theory of laser: coherence

- Although the analysis above addressed the threshold conditions for the laser, it does not provide any insight into the coherence properties of the radiation field.
- In fact, one may show that the radiation field generated by the laser cavity forms a **coherent or Glauber state**.
- The proof of this statement and the coherence properties that follow would take us on a considerable detour – see Part III quantum optics.
- However, we can gain some insight into the properties and physical manifestations of coherent states by looking at a toy example; but first we must define what we mean by a coherent state.

Coherent states

- A coherent state is defined as an eigenstate of the annihilation operator,

$$\boxed{a|\beta\rangle = \beta|\beta\rangle}$$

Since a is not Hermitian, β can take complex eigenvalues.

- The eigenstates are constructed from the harmonic oscillator ground state the by action of the unitary operator,

$$\boxed{|\beta\rangle = \hat{U}(\beta)|0\rangle, \qquad \hat{U}(\beta) = e^{\beta a^\dagger - \beta^* a}, \qquad \hat{U}^\dagger(\beta)\hat{U}(\beta) = \mathbb{I}}$$

- The proof follows from the identity (problem set I),

$$a\hat{U}(\beta) = \hat{U}(\beta)(a+\beta), \quad \text{i.e.} \quad a\hat{U}(\beta)|0\rangle = \beta\hat{U}(\beta)|0\rangle$$

i.e. $\hat{U}$ is a translation operator, $\hat{U}^\dagger(\beta)a\hat{U}(\beta) = a + \beta$.

Coherent states

$$\boxed{|\beta\rangle = \hat{U}(\beta)|0\rangle, \qquad \hat{U}(\beta) = e^{\beta a^\dagger - \beta^* a}}$$

- Since $\hat{U}(\beta) = e^{\beta a^\dagger - \beta^* a} = e^{-|\beta|^2/2} e^{\beta a^\dagger} e^{-\beta^* a}$ and $e^{-\beta^* a}|0\rangle = |0\rangle$, we can write

$$\boxed{|\beta\rangle = e^{-|\beta|^2/2} e^{\beta a^\dagger}|0\rangle}$$

- With $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$, we can write

$$|\beta\rangle = \sum_n e^{-|\beta|^2/2} \frac{\beta^n}{\sqrt{n!}}|n\rangle$$

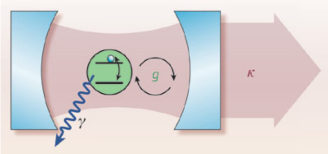
showing that the probability of observing n excitations

$$\boxed{P_n = |\langle n|\beta\rangle|^2 = e^{-|\beta|^2} \frac{|\beta|^{2n}}{n!}}$$

is a Poisson distribution with average occupation, $\langle\beta|a^\dagger a|\beta\rangle = |\beta|^2$.

Driven quantum harmonic oscillator

But how can we prepare a system in a coherent state?



- Consider a single two-level atom resonantly coupled to a single cavity mode – the quantum Hamiltonian of the coupled system,

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + \hbar g(\sigma_- a + \sigma_+ a^\dagger)$$

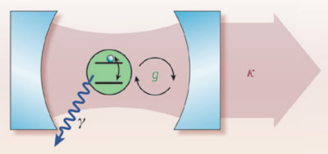
- When excitations of two level system are driven by an external pump, it can behave as a classical dipole source for the cavity mode leading to the driven harmonic oscillator Hamiltonian,

$$\hat{H} \simeq \hat{H}_{\mathrm{rad}} + V(t) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + i\hbar\left(f(t)a^\dagger - f^*(t)a\right)$$

where $f(t) = f_0 e^{-i\omega t}$

Driven quantum harmonic oscillator

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + i\hbar\left(f(t)a^\dagger - f^*(t)a\right)$$



$$f(t) = f_0 e^{-i\omega t}$$

- If photon system is prepared in ground state, $|0\rangle$, the perturbation drives the system into a coherent state.
- To understand how, let us turn to the interaction representation, $i\hbar\partial_t|\psi(t)\rangle_{\rm I} = V_{\rm I}|\psi(t)\rangle_{\rm I}$ where $|\psi(t)\rangle_{\rm I} = e^{i\hat{H}_0 t/\hbar}|\psi(t)\rangle_{\rm S}$. With $e^{i\omega t a^\dagger a} a e^{-i\omega t a^\dagger a} = e^{-i\omega t} a$,

$$V_{\rm I}(t) = e^{i\hat{H}_0 t/\hbar} i\hbar\left(f(t)a^\dagger - f^*(t)a\right) e^{-i\hat{H}_0 t/\hbar} = i\hbar\left(f_0 a^\dagger - f_0^* a\right)$$

- Since $V_{\rm I}(t)$ is time-independent, the time-evolution operator, defined by the equation $i\hbar\partial_t U_{\rm I}(t) = V_{\rm I} U_{\rm I}(t)$, is given simply by

$$\boxed{U_{\rm I}(t) = \exp\left[(f_0 a^\dagger - f_0^* a)t\right]}$$

Driven quantum harmonic oscillator

$$\boxed{U_{\rm I}(t) = \exp\left[(f_0 a^\dagger - f_0^* a)t\right]}$$

- Therefore, if the system was prepared in the ground state $|0\rangle$ at $t = 0$, at later times,

$$|\psi(t)\rangle_{\rm I} = \exp[(f_0 a^\dagger - f_0^* a)t]|0\rangle = e^{-|f_0|^2 t^2/2} e^{f_0 a^\dagger t}|0\rangle$$

- Reexpressed in the Schrödinger representation,

$$|\psi(t)\rangle_{\rm S} = e^{-i\hat{H}_0 t/\hbar}|\psi(t)\rangle_{\rm I} = e^{-|f_0|^2 t^2/2} e^{f_0 e^{-i\omega t} a^\dagger t}|0\rangle$$

- A classical oscillatory force drives a system prepared in the vacuum state into a coherent state with an excitation number which climbs as $|f_0|^2 t^2$.

Synopsis: Lectures 16-19

12. **Field theory: from phonons to photons:**

    From particles to fields: classical field theory of harmonic atomic chain; quantization of atomic chain; phonons. Classical theory of the EM field; waveguide; quantization of the EM field and photons.

13. **Time-dependent perturbation theory:**

    Rabi oscillations in two level systems; perturbation series; sudden approximation; harmonic perturbations and Fermi's Golden rule.

14. **Radiative transitions:**

    Light-matter interaction; spontaneous emission; absorption and stimulated emission; Einstein's A and B coefficents; dipole approximation; selection rules; †lasers.

Synopsis: Lectures 20-24

15. **Scattering theory**

    Elastic scattering; cross section; method of particle waves; Born approximation; scattering of identical particles.

16. **Relativistic quantum mechanics:**

    Klein-Gordon equation; Dirac equation; relativistic covariance and spin; free relativistic particles and the Klein paradox; antiparticles; coupling to EM field: minimal coupling and the connection to non-relativistic quantum mechanics; †field quantization.