Phase Transitions and Collective Phenomena

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Preface

The fundamental goal of statistical mechanics is to provide a framework in which the microscopic probabilistic description of systems with large numbers of degrees of freedom (such as the particles which constitute a gas) can be reconciled with the description at the macroscopic level (using equilibrium state variables such as pressure, volume and temperature). When we first meet these ideas they are usually developed in parallel with simple examples involving collections of weakly or non-interacting particles. However, strong interactions frequently induce transitions and lead to new equilibrium phases of matter. These phases exhibit their own characteristic fluctuations or elementary excitations known as collective modes. Although a description of these phenomena at the microscopic level can be quite complicated, the important large scale, or long-time “hydrodynamic” behaviour is often simple to describe. Phenomenological approaches based on this concept have led to certain quantum as well as classical field theories that over recent years have played a major role in shaping our understanding of condensed matter and high energy physics.

The goal of this course is to motivate this type of description; to establish and begin to develop a framework in which to describe critical properties associated with classical and quantum phase transitions; and, at the same time, to emphasise the importance and role played by symmetry and topology. Inevitably there is insufficient time to study such a wide field in any great depth. Instead, the aim will always be to develop fundamental concepts.

The phenomenological Ginzburg-Landau theory has played a pivotal rôle in the development of our understanding critical phenomena in both classical and quantum statistical mechanics, and much of our discussion will be based on it. The majority of the course will be involved in developing the important concept of universality in statistical mechanics and establish a general framework to describe critical phenomena — the scaling theory and the renormalisation group.
Synopsis

▷ **Introduction to Critical Phenomena:** Concept of Phase Transitions; Order Parameters; Response Functions; Universality. [1]

▷ **Ginzburg-Landau Theory:** Mean-Field Theory; Critical Exponents; Symmetry Breaking, Goldstone Modes, and the Lower Critical Dimension; Fluctuations and the Upper Critical Dimension; Importance of Correlation Functions; Ginzburg Criterion. [3]

▷ **Scaling:** Self-Similarity; The Scaling Hypothesis; Kadanoff’s Heuristic Renormalisation Group (RG); Gaussian Model; Fixed Points and Critical Exponent Identities; Wilson’s Momentum Space RG; Relevant, Irrelevant and Marginal Parameters; $\epsilon$-expansions. [4]

▷ **Topological Phase Transitions:** Continuous Spins and the Non-linear $\sigma$-model; XY-model; Algebraic Order; Topological defects, Confinement, the Kosterlitz-Thouless Transition and Superfluidity in Thin Films. [2]

Material indicated by a $^\dagger$ will be included if time allows.
Bibliography


This course follows no particular text but a number of books may be useful. Those which are particularly useful are marked by a “∗” in the list. I wish to thank Prof. Simons for an earlier version of these notes.
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Chapter 1

Critical Phenomena

The aim of this introductory chapter is to introduce the concept of a phase transition and to motivate the subject of statistical field theory. Here we introduce the concept of universality as applied to critical phenomena and define some of the notation used throughout these lectures.

1.1 Collective Phenomena: from Particles to Fields

It is rare in physics to find examples of interacting many-particle systems which admit to a full and accessible microscopic description. More useful is a hydrodynamic description of the collective long-wavelength behaviour which surrenders information at the microscopic scale. A familiar example is the Navier-Stokes equation of fluid dynamics. The averaged variables appropriate to these length and time scales are no longer the discrete set of particle degrees of freedom but rather slowly varying continuous fields describing the collective motion of a macroscopic set of particles. Familiar examples include magnetic spin-waves, and vibrational or phonon modes of an atomic lattice.

The most striking consequence of interactions among particles is the appearance of new phases of matter whose collective behaviour bears little resemblance to that of a few particles. How do the particles then transform from one macroscopic state to another? Formally, all macroscopic properties can be deduced from the free energy or the partition function. However, since phase transitions typically involve dramatic changes in response functions they must correspond to singularities in the free energy. Since the canonical partition function of a finite collection of particles is always analytic, phase transitions can only be associated with infinitely many particles, i.e. the thermodynamic limit. The study of phase transitions is thus related to finding the origin of various singularities in the free energy and characterising them.

Consider the classical equilibrium statistical mechanics of a regular lattice of ‘one-component’ or Ising ferromagnet (i.e. spin degrees of freedom which can take only two values: ±1). When viewed microscopically, the development of magnetic moments on the atomic lattice sites of a crystal and the subsequent ordering of the moments is a complex process involving the cooperative phenomena of many interacting electrons.
However, remarkably, the thermodynamic properties of different macroscopic ferromagnetic systems are observed to be the same — e.g. temperature dependence of the specific heat, susceptibility, etc. Moreover, the thermodynamic critical properties of completely different physical systems, such as an Ising ferromagnet and a liquid at its boiling point, show the same dependence on, say, temperature. What is the physical origin of this **Universality**?

Suppose we take a ferromagnetic material and measure some of its material properties such as magnetisation. Dividing the sample into two roughly equal halves, keeping the internal variables like temperature and magnetic field the same, the macroscopic properties of each piece will then be the same as the whole. The same holds true if the process is repeated. But eventually, after many iterations, something different must happen because we know that the magnet is made up of electrons and ions. The characteristic length scale at which the overall properties of the pieces begins to differ markedly from those of the original defines a **correlation length**. It is the typical length scale over which the fluctuations of the microscopic degrees of freedom are correlated.

![Phase diagram of the Ising ferromagnet showing the average magnetisation $M$ as a function of magnetic field $H$ and temperature $T$.](image)

**Figure 1.1:** Phase diagram of the Ising ferromagnet showing the average magnetisation $M$ as a function of magnetic field $H$ and temperature $T$. Following trajectory 1 by changing the magnetic field at constant temperature $T < T_c$, the sample undergoes a first order phase transition from a phase in which the average magnetisation is positive (i.e. ‘spin-up’) to a phase in which the average is negative (i.e. ‘spin-down’). Secondly, by changing the temperature at fixed zero magnetic field, the system undergoes a second order phase transition at $T = T_c$ whereupon the average magnetisation grows continuously from zero. This second order transition is accompanied by a **spontaneous symmetry breaking** in which the system chooses to be in either an up or down-spin phase. (Contrast this phase diagram with that of the liquid-gas transition — magnetisation $M \rightarrow$ density $\rho$, and magnetic field $H \rightarrow$ pressure.) The circle marks the region in the vicinity of the critical point where the correlation length is large as compared to the microscopic scales of the system, and ‘Ginzburg-Landau theory’ applies.

Now experience tells us that a ferromagnet may abruptly change its macroscopic behaviour when the external conditions such as the temperature or magnetic field are varied. The points at which this happens are called **critical points**, and they mark a
phase transition from one state to another. In the ferromagnet, there are essentially two ways in which the transition can occur (see Fig. 1.1). In the first case, the two states on either side of the critical point (spin up) and (spin down) coexist at the critical point. Such transitions, involve discontinuous behaviour of thermodynamic properties and are termed first-order (cf. melting of a three-dimensional solid). The correlation length at such a first-order transition is generally finite.

In the second case, the transition is continuous, and the correlation length becomes effectively infinite. Fluctuations become correlated over all distances, which forces the whole system to be in a unique, critical, phase. The two phases on either side of the transition (paramagnetic and ferromagnetic) must become identical as the critical point is approached. Therefore, as the correlation length diverges, the magnetisation goes smoothly to zero. The transition is said to be second-order.

The divergence of the correlation length in the vicinity of a second order phase transition suggests that properties near the critical point can be accurately described within an effective theory involving only long-range collective fluctuations of the system. This invites the construction of a phenomenological Hamiltonian or Free energy which is constrained only by the fundamental symmetries of the system. Such a description goes under the name of Ginzburg-Landau theory. Although the detailed manner in which the material properties and microscopic couplings of the ferromagnet influence the parameters of the effective theory might be unknown, qualitative properties such as the scaling behaviour are completely defined.

From this observation, we can draw important conclusions: critical properties in the vicinity of a both classical and quantum second order phase transitions fall into a limited number of universality classes defined not by detailed material parameters, but by the fundamental symmetries of the system. When we study the critical properties of the Ising transition in a one-component ferromagnet, we learn about the nature of the liquid-gas transition! (See below.) Similarly, in the jargon of statistical field theory, a superconductor, with its complex order parameter, is in the same universality class as the two-component or ‘XY Heisenberg’ ferromagnet. The analyses of critical properties associated with different universality classes is the subject of Statistical field theory.

1.2 Phase Transitions

With these introductory remarks in mind, let us consider more carefully the classic example of a phase transition involving the condensation of a gas into a liquid. The phase diagram represented in Fig. 1.2a exhibits several important features and generic features of a second order phase transition:

1. In the \((T, P)\) plane, the phase transition occurs along a line that terminates at a critical point \((T_c, P_c)\).

2. In the \((P, v \equiv V/N)\) plane, the transition appears as a coexistence interval, corresponding to a mixture of gas and liquid of densities \(\rho_g = 1/v_g\) and \(\rho_l = 1/v_l\) at temperatures \(T < T_c\).
CHAPTER 1. CRITICAL PHENOMENA

3. Due to the termination of the coexistence line, it is possible to go from the gas phase to the liquid phase continuously (without a phase transition) by going around the critical point. Thus there are no fundamental differences between the liquid and gas phases (i.e. there is no change of fundamental symmetry).

From a mathematical perspective, the free energy of this system is an analytic function in the \((P, T)\) plane except for some form of branch cut along the phase boundary. Observations in the vicinity of the critical point further indicate that:

4. The difference between the densities of the coexisting liquid and gas phases vanishes on approaching \(T_c\), i.e. \(\rho_l \to \rho_g\) as \(T \to T_c^-\).

5. The pressure versus volume isotherms become progressively more flat on approaching \(T_c\) from the high temperature side. This implies that the isothermal compressibility, the rate of change of density with pressure, \(\kappa_T = -(1/V)\partial V/\partial P|_T\) diverges as \(T \to T_c^+\).

6. The fluid appears “milky” close to criticality. This phenomenon, known as critical opalescence suggests the existence of collective fluctuations in the gas at long enough wavelengths to scatter visible light. These fluctuations must necessarily involve many particles, and a coarse-graining procedure must be appropriate to their description.
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How does this phase diagram compare with the phase transition that occurs between paramagnetic and ferromagnetic phases of certain substances such as iron or nickel. These materials become spontaneously magnetised below a Curie temperature, \( T_c \). Redrawn in cross-section, the phase diagram of Fig. 1.1 is shown in Fig. 1.2b. There is a discontinuity in magnetisation as the magnetic field, \( H \) goes through zero, and the magnetisation isotherms, \( M(H) \) have much in common with the condensation problem. In both cases, a line of discontinuous transitions terminates at a critical point, and the isotherms exhibit singular behaviour in the vicinity of this point. The phase diagram is simpler in appearance because the symmetry \( H \to -H \) ensures that the critical point occurs at \( H_c = M_c = 0 \).

In spite of the apparent similarities between the magnetic and liquid-gas transition, our intuition would suggest that they are quite different. Magnetic transitions are usually observed to be second-order — that is, the magnetisation \( m \), which plays the role of an order parameter, rises continuously from zero below the transition. On the other hand, our everyday experience of boiling a kettle of water shows the liquid-gas transition to be first-order — that is, the order parameter, corresponding to the difference between the actual density and the density at the critical point, \( \rho - \rho_c \) jumps discontinuously at the critical point with an accompanying absorption of latent heat of vapourisation (implying a discontinuous jump in the entropy of the system: \( Q_L = T_c \Delta S \)).

However, the perceived difference in behaviour simply reflects different paths through the transition in the two cases. In a ferromagnet, the natural experimental path (\( b \to c \to d \) in Fig. 1.3a) is one in which the external magnetic field takes the value \( H = 0 \). For \( T > T_c \), the average magnetisation is zero, while for \( T < T_c \) the magnetisation grows continuously from zero. In a liquid, the natural path is one in which temperature is varied at constant pressure (\( b' \to c' \to d' \) in Fig. 1.3b). Along this path, there is a discontinuous change in the density. This is the first-order boiling transition.

A path in the ferromagnetic \((H, T)\) plane analogous to the constant pressure path in a fluid is shown in Fig. 1.3c: Along this path \( m \) is negative from \( b' \to c' \) and then jumps discontinuously to a positive value as the coexistence line is crossed and remains positive from \( c' \to d' \). It is clear that the path in a fluid that most closely resembles the \( H = 0 \) path in a magnet, which shows a second-order transition, is the one with density fixed at its critical value \( \rho_c \), i.e. the critical isochore (\( b \to c \to d \) in Fig. 1.3d).

1.3 Critical Behaviour

The singular behaviour in the vicinity of the critical point is characterised by a set of critical exponents. These exponents describe the non-analyticity of various thermodynamic functions. Remarkably transitions as different as the liquid/gas and ferromagnetic transition can be described by the same set of critical exponents and are said to belong to the same Universality class.
1.3.1 Significance of Power laws

INFO: Prior to defining physically relevant critical exponents, we take a mathematical digression in probability theory. First consider an exponential probability density function

\[ p(x) = ae^{-ax} \]

for \( x > 0 \). The mean \( \langle x \rangle = 1/a \) and mean squared \( \langle x^2 \rangle = 2/a \). So, for this distribution the typical scale is \( 1/a \) with typical fluctuations of \( 1/a \). Now, consider a power law distribution

\[ p(x) \sim \frac{1}{x^{1+\mu}}. \]

For \( \mu \leq 1 \) the mean diverges \( \langle x \rangle \to \infty \), which implies that there is no typical size. Also for \( \mu \leq 2 \), \( \langle x^2 \rangle \to \infty \) so fluctuations are also unbounded. Such power laws distributions have no typical size and are thus scale invariant. They also have fluctuations or structure at all length scales.

Those critical exponents most commonly encountered are defined below.

1.3.2 Order Parameter

By definition, there is more than one equilibrium phase on a coexistence line. As mentioned above, the order parameter is a thermodynamic function that is different in each
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Figure 1.4: Critical behaviour of the magnetisation and response functions close to the ferromagnetic transition.

phase, and hence can be used to distinguish between them. For a (uniaxial) magnet, the order parameter is provided by the total magnetisation $M(H,T)$, or magnetisation density,

$$m(H,T) = \frac{M(H,T)}{V}.$$  

In zero field, $m$ vanishes for a paramagnet and is non-zero in a ferromagnet (see Fig. 1.4), i.e.

$$m(T,H \to 0^+) \propto \begin{cases} 0 & T > T_c, \\ |t|^\beta & T < T_c, \end{cases}$$

where $t = (T - T_c)/T_c$ denotes the reduced temperature. The singular behaviour of the order parameter along the coexistence line is therefore characterised by a critical exponent $\beta$. The singular behaviour of $m$ along the critical isotherm is governed by another exponent, $\delta$ (see Fig. 1.2)

$$m(T = T_c, H) \propto H^{1/\delta}.$$  

The two phases along the liquid-gas coexistence line are differentiated by their density allowing one to define $\rho - \rho_c$, where $\rho_c$ denotes the critical density, as the order parameter.

1.3.3 Response Functions

The critical system is highly sensitive to external perturbations: for example, at the liquid-gas critical point, the compressibility $\kappa_T = -(1/V)\partial V/\partial P|_T$ becomes infinite. The divergence of the compressibility or, more generally, the susceptibility of the system (i.e. the response of the order parameter to a field conjugate to it) is characterised by another critical exponent $\gamma$. For example, in a magnet, the analogous quantity is the zero-field susceptibility

$$\chi_{\pm}(T,H \to 0^+) = \frac{\partial m}{\partial H} \bigg|_{H=0^+} \propto |t|^{-\gamma_{\pm}}.$$
where, in principle, two exponents $\gamma_+$ and $\gamma_-$ are necessary to describe the divergences on the two sides of the phase transition. Actually, in almost all cases, the same singularity governs both sides, and $\gamma_+ = \gamma_- = \gamma$.

Similarly, the **heat capacity** represents the thermal response function, and its singularities at zero field are described by the exponent $\alpha$,

$$C_\pm = \frac{\partial E}{\partial T} \propto |t|^{-\alpha \pm},$$

where $E$ denotes the internal energy and, again, the exponents usually coincide $\alpha_+ = \alpha_- = \alpha$.

### 1.3.4 Long-range Correlations

Since the response functions are related to equilibrium fluctuations, their divergence in fact implies that fluctuations are correlated over long distances. To see this let us consider the magnetic susceptibility of, say, the **Ising ferromagnet**. The latter describes a lattice of scalar or one-component spins which interact ferromagnetically with their neighbours. Starting with the microscopic Ising Hamiltonian

$$H_{\text{Ising}} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j,$$

where $\{\sigma_i = \pm 1\}$ denotes the set of Ising spins, $M = \sum_i \sigma_i$ represents the total magnetisation, and sum $\sum_{\langle ij \rangle}$ runs over neighbouring lattice sites, the total partition function\(^1\) takes the form

$$Z(T, h) = \sum_{\{\sigma_i\}} e^{-\beta (H_{\text{Ising}} - hM)}.$$

Here we have included an external magnetic field $h$, and the sum extends over the complete set of microstates $\{\sigma_i\}$. From $Z$, the thermal average magnetisation can be obtained from the equation

$$\langle M \rangle \equiv \frac{\partial \ln Z}{\partial (\beta h)} = 1 \sum_{\{\sigma_i\}} Me^{-\beta (H_{\text{Ising}} - hM)}.$$

Taking another derivative one obtains the susceptibility

$$\chi(T, h) = \frac{1}{V} \frac{\partial \langle M \rangle}{\partial h} = \frac{\beta}{V} \left\{ \frac{1}{Z} \sum_{\{\sigma_i\}} M^2 e^{-\beta (H_{\text{Ising}} - hM)} - \left( \frac{1}{Z} \sum_{\{\sigma_i\}} Me^{-\beta (H_{\text{Ising}} - hM)} \right)^2 \right\},$$

from which identifies the relation

$$\frac{V \chi}{\beta} = \text{var}(M) \equiv \langle M^2 \rangle - \langle M \rangle^2.$$

\(^1\)Throughout these notes $1/k_B T$ and the symbol $\beta$ (not to be mistaken for the order parameter exponent) will be used interchangeably.
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Now the overall magnetisation can be thought of as arising from separate contributions from different parts of the system, i.e. taking a continuum limit

\[ M = \int d\mathbf{x} \, m(\mathbf{x}), \]

where \( m(\mathbf{x}) \) represents the “local” magnetisation. Substituting this definition into the equation above we obtain

\[ k_B T \chi = \frac{1}{V} \int d\mathbf{x} \int d\mathbf{x}' \left[ \langle m(\mathbf{x})m(\mathbf{x}') \rangle - \langle m(\mathbf{x}) \rangle \langle m(\mathbf{x}') \rangle \right]. \]

Since the system is symmetric under spatial translation, \( \langle m(\mathbf{x}) \rangle \) is a constant \( m \) independent of position, while \( \langle m(\mathbf{x})m(\mathbf{x}') \rangle = G(\mathbf{x} - \mathbf{x}') \) depends only on the spatial separation. Thus, in terms of the ‘connected’ correlation function defined as \( G_c(\mathbf{x}) = \langle m(\mathbf{x})m(\mathbf{x}'(0)) \rangle_c = \langle m(\mathbf{x})m(\mathbf{x}(0)) \rangle_m - m^2 \), the susceptibility is given by

\[ k_B T \chi = \int d\mathbf{x} \langle m(\mathbf{x})m(0) \rangle_c. \]

The connected correlation function is a measure of how the local fluctuations in one part of the system affect those in another. Typically such influences occur over a characteristic distance \( \xi \) known as the correlation length (see Fig. 1.4). (In many cases, the connected correlation function \( G_c(\mathbf{x}) \) decays exponentially \( \exp\left[-|\mathbf{x}|/\xi\right] \) at separations \( |\mathbf{x}| > \xi \).) If \( g \sim m^2 \) denotes a typical value of the correlation function for \( |\mathbf{x}| < \xi \), it follows that \( k_B T \chi < g\xi^d \) where \( d \) denotes the dimensionality; the divergence of \( \chi \) necessarily implies the divergence of \( \xi \). This divergence of the correlation length also explains the observation of critical opalescence. The correlation function can be measured by scattering probes, and its divergence

\[ \xi_{\pm}(T, H = 0) \propto |t|^{-\nu_{\pm}} \]

is controlled by exponents \( \nu_+ = \nu_- = \nu \).

This completes our preliminary survey of phase transitions and critical phenomena. We found that the singular behaviour of thermodynamic functions at a critical point (the termination of a coexistence line) can be characterised by a set of critical exponents. Experimental observations indicate that these exponents are universal, independent of the material, and to some extent of the nature of the transition. Moreover the divergence of the response functions, as well as fluctuations, indicate that correlations become long-ranged in the vicinity of this point. The fluctuations responsible for the correlations involve many particles, and their description calls for a “coarse-graining” approach. In the next chapter we will exploit this idea to construct a statistical field theory which reveals the origin of the universality. To do so it will be convenient to frame our discussion in the language of the magnetic system whose symmetry properties are transparent. The results, however, have a much wider range of validity.
Chapter 2

Ginzburg-Landau Phenomenology

The divergence of the correlation length in the vicinity of a second-order phase transition indicates that the properties of the critical point are insensitive to microscopic details of the system. This redundancy of information motivates the search for a phenomenological description of critical phenomena which is capable of describing a wide range of model systems. In this chapter we introduce and investigate such a phenomenology known as the Ginzburg-Landau theory. Here we will explore the ‘mean-field’ properties of the equilibrium theory and perturbatively investigate the influence of spatial fluctuations.

2.1 Ginzburg-Landau Theory

Consider the magnetic properties of a metal, say iron, close to its Curie point. The microscopic origin of magnetism is quantum mechanical, involving such ingredients as itinerant electrons, their spin, and the exclusion principle. Clearly a microscopic description is complicated, and material dependent. Such a theory would be necessary if we are to establish which elements are likely to produce ferromagnetism. However, if we accept that such behaviour exists, a microscopic theory is not necessarily the most useful way to describe its disappearance as a result of thermal fluctuations. This is because the (quantum) statistical mechanics of a collection of interacting electrons is excessively complicated. However, the degrees of freedom which describe the transition are long-wavelength collective excitations of spins. We can therefore “coarse-grain” the magnet to a scale much larger than the lattice spacing, and define a magnetisation vector field \( \mathbf{m}(\mathbf{x}) \), which represents the average of the elemental spins in the vicinity of a point \( \mathbf{x} \). It is important to emphasise that, while \( \mathbf{x} \) is treated as a continuous variable, the functions \( \mathbf{m} \) do not exhibit any variations at distances of the order of the lattice spacing \( a \), i.e. their Fourier transforms involve wavevectors with magnitude less than some upper cut-off \( \Lambda \sim 1/a \).

In describing other types of phase transitions, the role of \( \mathbf{m}(\mathbf{x}) \) is played by the appropriate order parameter. For this reason it is useful to examine a generalised magnetisation vector field involving \( n \)-component magnetic moments existing in a \( d \)-dimensional space,
i.e. \( \mathbf{x} \equiv (x_1, \cdots, x_d) \in \mathbb{R}^d \) (space), \( \mathbf{m} \equiv (m_1, \cdots, m_n) \in \mathbb{R}^n \) (spin).

Some specific problems covered in this framework include:

- \( n = 1 \): Liquid-gas transitions; binary mixtures; and uniaxial magnets;
- \( n = 2 \): Superfluidity; superconductivity; and planar magnets;
- \( n = 3 \): Classical isotropic magnets.

While most applications occur in three-dimensions, there are also important phenomena on surfaces \( (d = 2) \), and in wires \( (d = 1) \). (Relativistic field theory is described by a similar structure, but in \( d = 4 \).)

A general coarse-grained Hamiltonian can be constructed on the basis of appropriate symmetries:

1. **Locality**: The Hamiltonian should depend on the local magnetisation and short range interactions expressed through gradient expansions:

   \[
   \beta H = \int d\mathbf{x} \, f[\mathbf{m}(\mathbf{x}), \nabla \mathbf{m}, \cdots]
   \]

2. **Rotational Symmetry**: Without magnetic field, the Hamiltonian should be isotropic in space and therefore invariant under rotations, \( \mathbf{m} \mapsto R_n \mathbf{m} \).

   \[
   \beta H[\mathbf{m}] = \beta H[R_n \mathbf{m}].
   \]

3. **Translational and Rotational Symmetry in \( \mathbf{x} \)**: This last constraint finally leads to a Hamiltonian of the form

   \[
   \beta H = \int d\mathbf{x} \left[ \frac{1}{2} \mathbf{m}^2 + u \mathbf{m}^4 + \cdots \\
   + \frac{K}{2} (\nabla \mathbf{m})^2 + \frac{L}{2} (\nabla^2 \mathbf{m})^2 + \frac{N}{2} \mathbf{m}^2 (\nabla \mathbf{m})^2 + \cdots - \mathbf{h} \cdot \mathbf{m} \right].
   \tag{2.1}
   \]

Eq. (2.1) is known as the **Ginzburg-Landau Hamiltonian**. It depends on a set of phenomenological parameters \( t, u, K, \) etc. which are non-universal functions of microscopic interactions, as well as external parameters such as temperature, and pressure.\(^1\)

\(^1\)It is essential to appreciate the latter point, which is usually the source of much confusion. The probability for a particular configuration of the field is given by the Boltzmann weight \( \exp[-\beta H[\mathbf{m}(\mathbf{x})]] \). This does not imply that all terms in the exponent are proportional to \( (k_B T)^{-1} \). Such dependence holds only for the true microscopic Hamiltonian. The Ginzburg-Landau Hamiltonian is more accurately regarded as an effective free energy obtained by integrating over the microscopic degrees of freedom (coarse-graining), while constraining their average to \( \mathbf{m}(\mathbf{x}) \). It is precisely because of the difficulty of carrying out such a first principles program that we postulate the form of the resulting effective free energy on the basis of symmetries alone. The price we pay is that the phenomenological parameters have an unknown functional dependence on the original microscopic parameters, as well as on external constraints such as temperature (since we have to account for the entropy of the short distance fluctuations lost in the coarse-graining procedure).
2.2 Landau Mean-Field Theory

The original problem has been simplified considerably by focusing on the coarse-grained magnetisation field described by the Ginzburg-Landau Hamiltonian. The various thermodynamic functions (and their singular behaviour) can now be obtained from the corresponding partition function

\[ Z[T, h] = \int Dm(x) \, e^{-\beta H[m, h]} \]  

(2.2)

Since the degrees of freedom appearing in the Hamiltonian are functions of \(x\), the symbol \(\int Dm(x)\) refers to the functional integral. As such, it should be regarded as a limit of discrete integrals, i.e., for a one-dimensional Hamiltonian,

\[ \int Dm(x) \, z[m(x), \partial m, \cdots] = \lim_{a \rightarrow 0, N \rightarrow \infty} \int_{[-N, N]} \prod_{i=1}^{N} dm_i \, z[m_i, (m_{i+1} - m_i)/a, \cdots]. \]

In general, evaluating the functional integral is not straightforward. However, we can obtain an estimate of \(Z\) by applying a saddle-point or mean-field approximation to the functional integral

\[ Z[T, h] \equiv e^{-\beta F[T, h]}, \quad \beta F[T, h] \simeq \min_m [\beta H[m, h]], \]

where \(\min_m [\beta H[m, h]]\) represents the minimum of the function with respect to variations in \(m\). Such an approach is known as Landau mean-field theory. For \(K > 0\), the minimum free energy occurs for a uniform vector field \(m(x) \equiv \bar{m}\hat{e}_h\), where \(\hat{e}_h\) points along the direction of the magnetic field, and \(\bar{m}\) is obtained by minimizing the Landau free energy density

\[ f(m, h) \equiv \beta F = \frac{\beta F}{V} = \frac{1}{2} m^2 + um^4 - hm \]

In the vicinity of the critical point \(\bar{m}\) is small, and we are justified in keeping only the lowest powers in the expansion of \(f(m, h)\). The behaviour of \(f(m, h)\) depends sensitively on the sign of \(t\) (see Fig. 2.1).

1. For \(t > 0\) the quartic term can be ignored, and the minimum occurs for \(\bar{m} \simeq h/t\). The vanishing of the magnetisation as \(h \rightarrow 0\) signals paramagnetic behaviour, and the zero-field susceptibility \(\chi \sim 1/t\) diverges as \(t \rightarrow 0\).

2. For \(t < 0\) a quartic term with a positive value of \(u\) is required to ensure stability (i.e. to keep the magnetisation finite). The function \(f(m, h)\) now has degenerate minima at a non-zero value of \(\bar{m}\). At \(h = 0\) there is a spontaneous breaking of rotational symmetry in spin space indicating ordered or ferromagnetic behaviour. Switching on an infinitesimal field \(h\) leads to a realignment of the magnetisation along the field direction and breaks the degeneracy of the ground state.
Thus a saddle-point evaluation of the Ginzburg-Landau Hamiltonian suggests paramagnetic behaviour for \( t > 0 \), and ferromagnetic behaviour for \( t < 0 \). Without loss of generality (i.e. by adjusting the scale of the order parameter), we can identify the parameter \( t \) with the reduced temperature \( t = (T - T_c)/T_c \). More generally, we can map the phase diagram of the Ginzburg-Landau Hamiltonian to that of a magnet by setting

\[
\begin{align*}
t(T, \cdots) &= (T - T_c)/T_c + O(T - T_c)^2, \\
u(T, \cdots) &= u_0 + u_1(T - T_c) + O(T - T_c)^2, \\
K(T, \cdots) &= K_0 + K_1(T - T_c) + O(T - T_c)^2,
\end{align*}
\]

where \( u_0, K_0 \) are unknown positive constants depending on material properties of the system. With this identification, let us determine some of the thermodynamical properties implied by the mean-field analysis.

\pmb{Magnetisation:} An explicit expression for the average magnetisation \( \bar{m} \) can be found from the stationary condition

\[
\left. \frac{\partial f}{\partial m} \right|_{m=\bar{m}} = 0 = t\bar{m} + 4u\bar{m}^3 - h.
\]

In zero magnetic field we find

\[
\bar{m} = \begin{cases} 
0 & t > 0, \\
\sqrt{-t/4u} & t < 0,
\end{cases}
\]

which implies a universal exponent of \( \beta = 1/2 \), while the amplitude is material dependent.

\pmb{Heat Capacity:} For \( h = 0 \), the free energy density is given by

\[
f(m, h = 0) \equiv \frac{\beta F}{V} \bigg|_{h=0} = \begin{cases} 
0 & t > 0, \\
-t^2/16u & t < 0.
\end{cases}
\]
Thus, by making use of the identities
\[ E = -\frac{\partial \ln Z}{\partial \beta}, \quad \frac{\partial}{\partial \beta} = -k_B T^2 \frac{\partial}{\partial T} \simeq -k_B T_c \frac{\partial}{\partial T}, \]
the singular contribution to the heat capacity is found to be
\[ C_{\text{sing.}} = \frac{\partial E}{\partial T} = \begin{cases} 0, & t > 0, \\ k_B/8u & t < 0. \end{cases} \]
This implies that the specific heat exponents \( \alpha_+ = \alpha_- = 0 \). In this case we observe only a discontinuity in the singular part of the specific heat. However, notice that treating, by higher orders, we can in principle obtain non-zero critical exponents.

\[ \text{\textbf{Susceptibility:}} \quad \text{The magnetic response is characterised by the (longitudinal) susceptibility} \]
\[ \chi_l \equiv \frac{\partial \tilde{m}}{\partial h} \bigg|_{h=0}, \quad \chi_l^{-1} = \frac{\partial h}{\partial \tilde{m}} \bigg|_{h=0} = t + 12u\tilde{m}^2 \bigg|_{h=0} = \begin{cases} t, & t > 0, \\ -2t, & t < 0, \end{cases} \]
which, as a measure of the variance of the magnetisation, must be positive. From this expression, we can deduce the critical exponents \( \gamma_+ = \gamma_- = 1 \). Although the amplitudes are parameter dependent, their ratio \( \chi_l^+/\chi_l^- = 2 \) is also universal.

\[ \text{\textbf{Equation of State:}} \quad \text{Finally, on the critical isotherm,} \quad t = 0, \quad \text{the magnetisation behaves as} \]
\[ \tilde{m} = \left( \frac{h}{4u} \right)^{1/3} \sim h^{1/3}. \]
giving the exponent \( \delta = 3 \).

This completes our survey of the critical properties of the Ginzburg-Landau theory in the Landau mean-field approximation. To cement these ideas one should attempt to find the mean-field critical exponents associated with a tricritical point (see, for example, Question 2 of problem set 2.10). To complement these notes it is also useful to refer to Section 4.2 (p151–154) of Chaikin and Lubensky on Landau theory.

Landau mean-field theory accommodates only the minimum energy configuration. To test the validity of this approximation scheme, and to determine spatial correlations it is necessary to take into account configurations of the field \( m(\mathbf{x}) \) involving spatial fluctuations. However, before doing so, it is first necessary to acquire some familiarity with the method of Gaussian functional integration, the basic machinery of statistical (and quantum) field theory.
2.3 Gaussian and Functional Integration

INFO: Before defining the Gaussian functional integral, it is useful to recall some results involving integration over discrete variables. We begin with the Gaussian integral involving a single (real) variable $\phi$,

$$Z_1 = \int_{-\infty}^{\infty} d\phi \exp \left[ -\frac{\phi^2}{2G} + h\phi \right] = \sqrt{2\pi G} \exp \left[ \frac{Gh^2}{2} \right].$$

Now derivatives of $Z_1$ on $h$ generate Gaussian integrals involving powers of $\phi$. Thus, if the integrand represents the probability distribution of a random variable $\phi$, logarithmic derivatives can be used to generate moments $\langle \phi \rangle$. In particular,

$$\langle \phi \rangle \equiv \frac{\partial \ln Z_1}{\partial h} = hG.$$

Subjecting $\ln Z_1$ to a second derivative obtains (exercise)

$$\frac{\partial^2 \ln Z_1}{\partial h^2} = \langle \phi^2 \rangle - \langle \phi \rangle^2 = G.$$

Note that, in general, the second derivative does not simply yield the second moment. In fact it obtains an object known as the ‘second cumulant’, the physical significance of which will become clear later. However, in the present case, it is simple to deduce from the expansion, $\langle \phi \rangle = hG$, and $\langle \phi^2 \rangle = h^2G^2 + G$.

Higher moments are more conveniently expressed by the cumulant expansion$^2$

$$\langle \phi^r \rangle_c = \frac{\partial^r}{\partial k^r} \left|_{k=0} \right. \ln \langle e^{k\phi} \rangle.$$

Applied to the first two cumulants, one obtains $\langle \phi \rangle_c = \langle \phi \rangle = hG$, and $\langle \phi^2 \rangle_c = \langle \phi^2 \rangle - \langle \phi \rangle^2 = G$ (as above), while $\langle \phi^r \rangle_c = 0$ for $r > 2$. The average $\langle e^{k\phi} \rangle$ is known as the moment generating function.

Gaussian integrals involving $N$ (real) variables

$$Z_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp \left[ -\frac{1}{2} \phi^T G^{-1} \phi + \mathbf{h} \cdot \phi \right], \quad (2.3)$$

can be reduced to a product of $N$ one-dimensional integrals by diagonalising the (real symmetric) matrix $G^{-1}$. (Convergence of the Gaussian integration is assured only when the eigenvalues are positive definite.) Denoting the unitary matrices that diagonalise $G$ by $U$, the matrix

$^2$The moments are related to the cumulants by the identity

$$\langle \phi^n \rangle = \sum_P \prod_\alpha \langle \phi^{n_\alpha} \rangle_c,$$

where $\sum_P$ represents the sum over all partitions of the product $\phi^n$ into subsets $\phi^{n_\alpha}$ labelled by $\alpha$.  


\( \tilde{G}^{-1} = U \tilde{G}^{-1} U^{-1} \) represents the diagonal matrix of eigenvalues. Making use of the identity (i.e. completing the square)

\[
\frac{1}{2} \phi^T \tilde{G}^{-1} \phi - h \cdot \phi = \chi^T \tilde{G}^{-1} \chi - \frac{1}{2} h^T U^{-1} \tilde{G} U h,
\]

where \( \chi = U \phi - \tilde{G} U h \), and changing integration variables (since the transformation is unitary, the corresponding Jacobian is unity) we obtain

\[
Z_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\chi_i \exp \left[-\frac{1}{2} \chi^T \tilde{G}^{-1} \chi + \frac{1}{2} h^T U^{-1} \tilde{G} U h \right],
\]

\[
= \det(2\pi G)^{1/2} \exp \left[\frac{1}{2} h^T G h \right].
\]

Regarding \( Z_N \) as the partition function of a set of \( N \) Gaussian distributed random variables, \( \{\phi_i\} \), the corresponding cumulant expansion is generated by

\[
\langle \phi_i \cdots \phi_j \rangle_c = \frac{\partial}{\partial k_i} \cdots \frac{\partial}{\partial k_j} \left|_{k=0} \ln \langle e^{k \cdot \phi} \rangle \right.,
\]

where the moment generating function is equal to

\[
\langle e^{k \cdot \phi} \rangle = \exp \left[ h^T G k + \frac{1}{2} k^T G k \right].
\]

Applying this result we find that the first two cumulants are given by

\[
\langle \phi_i \rangle_c = \sum_j G_{ij} h_j, \quad \langle \phi_i \phi_j \rangle_c = G_{ij},
\]

while, as for the case \( N = 1 \), cumulants higher than the second vanish. The latter is a unique property of Gaussian distributions. Applying Eq. (2.5), we can further deduce the important result that for any linear combination of Gaussian distributed variables \( A = a \cdot \phi \),

\[
\langle e^{A} \rangle = e^{\langle A \rangle_c + \langle A^2 \rangle_c / 2}.
\]

Now Gaussian functional integrals are a limiting case of the above. Consider the points \( i \) as the sites of a \( d \)-dimensional lattice and let the spacing go to zero. In the continuum limit, the set \( \{\phi_i\} \) translates to a function \( \phi(x) \), and the matrix \( G_{ij}^{-1} \) is replaced by an operator kernel or propagator \( G^{-1}(x, x') \). The natural generalisation of Eq. (2.4) is

\[
\int D\phi(x) \exp \left[-\frac{1}{2} \int dx \int dx' \phi(x) G^{-1}(x, x') \phi(x') + \int dx h(x) \phi(x) \right]
\]

\[
\propto (\det G)^{1/2} \exp \left[\frac{1}{2} \int dx \int dx' h(x) G(x, x') h(x') \right],
\]

where the inverse kernel \( G(x, x') \) satisfies the equation

\[
\int dx' G^{-1}(x, x') G(x', x'') = \delta^d(x - x').
\]
The notation $D\phi(x)$ is used to denote the measure of the functional integral. Although the constant of proportionality, $(2\pi)^N$ left out is formally divergent in the thermodynamic limit, it does not affect averages that are obtained from derivatives of such integrals. For Gaussian distributed functions, Eq. (2.6) then generalises to

$$\langle \phi(x) \rangle_c = \int dx \, G(x, x') \, h(x'), \quad \langle \phi(x)\phi(x') \rangle_c = G(x, x').$$

Later, in dealing with small fluctuations in the Ginzburg-Landau Hamiltonian we will frequently encounter the quadratic form,

$$\beta H[\phi] = \frac{1}{2} \int dx \, [(\nabla \phi)^2 + \xi^{-2} \phi^2] = \frac{1}{2} \int dx \int dx' \, \phi(x') \delta^d(x-x')(-\nabla^2 + \xi^{-2}) \, \phi(x), \quad (2.8)$$

which (integrating by parts) implies an operator kernel

$$G^{-1}(x, x') = K \delta^d(x-x')(-\nabla^2 + \xi^{-2}).$$

Substituting into Eq. (2.7) and integrating we obtain $(-\nabla^2+\xi^{-2})G(x) = \delta^d(x)$. The propagator can thus be identified as nothing but the Green’s function.

In the present case, translational invariance of the propagator suggests the utility of the Fourier representation

$$\phi(x) = \sum_q \phi_q \, e^{i q \cdot x}, \quad \phi_q = \frac{1}{L^d} \int_0^L dx \, \phi(x) \, e^{-i q \cdot x},$$

where $q = (q_1, \cdots q_d)$, with the Fourier elements taking values $q_i = 2\pi m/L$, $m$ integer. In this representation, making use of the identity $\int_0^L dx \, e^{-i(q+q') \cdot x} = L^d \delta_{q, -q'}$, the quadratic form above becomes diagonal in $q^4$

$$\beta H[\phi] = \frac{1}{2} \sum_q (q^2 + \xi^{-2})|\phi_q|^2.$$

---

3Here the system is supposed to be confined to a square box of dimension $d$ and volume $L^d$. In the thermodynamic limit $L \to \infty$, the Fourier series becomes the transform

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dq}{(2\pi)^d} \phi(q) \, e^{iq \cdot x}, \quad \phi(q) = \int_{-\infty}^{\infty} dx \, \phi(x) \, e^{-iq \cdot x}.$$  

Similarly,

$$\int_{-\infty}^{\infty} dx \, e^{i(q+q') \cdot x} = (2\pi)^d \delta^d(q + q'), \quad \int_{-\infty}^{\infty} \frac{dq}{(2\pi)^d} \, e^{-i(q+q') \cdot x} = \delta^d(x + x').$$

In the formulae above, the arrangements of $(2\pi)^d$ is not occasional. In defining the Fourier transform, it is wise to declare a convention and stick to it. The convention chosen here is one in which factors of $(2\pi)^d$ are attached to the $q$ integration, and the $\delta$-function in $q$.

4Similarly, in the thermodynamic limit, the Hamiltonian takes the form

$$\beta H[\phi] = \int_{-\infty}^{\infty} \frac{dq}{(2\pi)^d} \frac{1}{2} (q^2 + \xi^{-2})|\phi(q)|^2.$$
2.4 SYMMETRY BREAKING: GOLDSTONE MODES

where, since $\phi(x)$ is real, $\phi_{-q} = \phi_q^*$. The corresponding propagator is given by $G(q) = (q^2 + \xi^{-2})^{-1}$. Thus in real space, the correlation function is given by

$$G(x, x') \equiv \langle \phi(x)\phi(x') \rangle_c = \sum_q e^{i q \cdot (x - x')} G(q).$$

Here we have kept $L$ finite and the modes discrete to emphasize the connection between the discrete Gaussian integrations $Z_N$ and the functional integral. Hereafter, we will focus on the thermodynamic limit $L \to \infty$.

---

### 2.4 Symmetry Breaking: Goldstone Modes

With these important mathematical preliminaries, we return to the consideration of the influence of spatial fluctuations on the stability of the mean-field analysis. Even for $h = 0$, when $\beta H$ has full rotational symmetry, the ground state of the Ginzburg-Landau Hamiltonian is ordered along some given direction for $T < T_c$ — a direction of ‘magnetisation’ is specified. One can say that the onset of long-range order is accompanied by the spontaneous breaking of the rotational symmetry. The presence of a degenerate manifold of ground states obtained by a global rotation of the order parameter implies the existence of low energy excitations corresponding to a slowly varying rotations in the spin space. Such excitations are characteristic of systems with a broken continuous symmetry and are known as Goldstone modes. In magnetic systems the Goldstone modes are known as spin-waves, while in solids, they are the vibrational or phonon modes.

The influence of Goldstone modes can be explored by treating fluctuations within the framework of the Ginzburg-Landau theory. For a fixed magnitude of the $n$-component order parameter or, in the spin model, the magnetic moment $\mathbf{m} = \bar{m} \hat{e}_h$, the transverse fluctuations can be parametrized in terms of a set of $n - 1$ angles. One-component, or Ising spins have only a discrete symmetry and possess no low energy excitations. Two-component, or XY-spins, where the moment lies in a plane, are defined by a single angle $\theta$, $\mathbf{m} = \bar{m}(\cos \theta, \sin \theta)$ (cf. the complex phase of ‘superfluid’ order parameter). In this case the Ginzburg-Landau free energy functional takes the form

$$\beta H[\theta(x)] = \beta H_0 + \frac{K}{2} \int d\mathbf{x} (\nabla \theta)^2,$$

where $K = K \bar{m}^2 / 2$.

Although superficially quadratic, the multi-valued nature of the transverse field $\theta(x)$ makes the evaluation of the partition function problematic. However, at low temperatures, taking the fluctuations of the fields to be small $\theta(x) \ll 2\pi$, the functional integral can be taken as Gaussian. Following on from our discussion of the Gaussian functional integral, the operator kernel or propagator can be identified simply as the Laplacian operator. The latter is diagonalised in Fourier space, and the corresponding degrees of freedom are associated with spin-wave modes.
Then, employing the results of the previous section, we immediately find the average phase vanishes $\langle \theta(x) \rangle = 0$, and the correlation function takes the form

$$G(x, x') \equiv \langle \theta(x)\theta(x') \rangle = -\frac{C_d(x - x')}{K}, \quad \nabla^2 C_d(x) = \delta^d(x)$$

where $C_d$ denotes the Coulomb potential for a $\delta$-function charge distribution. Exploiting the symmetry of the field, and employing Gauss’ law, $\int_V d\mathbf{x} \nabla^2 C_d(x) = \oint d\mathbf{S} \cdot \nabla C_d$, we find that $C_d$ depends only on the radial coordinate $x$, and

$$\frac{dC_d}{dx} = \frac{1}{x^{d-1}S_d}, \quad C_d(x) = \frac{x^{2-d}}{(2-d) S_d} + \text{const.,} \quad (2.10)$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of a unit $d$-dimensional ball.\(^5\) Hence

$$\langle [\theta(x) - \theta(0)]^2 \rangle = 2 \left[ \langle \theta(0)^2 \rangle - \langle \theta(x)\theta(0) \rangle \right] \bigg|_{|x| > a} = \frac{2(|x|^{2-d} - a^{2-d})}{K(2-d) S_d},$$

where the cut-off, $a$ is of the order of the lattice spacing. Note that the case where $d = 2$, the combination $|x|^{2-d}/(2 - d)$ must be interpreted as $\ln |x|$.

The long distance behaviour changes dramatically at $d = 2$. For $d > 2$, the phase fluctuations approach some finite constant as $|x| \to \infty$, while they become asymptotically large for $d \leq 2$. Since the phase is bounded by $2\pi$, it implies that long-range order (predicted by the mean-field theory) is destroyed. This result becomes more apparent by examining the effect of phase fluctuations on the two-point correlation function,

$$\langle \mathbf{m}(x) \cdot \mathbf{m}(0) \rangle = \bar{m}^2 \text{Re} \left\langle e^{i[\theta(x) - \theta(0)]} \right\rangle.$$

(Since amplitude fluctuations are neglected, we are in fact looking at the transverse correlation function.) For Gaussian distributed variables we have already seen that $\langle \exp[\alpha \theta] \rangle = \exp[\alpha^2 \langle \theta^2 \rangle/2]$. We thus obtain

$$\langle \mathbf{m}(x) \cdot \mathbf{m}(0) \rangle = \bar{m}^2 \exp \left[ -\frac{1}{2} \langle [\theta(x) - \theta(0)]^2 \rangle \right] = \bar{m}^2 \exp \left[ -\frac{(x^{2-d} - a^{2-d})}{K(2-d) S_d} \right],$$

\(^5\)An important consequence of Eq. (2.10) is the existence of an unphysical ultraviolet divergence of the theory (i.e. $x \to 0 \leftrightarrow q \to \infty$) in dimensions $d \geq 2$. In the present case, this divergence can be traced to the limited form of the effective free energy which accommodates short-range fluctuations of arbitrary magnitude. In principle one can account for the divergence by introducing additional terms in the free energy which control the short-range behaviour more precisely. Alternatively, and in keeping with the philosophy that lies behind the Ginzburg-Landau theory, we can introduce a short-length scale cut-off into the theory, a natural candidate being the “lattice spacing” $a$ of the coarse-grained free energy. Note, however, that were the free energy a microscopic one — i.e. a free field theory — we would be forced to make sense of the ultraviolet divergence. Indeed finding a renormalisation scheme to control ultraviolet aspects of the theory is the subject of high energy quantum field theory. In condensed matter physics our concern is more naturally with the infrared, long-wavelength divergence of the theory which, in the present case (2.11), appear in dimensions $d \leq 2$. 

implying a power-law decay of correlations in $d = 2$, and an exponential decay in $d < 2$.

$$\lim_{|x| \to \infty} \langle \mathbf{m}(x) \cdot \mathbf{m}(0) \rangle = \begin{cases} \bar{m}^2 & d > 2, \\ 0 & d \leq 2. \end{cases}$$

The saddle-point approximation to the order parameter, $\bar{m}$ was obtained by neglecting fluctuations. The result above demonstrates that the inclusion of phase fluctuations leads to a reduction in the degree of order in $d = 2$, and to its complete destruction in $d < 2$. This result typifies a more general result known as the Mermin-Wagner Theorem (N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1967)). The theorem states that there is no spontaneous breaking of a continuous symmetry in systems with short-range interactions in dimensions $d \leq 2$. Corollaries to the theorem include:

- The borderline dimensionality of two, known as the lower critical dimension $d_l$, has to be treated carefully. As we shall show in Chapter 5, there is in fact a phase transition for the two-dimensional XY-model (or superfluid), although there is no true long-range order.

- There are no Goldstone modes when the broken symmetry is discrete (e.g., for $n = 1$). In such cases long-range order is possible down to the lower critical dimension of $d_l = 1$.

### 2.5 Fluctuations, Correlations & Susceptibilities

Our study of Landau mean-field theory showed that the most probable configuration was spatially uniform with $\mathbf{m}(x) = \bar{m} \hat{e}_1$, where $\hat{e}_1$ is a unit vector ($\bar{m}$ is zero for $t > 0$, and equal to $\sqrt{-t/4\pi}$ for $t < 0$). The role of small fluctuations around such a configuration can be examined by setting

$$\mathbf{m}(x) = [\bar{m} + \phi_l(x)] \hat{e}_1 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}(x) \mathbf{\hat{e}}_{\alpha},$$

where $\phi_l$ and $\phi_t$ refer respectively to fluctuations longitudinal and transverse to the axis of order $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$.

After substitution into the Ginzburg-Landau Hamiltonian, a quadratic expansion of the free energy functional with

$$\begin{align*}
(\nabla \mathbf{m})^2 &= (\nabla \phi_l)^2 + (\nabla \phi_t)^2, \\
\mathbf{m}^2 &= \bar{m}^2 + 2\bar{m}\phi_l + \phi_l^2 + \phi_t^2, \\
\mathbf{m}^4 &= \bar{m}^4 + 4\bar{m}^3\phi_l + 6\bar{m}^2\phi_l^2 + 2\bar{m}\phi_l^4 + O(\phi_l^3, \phi_l\phi_t^2),
\end{align*}$$

where $\phi_l$ and $\phi_t$ refer respectively to fluctuations longitudinal and transverse to the axis of order $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1$. The transverse fluctuations can take place along any of the $n - 1$ directions perpendicular to $\hat{e}_1. $
Chapter 2. Ginzburg-Landau Phenomenology

generates the perturbative expansion of the Hamiltonian

\[ \beta H = V \left( \frac{1}{2} \bar{m}^2 + u \bar{m}^4 \right) + \int d\mathbf{x} \left[ \frac{K}{2} (\nabla \phi_l)^2 + \frac{t + 12u \bar{m}^2}{2} \phi_l^2 \right] \]

\[ + \int d\mathbf{x} \left[ \frac{K}{2} (\nabla \phi_t)^2 + \frac{t + 4u \bar{m}^2}{2} \phi_t^2 \right] + O(\phi^3_l, \phi_l \phi_t^2). \] (2.11)

For spatially uniform fluctuations, one can interpret the prefactors of the quadratic terms in \( \phi \) as “masses” or “restoring forces” (cf. the action of a harmonic oscillator). These effective masses for the fluctuations can be associated with a length scale defined by

\[ \frac{K}{\xi_l^2} \equiv t + 12u \bar{m}^2 = \begin{cases} t & t > 0, \\ -2t & t < 0, \end{cases} \]

\[ \frac{K}{\xi_t^2} \equiv t + 4u \bar{m}^2 = \begin{cases} t & t > 0, \\ 0 & t < 0. \end{cases} \] (2.12)

(The physical significance of the length scales \( \xi_l \) and \( \xi_t \) will soon become apparent.) Note that there is no distinction between longitudinal and transverse components in the paramagnet phase \((t > 0)\), while below the transition \((t < 0)\), there is no restoring force for the transverse fluctuations (a consequence of the massless Goldstone degrees of freedom discussed previously).

To explore spatial fluctuations and correlation functions, it is convenient to switch to the Fourier representation, wherein the Hamiltonian becomes diagonal (cf. discussion of Gaussian functional integration). After the change of variables

\[ \phi(\mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\mathbf{q}}{(2\pi)^d} e^{i\mathbf{q} \cdot \mathbf{x}} \phi(\mathbf{q}), \]

the quadratic Hamiltonian becomes separable into longitudinal and transverse modes,

\[ \beta H[\phi_l, \phi_t] = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{K}{2} \left[ (\mathbf{q}^2 + \xi_l^{-2}) |\phi_l(\mathbf{q})|^2 + (\mathbf{q}^2 + \xi_t^{-2}) |\phi_t(\mathbf{q})|^2 \right]. \]

Thus, each mode behaves as a Gaussian distributed random variable with zero mean, while the two-point correlation function assumes the form of a Lorentzian,

\[ \langle \phi_\alpha(\mathbf{q}) \phi_\beta(\mathbf{q'}) \rangle = \delta_{\alpha\beta} (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q'}) G_\alpha(\mathbf{q}), \quad G^{-1}_\alpha(\mathbf{q}) = K(\mathbf{q}^2 + \xi_l^{-2}) \] (2.13)

where the indices \( \alpha, \beta \) denote longitudinal and transverse components. In fact, this equation describing correlations of an order parameter in the vicinity of a critical point was first proposed by Ornstein and Zernike as a means to explain the phenomenon of critical opalescence in the light scattering from a fluid in the vicinity of a liquid-gas transition. To understand the mechanism by which \( \xi \) sets the characteristic length scale of fluctuations let us consider the scattering amplitude.
2.5. FLUCTUATIONS, CORRELATIONS & SUSCEPTIBILITIES

In the case of the ferromagnetic model, the two-point correlation function of magnetisation can be observed directly using spin-polarised scattering experiments. The scattering amplitude is related to the Fourier density of scatterers $S(q) \propto \langle |m(q)|^2 \rangle$ (see Fig. 2.2). The Lorentzian form predicted above usually provides an excellent fit to scattering line shapes away from the critical point. Eq. (2.12) indicates that in the ordered phase longitudinal scattering still gives a Lorentzian form (on top of a $\delta$-function at $q = 0$ due to the spontaneous magnetisation), while transverse scattering always grows as $1/q^2$.

The same power law decay is also predicted to hold at the critical point, $t = 0$. In fact, actual experimental fits yield a power law of the form

$$S(q, T = T_c) \propto \frac{1}{|q|^{2-\eta}},$$

with a small positive value of the universal exponent $\eta$.

Turning to real space, we find that the average magnetisation is left unaffected by fluctuations, $\langle \phi_\alpha(x) \rangle = \langle m_\alpha(x) - \bar{m}_\alpha \rangle = 0$, while the connected part of the two-point correlation function takes the form $G_{c\beta}(x, x') \equiv \langle (m_\alpha(x) - \bar{m}_\alpha)(m_\beta(x') - \bar{m}_\beta) \rangle = \langle \phi_\alpha(x)\phi_\beta(x') \rangle$ where

$$\langle \phi_\alpha(x)\phi_\beta(x') \rangle = -\frac{\delta_{\alpha\beta}}{K} I_d(x - x', \xi_\alpha), \quad I_d(x, \xi) = -\int \frac{dq}{(2\pi)^d} \frac{e^{iq\cdot x}}{q^2 + \xi^{-2}}. \quad (2.14)$$

The detailed profile of this equation\(^6\) is left as an exercise, but leads to the asymptotics

\(^6\)This Fourier transform is discussed in Chaikin and Lubensky p 156. However, some clue to understanding the form of the transform can be found from the following: Expressed in terms of the modulus $q$ and $d-1$ angles $\theta_d$, the $d$-dimensional integration measure takes the form

$$dq = q^{d-1}dq \sin^{d-2}\theta_{d-1} d\theta_{d-1} \sin^{d-3}\theta_{d-2} d\theta_{d-2} \cdots d\theta_1,$$

where $0 < \theta_k < \pi$ for $k > 1$, and $0 < \theta_1 < 2\pi$. Thus, by showing that

$$I_d(x, \xi) = -\frac{1}{(2\pi)^{d/2}|x|^{d/2-1}} \int_0^{1/\alpha} \frac{q^{d/2}dq}{q^2 + \xi^{-2}} J_{d/2-1}(q|x|),$$
CHAPTER 2. GINZBURG-LANDAU PHENOMENOLOGY

Figure 2.3: Decay of the two-point correlation of magnetisation, and the divergence of the longitudinal and transverse susceptibility in the vicinity of $T_c$.

(see Fig. 2.5)

$$I_d(x, \xi) \simeq \begin{cases} C_d(x) = \frac{|x|^{2-d}}{(2-d)S_d} & |x| \ll \xi, \\ \exp[-|x|/\xi] & |x| \gg \xi. \end{cases}$$

From the form of this equation we can interpret the length scale $\xi$ as the correlation length.

Using Eq. (2.12) we see that close to the critical point the longitudinal correlation length behaves as

$$\xi_l = \begin{cases} (K/t)^{1/2} & t > 0, \\ (-K/2t)^{1/2} & t < 0. \end{cases}$$

The singularities can be described by $\xi_{\pm} \simeq \xi_0 B_{\pm} |t|^{-\nu_{\pm}}$, where $\nu_{\pm} = 1/2$ and the ratio $B_+ / B_- = \sqrt{2}$ are universal, while $\xi_0 \propto \sqrt{K}$ is not. The transverse correlation length is equivalent to $\xi_l$ for $t > 0$, while it is infinite for all $t < 0$. Eq. (2.15) implies that precisely at $T_c$, the correlations decay as $1/|x|^{d-2}$. Again, the decay of the exponent is usually given by $1/|x|^{d-2-\eta}$.

These results imply a longitudinal susceptibility of the form (see Fig. 2.5)

$$\chi_l \propto \int dx \frac{G_l^t(x)}{\langle \phi_l(x) \phi_l(0) \rangle} \propto \int_0^{\xi_l} \frac{dx}{|x|^{d-2}} \propto \xi_l^2 \simeq A_{\pm} t^{-1}$$

The universal exponents and amplitude ratios are again recovered from this equation. For $T < T_c$ there is no upper cut-off length for transverse fluctuations, and the divergence of one can obtain Eq. (2.15) by asymptotic expansion. A second approach is to present the correlator as

$$I_d(x, \xi) = -\int_0^\infty dt \int \frac{d\mathbf{q}}{(2\pi)^d} e^{i\mathbf{q} \cdot \mathbf{x} - t(\mathbf{q}^2 + \xi^2)},$$

integrate over $\mathbf{q}$, and employ a saddle-point approximation.
2.6. COMPARISON OF THEORY AND EXPERIMENT

The transverse susceptibility can be related to the system size $L$, as

$$\chi_t \propto \int d\mathbf{x} \frac{G_t(x)}{|x|^{d-2}} \propto \int_0^L \frac{d\mathbf{x}}{|\mathbf{x}|^{d-2}} \propto L^2$$

(2.16)

2.6 Comparison of Theory and Experiment

The validity of the mean-field approximation is assessed in the table below by comparing the results with (approximate) exponents for $d = 3$ from experiment.

<table>
<thead>
<tr>
<th>Transition type</th>
<th>Material</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ferromag. ($n = 3$)</td>
<td>Fe, Ni</td>
<td>$-0.1$</td>
<td>$0.34$</td>
<td>$1.4$</td>
<td>$0.7$</td>
</tr>
<tr>
<td>Superfluid ($n = 2$)</td>
<td>He$^4$</td>
<td>$0$</td>
<td>$0.3$</td>
<td>$1.3$</td>
<td>$0.7$</td>
</tr>
<tr>
<td>Liquid-gas ($n = 1$)</td>
<td>CO$_2$, Xe</td>
<td>$0.11$</td>
<td>$0.32$</td>
<td>$1.24$</td>
<td>$0.63$</td>
</tr>
<tr>
<td>Superconductors</td>
<td></td>
<td>$1/2$</td>
<td>$1$</td>
<td>$1/2$</td>
<td></td>
</tr>
</tbody>
</table>

The discrepancy between the mean-field results and experiment signal the failure of the saddle-point approximation. Indeed, the results suggest a dependence of the critical exponents on $n$ (and $d$). Later we will try to explore ways of going beyond the mean-field approximation. However, before doing so, the experimental results above leave a dilemma. Why do the critical exponents obtained from measurements of the superconducting transition agree so well with mean field theory? Indeed, why do they differ from other transitions which apparently belong to the same universality class? To understand the answer to these questions, it is necessary to examine more carefully the role of fluctuations on the saddle-point.

2.7 Fluctuation Corrections to the Saddle-Point

We are now in a position to determine the corrections to the saddle-point from fluctuations at quadratic order. To do so, it is necessary to determine the fluctuation contribution to the free energy. Applying the matrix (or functional) identity $\ln \det \mathbf{G}^{-1} = -\text{tr} \ln \mathbf{G}$ to Eq. (2.11) we obtain the following estimate for the free energy density

$$f = -\frac{\ln Z}{V} = \frac{t}{2} \bar{m}^2 + w \bar{m}^4 + \frac{1}{2} \int \frac{dq}{(2\pi)^d} \ln[K(q^2 + \xi_t^{-2})]$$

$$+ \frac{n-1}{2} \int \frac{dq}{(2\pi)^d} \ln[K(q^2 + \xi_t^{-2})].$$
Inserting the dependence of the correlation lengths on reduced temperature, the singular component of the heat capacity is given by

\[
C_{\text{sing.}} \propto -\frac{\partial^2 f}{\partial t^2} = \begin{cases} 
0 + \frac{n}{2} \int \frac{dq}{(2\pi)^d} \left( \frac{1}{(Kq^2 + t)^2} \right) & t > 0, \\
\frac{1}{8n} + 2 \int \frac{dq}{(2\pi)^d} \left( \frac{1}{(Kq^2 - 2t)^2} \right) & t < 0.
\end{cases}
\] (2.17)

The behaviour of the integral correction changes dramatically at \(d = 4\). For \(d > 4\) the integral diverges at large \(q\) and is dominated by the upper cut-off \(\Lambda \approx 1/a\), while for \(d < 4\), the integral is convergent in both limits. It can be made dimensionless by rescaling \(q\) by \(\xi^{-1}\), and is hence proportional to \(\xi^{4-d}\). Therefore

\[
\delta C \simeq \frac{1}{K^2} \begin{cases} 
a^{4-d} & d > 4, \\
\xi^{4-d} & d < 4.
\end{cases}
\] (2.18)

In dimensions \(d > 4\) fluctuation corrections to the heat capacity add a constant term to the background on each side of the transition. However, the primary form of the discontinuity in \(C_{\text{sing.}}\) is unchanged. For \(d < 4\), the divergence of \(\xi \propto t^{-1/2}\) at the transition leads to a correction term that dominates the original discontinuity. Indeed, the correction term suggests an exponent \(\alpha = (4-d)/2\). But even this is not reliable — a treatment of higher order corrections will lead to yet more severe divergences. In fact the divergence of \(\delta C\) implies that the conclusions drawn from the saddle-point approximation are simply no longer reliable in dimensions \(d < 4\). One says that Ginzburg-Landau models which belong to this universality class exhibit an **upper critical dimension** \(d_u\) of four. Although we reached this conclusion by examining the heat capacity the same conclusion would have been reached for any physical quantity such as magnetisation, or susceptibility.

### 2.8 Ginzburg Criterion

We have thus established the importance of fluctuations as the probable reason for the failure of the saddle-point approximation to correctly describe observed exponents. How,
2.9 Summary

A summary of our findings for the Ginzburg-Landau Hamiltonian based on mean-field theory and Gaussian fluctuations is shown in Fig. 2.5.

therefore, it is possible to account for materials such as superconductors in which the exponents agree well with mean-field theory?

Eq. (2.18) suggests that fluctuations become important when the correlation length begins to diverge. Within the saddle-point approximation, the correlation length diverges as $\xi \approx \xi_0 |t|^{-1/2}$, where $\xi_0 \approx \sqrt{K}$ represents the microscopic length scale. The importance of fluctuations can be assessed by comparing the two terms in Eq. (2.17), the saddle-point discontinuity $\Delta C_{sp} \propto 1/u$, and the correction, $\delta C$. Since $K \propto \xi_0^d$, and $\Delta C \propto \xi_0^{d/(4-d)}$, fluctuations become important when

$$\left( \frac{\xi_0}{a} \right)^{d} t^{(d-4)/2} \gg \left( \frac{\Delta C_{sp}}{k_B} \right) \quad \Rightarrow \quad |t| \ll t_G \approx \frac{1}{[\xi_0/a]^d (\Delta C_{sp}/k_B)^{2/(4-d)}}.$$ 

This inequality is known as the **Ginzburg Criterion**. Naturally, in $d < 4$ it is satisfied sufficiently close to the critical point. However, the resolution of the experiment may not be good enough to get closer than the Ginzburg reduced temperature $t_G$. If so, the apparent singularities at reduced temperatures $t > t_G$ may show saddle-point behaviour.

In principle, $\xi_0$ can be deduced experimentally from scattering line shapes. It has to approximately equal the size of the units that undergo ordering at the phase transition. For the liquid-gas transition, $\xi_0$ can be estimated by $v_c^{1/3}$, where $v_c$ is the critical atomic volume. In superfluids, $\xi_0$ is approximately equal to the thermal wavelength $\lambda(T)$. Taking $\Delta C_{sp}/k_B \sim 1$ per particle, and $\lambda \sim 2-3\AA$ we obtain $t_G \sim 10^{-1}-10^{-2}$, a value accessible in experiment. However, for a superconductor, the underlying length scale is the separation of Cooper pairs which, as a result of Coulomb repulsion, typically gives $\xi_0 \approx 10^3\AA$. This implies $t_G \sim 10^{-16}$, a degree of resolution inaccessible by experiment.

The Ginzburg criterion allows us to restore some credibility to the mean-field theory. As we will shortly see, a theoretical estimate of the critical exponents below the upper critical dimension is typically challenging endeavour. Yet, for many purposes, a good qualitative understanding of the thermodynamic properties of the experimentally relevant regions of the phase diagram can be understood from the mean-field theory alone.
2.10 Problem Set I

2.10.1 Questions on Ginzburg-Landau Theory

Connecting Van der Waals equation of state and Ginzburg-Landau Theory:
This is an optional question examining the connection between the Van der Waals gas, the law of corresponding states and Landau theory.

1. Consider the Van der Waals equation of state

\[ P = \frac{NkT}{V - Nb} - \frac{N^2a}{V^2}. \]

(a) Why is the critical point given by

\[ \frac{\partial P}{\partial V} = \frac{\partial^2 P}{\partial V^2} = 0? \]

(b) Show that (or trust)

\[ V_c = 3Nb, \quad P_c = \frac{a}{27b^2}, \quad kT_c = \frac{8a}{27b}, \]

and obtain the law of corresponding states

\[ p = \frac{8\tau}{3v - 1} - \frac{3}{v^2}, \]

where \( p = P/P_c, \ v = V/V_c \) and \( \tau = T/T_c. \)

(c) Expand this equation up to cubic order in \( \phi \) and \( t \) where \( v = 1 + \phi \) and \( \tau = 1 + t. \)

(d) Show that the critical exponent \( \delta = 3 \) and use the Maxwell construction to show that the critical exponent \( \beta = 1/2 \) where the order parameter is \( \phi_{\text{gas}} - \phi_{\text{liq}}. \)

(The Maxwell construction implies drawing a straight line through the unphysical region of the \( P-V \) curve so that there are equal areas above and below.)

(e) Assuming \( P = -\frac{\partial F}{\partial V}|_T \), obtain the Gibbs free energy \( G = F + PV \) and compare with Landau theory.

Discontinuous Transitions: In lectures we focussed on the study of Landau theory of second order phase transitions in which the order parameter goes to zero continuously. When the order parameter vanishes discontinuously, the transition is said to be first order. Amongst those first order transitions most commonly encountered in Landau theory there includes the following model:
2. **Tricritical Point**: In class we examined the Ginzburg-Landau Hamiltonian

\[
\beta H = \int d\mathbf{x} \left[ \frac{t}{2} \mathbf{m}^2 + um^4 + vm^6 + \frac{K}{2} (\nabla \mathbf{m})^2 - h \cdot \mathbf{m} \right],
\]

with \( u > 0 \) and \( v = 0 \). If \( u < 0 \), then a positive \( v \) is necessary to ensure stability.

(a) By sketching the free energy \( F(m) \) for various values of \( t \), show that there is a first order transition for \( u < 0 \) and \( h = 0 \).

(b) Calculate \( \bar{t} \) and the discontinuity in \( \bar{m} \) at the transition.

(c) For \( h = 0 \) and \( v > 0 \) plot the phase boundary in the \((u, t)\) plane, identifying the phases, and the order of the phase transitions.

(d) The special point \( u = t = 0 \), separating first and second order phase boundaries, is called a tricritical point. For \( u = 0 \) calculate the exponents \( \alpha, \beta, \gamma, \) and \( \delta \). (A discussion of the tricritical point and multi-critical points in general can be found on p. 172 in Chaikin and Lubensky — although you should attempt to complete the question yourself before resorting to the text!)

[Recall: \( C \sim t^{-\alpha}; \bar{m} \sim t^\beta; \chi \sim t^{-\gamma}; \) and \( \bar{m} \sim h^{1/\delta} \).]

---

**Fluctuations**: The final set of questions on the problem set are concerned with studying the fluctuation corrections to the mean-field.

---

3. Following on from the previous question, taking the Ginzburg-Landau Hamiltonian from above with \( u = 0 \):

(a) Calculate the heat capacity singularity as \( t \to 0 \) using the saddle-point approximation.

(b) Setting

\[ \mathbf{m}(\mathbf{x}) = (\bar{m} + \phi_l(\mathbf{x}))\hat{e}_l + \sum_{\alpha=2}^n \phi^\alpha_l(\mathbf{x})\hat{e}_\alpha, \]

expand \( \beta H \) to quadratic order in longitudinal and transverse fluctuations \( \phi \).

(c) Following the Fourier analysis developed in the lectures, and making use of the integral identity below, obtain an estimate for the longitudinal and transverse correlation functions \( \langle \phi_l, \phi_l(0) \rangle \).

(d) Taking into account the leading contribution from fluctuations obtain the first correction to the saddle-point free energy.

(e) From this result, obtain the leading fluctuation corrections to the heat capacity.
(f) By comparing the results from parts (a) and (e) obtain the Ginzburg criterion for mean-field theory to apply, and show that, for the tricritical point, the upper critical dimension \(d_u = 3\).

(g) A generalised multi-critical point is described by replacing the term \(u m^6\) with \(u_2 n^m\). Using only power counting arguments, show that the upper critical dimension of the multi-critical point is \(d_u = 2n/(n - 1)\).

\[
- \int \frac{dq}{(2\pi)^d} \frac{e^{iq\cdot x}}{q^2 + \xi^{-2}} \simeq \begin{cases} \frac{|x|^{2-d}}{(2-d)\xi^{d}}, & |x| \ll \xi, \\ \frac{e^{(3-d)/2}}{(2-d)\xi|d-1|/2} \exp(-|x|/\xi), & |x| \gg \xi. \end{cases}
\]

4. **Spin Waves**: In the XY-model of magnetism, a unit two-component vector \(S = (S_x, S_y)\) (with \(S^2 = 1\)), is placed on each site of a \(d\)-dimensional lattice. There is an interaction that tends to keep nearest neighbours parallel, i.e. a Hamiltonian

\[
\beta H = -K \sum_{\langle ij \rangle} S_i \cdot S_j.
\]

[The notation \(\sum_{\langle ij \rangle}\) is conventionally used to indicate a sum over all nearest neighbour pairs \((i, j)\).]

(a) Rewrite the partition function \(Z = \int \prod_i dS_i \exp[-\beta H]\) as an integral over the set of angles \(\{\theta_i\}\) between the spins \(\{S_i\}\) and some arbitrary axis.

(b) At low temperatures (i.e. \(K \gg 1\)), the angles \(\{\theta_i\}\) vary slowly from site to site. In this case expand \(\beta H\) to obtain a quadratic expansion in \(\{\theta_i\}\).

(c) For \(d = 1\) consider \(L\) sites with periodic boundary conditions (i.e. forming a chain). Find the normal (spin-wave) modes \(\theta_q\) that diagonalise the quadratic form (by Fourier transformation), and show that the corresponding eigenvalue spectrum, the *dispersion relation*, is given by \(K(q) = 2K(1 - \cos q)\). [Hint: Compare this result to the phonon dispersion curve obtained from a weakly coupled chain of oscillators — the phonon modes of a lattice.]

(d) Generalise the results from (c) to a \(d\)-dimensional simple cubic lattice with periodic boundary conditions.

(e) Obtain an estimate of the contribution of these modes to the free energy in the form of an integral. (Evaluate the classical partition function, i.e. do not quantise the modes.) Without performing the integration explicitly (i.e. by examining the
temperature dependence alone), determine the contribution of these modes to the specific heat. Show that, at high temperatures, this result is in accord with the equipartition theorem.

(f) Find an expression for \( \langle S_0 \cdot S_x \rangle = \text{Re} \langle \exp[i(\theta_x - \theta_0)] \rangle \) in the form of an integral. Convince yourself that for \( |x| \to \infty \), only \( q \to 0 \) modes contribute appreciably to this expression, and hence obtain an expression for the asymptotic limit in dimensions 1, 2, and 3. [Hint: Note that this calculation is simply the discrete analogue of the continuum calculation made in the text.]
2.10.2 Answers

1. (a) At high temperatures \((T > T_c)\), the Van der Waals equation of state \(P(V,T)\) resembles that of an ideal gas. Below a certain critical temperature \(T_c\), the isotherms develop unphysical regions where \(\partial P/\partial V > 0\). The onset of this occurs when both the first and second derivatives are zero.

(b) The critical values for the pressure, volume and temperature may be found by explicitly evaluating the first and second derivatives and setting them to zero. Alternatively, one can note that at the critical temperature the equation of state reduces to

\[ P_c(V - V_c)^3 = 0. \]

The required critical values are now readily deducible from the coefficients of the cubic. The law of corresponding states is merely a rewriting of the Van der Waals equation in reduced coordinates.

(c) Expansion the new variables gives

\[ p \approx 1 + 4t - 6t\phi + 9t\phi^2 - \frac{3}{2}\phi^3. \quad \text{(2.19)} \]

(d) The exponent \(\delta\) relates the pressure to the volume at the critical temperature. Given the first and second derivatives are zero, it is anticipated that the leading order Laurent expansion of \(p(\phi)\) to be cubic. This is confirmed when we set \(t = 0\) in Eq. (2.19),

\[ p - 1 = -\frac{3}{2}\phi^3. \]

Hence \(\delta = 3\).

2. (a) For \(h = 0\) and \(t > 0\), the Landau Hamiltonian

\[ f(m, h = 0) \equiv \frac{\beta F}{V} = \frac{t}{2}m^2 + um^4 + vm^6 \]

has a minimum at \(\bar{m} = 0\). In addition, for a range of parameters \(u\) and \(v\), the free energy can have minima at non-zero \(\bar{m}\). Differentiating, extrema are found at

\[ \frac{\partial f}{\partial \bar{m}^2} = 3vm^4 + 2um^2 + \frac{t}{2} = 0, \quad \frac{\partial f}{\partial \bar{m}^4} = \frac{t}{6} \pm \frac{1}{96}(4u^2 - 6vt)^{1/2}. \]

Firstly, for \(u > 0\), minima exist at \(m_+\) if \(t > 0\). In this case, the line set by \(t = 0\) and \(u > 0\) separates an ordered from a disordered phase by a line of second order critical points. If, on the other hand, \(u < 0\), and \(t < 2u^2/3v\), two maxima occur at \(\pm m_-\), and minima occur at \(\pm m_+\). Finally, for \(t = \bar{t}\) (see below) a discontinuous transition occurs (see Fig. 2.6) below which the phase orders.

(b) To determine \(\bar{m}\) and \(\bar{t}\), we require the simultaneous solution of \(\frac{\partial f}{\partial \bar{m}^2} \bigg|_{\bar{m}^2} = 0\), \(f(\bar{m}^2) = f(0) = 0\). Doing so, we find

\[ \bar{t} = -u\bar{m}^2 = \frac{u^2}{2v}, \quad \bar{m}^2 = -\frac{u}{2v} = \frac{|u|}{2v}. \]
2.10. PROBLEM SET I

Figure 2.6: Sketch of the free energy as a function of $m$ for (1) $t > 2u^2/3v$, (2) $\bar{t} < t < 2u^2/3v$, (3) $t = \bar{t}$, and (4) $\bar{t} > t > 0$.

![Phase Diagram](image)

Figure 2.7: Phase diagram of the model.

(c) On the $(u, t)$ plane, the line $\bar{t} = u^2/2v$ ($u < 0$) is a first order phase boundary.

(d) $u = t = 0$ describes a **tricritical point**. To determine the critical exponents $\alpha$ and $\beta$ associated with the transition, we set $h = 0$ and seek the minimum

$$\frac{\partial f}{\partial m} \bigg|_{m = m_{eq}} = m_{eq}(t + 6vm_{eq}^4) = 0, \quad m_{eq} = \begin{cases} 0, t > 0, \\ (t^{-1} - 6v)^{1/4}, t < 0. \end{cases}$$

From this we find $\beta = 1/4$. Using $f[m_{eq}] \sim m_{eq}^6 \sim (-t)^{3/2}$, we obtain the heat capacity $C \sim \partial^2 f/\partial t^2 \sim t^{-1/2}$ implying $\alpha = 1/2$. To determine $\delta$, set $t = 0$, $h \neq 0$, from which we obtain

$$f[m_{eq}] = vm_{eq}^6 - hm_{eq}, \quad \frac{\partial f}{\partial m} \bigg|_{m = m_{eq}} = 0, \quad h = 6m_{eq}^5, \quad m_{eq} \sim h^{-1/3}.$$  

Thus we find $\delta = 5$. Finally, for finite $h$ and $t$, solving for $\partial f/\partial m = 0$ we find
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\[ h = tm_{eq} + 6vm_{eq}^5. \]

From this we obtain the susceptibility

\[ \chi = \frac{\partial m_{eq}}{\partial h} = (t + 30vm_{eq}^4)^{-1}, \]

implying that \( \chi \sim 1/|t| \) for \( t < 0 \) and \( t > 0 \). Thus we find the exponent \( \gamma = 1 \).

3. (a) In the saddle-point approximation, the Hamiltonian is minimized by a field configuration which is uniform, \( m = \bar{m} e \) with

\[ \bar{m} = \begin{cases} 0, & t > 0, \\ (-t/6v)^{1/4}, & t < 0. \end{cases} \]

From this result, we obtain the free energy per unit volume

\[ f = \frac{t}{2} \bar{m}^2 + v\bar{m}^6 = \begin{cases} 0, & t > 0, \\ -|t|^{3/2}/3(6v)^{1/2}, & t < 0. \end{cases} \]

This implies a heat capacity singularity,

\[ C = C_{s.p.} = -\frac{\partial^2 f}{\partial t^2} \approx -\frac{\partial^2 f}{\partial t^2} = \begin{cases} 0, & t > 0 \\ 1/4(-6vt)^{1/2}, & t < 0 \end{cases} \]

From this result we deduce that, below \( T_c \), there is a divergence of the heat capacity with an exponent \( \alpha = 1/2 \).

(b) Expanding to quadratic order in the longitudinal \( \phi_l \) and transverse \( \phi_t \) fluctuations around the mean field (cf. example in lecture notes), we find

\[ \beta H(\phi_l, \phi_t) - \beta H(0,0) = \frac{K}{2} \int d\mathbf{x} \left[ (\nabla \phi_l)^2 + \frac{\phi_l^2}{\xi_l^2} + (\nabla \phi_t)^2 + \frac{\phi_t^2}{\xi_t^2} \right] \]

where the correlation length is defined by

\[ \xi_l^{-2} = \begin{cases} t/K, & t > 0, \\ -4t/K, & t < 0. \end{cases}, \quad \xi_t^{-2} = \begin{cases} t/K, & t > 0, \\ 0, & t < 0. \end{cases} \]

(c) From this result we obtain the correlation function

\[ \langle \phi_\alpha(\mathbf{x})\phi_\beta(0) \rangle = -\frac{\delta_{\alpha\beta}}{K} I_d(\mathbf{x}, \xi_\alpha), \]

where

\[ I_d(\mathbf{x}, \xi_\alpha) = -\int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{e^{i\mathbf{q} \cdot \mathbf{x}}}{\mathbf{q}^2 + \xi_\alpha^{-2}}. \]
An explicit expression for the asymptotics of this integral can be found in the lecture notes. The key point is that, above the transition, the decay of both longitudinal and transverse correlation functions is exponential while below the transition, the decay of transverse correlation functions is power law (or logarithmic in two-dimensions).

(d) In the quadratic approximation, an integration over the spatial fluctuations generates a correction to the free energy of

$$\beta F = \beta H(0,0) + \begin{cases} \frac{n}{2} \int \frac{d^dq}{(2\pi)^d} \ln(Kq^2 + t), t > 0, \\ \frac{1}{16} \int \frac{d^dq}{(2\pi)^d} \ln(Kq^2 + 4|t|), t < 0. \end{cases}$$

where $n$ denotes the number of components of the order parameter. Note that this expression does not take explicitly into account contributions from the transverse degrees of freedom below $T_c$ since these contributions are independent of $t$ and therefore remain finite at the transition.

(e) From this result, we obtain the fluctuation correction to the heat capacity

$$C - C_{s.p.} \propto \begin{cases} n \int \frac{d^dq}{(2\pi)^d} \frac{1}{(Kq^2 + t)^2}, t > 0, \\ 16 \int \frac{d^dq}{(2\pi)^d} \frac{1}{(Kq^2 + 4|t|)^2}, t < 0. \end{cases}$$

Thus, for $d > 4$, integrals are dominated by large $q$, and $C - C_{s.p.}$ is finite. i.e. Mean field contribution ($|t|^{-1/2} \to \infty$) dominates. However, note that this does not imply that the upper critical dimension is 4! (see below).

For $d < 4$, the fluctuation contribution to the specific heat is given by

$$C_{fl.} = C - C_{s.p.} \propto K^{-d/2} |t|^{d/2 - 2}.$$

Note that, convergence of the integral at small $|q|$ admits the approximation

$$\int_0^{\Lambda = 1/\alpha} \frac{d^dq}{(2\pi)^d} \frac{1}{(Kq^2 + \alpha t)^2} \approx \frac{1}{\ell^2} \left( \frac{t}{K} \right)^{d/2} \int \frac{d^dz}{(z^2 + \alpha)^2}.$$

(f) Taking the ratio of the specific heat contributions

$$\frac{C_{fl.}}{C_{s.p.}} \propto |t|^{(d-3)/2} \frac{1}{\sqrt{K^d}/v},$$

we obtain the Ginzburg criterion. The mean field contribution dominates over the fluctuation contributions as $t \to 0$ provided $d > d_u = 3$, where $d_u$ represents the upper critical dimension.

For $d < 3$, the mean field result is still valid far enough from the critical point:

$$|t|^{3-d} \gg \frac{v}{K^d}, \quad |t| \gg \left( \frac{v}{K^d} \right)^{1/(3-d)}.$$
(g) In general, an interaction $u_{2n} m^{2n}$ leads to a mean-field order parameter which varies as $\bar{m} \propto |t|^{1/(2n-2)}$. Thus $F_{s,p.} \propto |t|^{n/(n-1)}$, $C_{s,p.} \propto |t|^{(2-n)/(n-1)}$, and $C_{fl.} = |t|^{d/2-2}$ for any $n$. From this we obtain the upper critical dimension $d_u = 2n/(n-1)$.

4. (a) Expressed in terms of the angle the spin makes to some fixed axis, the partition function takes the form

$$Z = \int \prod_i d\theta_i \exp \left[ K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right].$$

(b) For $K \gg 1$, an expansion around the ferromagnetically aligned ground state generates the partition function

$$Z = e^{KL^d} \int \prod_i d\theta_i \exp \left[ -\frac{K}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 \right],$$

where $L^d$ denotes the total number of sites. Since fluctuations are small $\langle (\theta_i - \theta_j)^2 \rangle \sim 1/K$ the contribution from higher order terms can be neglected.

(c,d) In this limit, the Hamiltonian is diagonalised by the Fourier decomposition

$$\theta_q = \frac{1}{L^d} \sum_{j=1}^{L^d} \theta_j e^{iq \cdot x_j},$$

with the periodic boundary condition $\theta_{L^d+1} = \theta_1$. ($q = (2\pi/L)n$ with $n$ integer). Moreover, since $\theta_j$ is real, $\theta_q = \theta_{-q}^*$. Substituting, and making use of the identity

$$\frac{1}{L^d} \sum_j e^{i(q+q') \cdot x_j} = \delta_{q,-q'},$$

we obtain

$$-\beta H = \frac{K}{2} \sum_q \theta_q \theta_{-q} \sum_\epsilon |1 - e^{-i\epsilon \cdot x_j}|^2 = \frac{1}{2} \sum_q K(q)|\theta_q|^2,$$

where $K(q) = 2K \sum_\alpha (1 - \cos q_\alpha)$. In the hydrodynamic, or long-wavelength limit $|q| \to 0$, the dispersion relation takes the form

$$K(q) = Kq^2.$$

These low energy degrees of freedom correspond to the Goldstone modes associated with the breaking of the continuous symmetry — the spin-wave modes.
(e) Integrating over the normal modes, the partition function takes the form

\[ Z = \prod_q \left( \frac{2\pi}{K(q)} \right)^{1/2}. \]

Taking the continuum limit, from this result, we obtain the free energy

\[ F = \frac{T}{2} \int \frac{d^d q}{(2\pi)^d} L^d \ln K(q) \]

where the factor \((L/2\pi)^d\) represents the continuum density of states. Thus, if \(K \propto 1/T\) (as it would be if the model were microscopic), we obtain

\[ F = -\frac{T}{2} (\ln T + \text{const.}) L^d, \quad C = -\frac{T}{L^d} \frac{\partial^2 F}{\partial T^2} = \frac{1}{2}, \]

This result is in accord with the equipartition theorem in which we find one spin degree of freedom per site.

(f) Evaluating the correlation function, we find

\[ \theta_x - \theta_0 = \frac{1}{L^d} \sum_q \theta_q \left( e^{i \langle q \cdot x \rangle} - 1 \right) \]

\[ \langle e^{i(\theta_x - \theta_0)} \rangle = \exp \left[ -\frac{1}{2} \langle (\theta_x - \theta_0)^2 \rangle \right] = \exp \left[ -\int \frac{d^d q}{(2\pi)^d} \frac{(1 - \cos(q \cdot x))}{2K(q)} \right]. \]

Taking \(|x| \to \infty\), a stationary phase approximation shows the integrals to be dominated by contribution from the interval in which \(|q| \sim 1/|x|\). Thus, for \(d = 1\) we obtain the asymptotic dependence (in this case it is possible to obtain the precise numerical prefactor)

\[ \lim_{|x| \to \infty} \langle S_x \cdot S_0 \rangle = \exp \left[ -\frac{|x|}{2K} \right], \]

in \(d = 2\) we find

\[ \lim_{|x| \to \infty} \langle S_x \cdot S_0 \rangle = |x|^{-1/2\pi K}, \]

and, in \(d \geq 3\),

\[ \lim_{|x| \to \infty} \langle S_x \cdot S_0 \rangle = \text{const.} \]
Chapter 3

The Scaling Hypothesis

Previously, we found that singular behaviour in the vicinity of a second order critical point was characterised by a set of critical exponents \( \{\alpha, \beta, \gamma, \delta, \ldots\} \). These power law dependencies of thermodynamic quantities are a symptom of scaling behaviour. Mean-field estimates of the critical exponents were found to be unreliable due to fluctuations. However, since the various thermodynamic quantities are related, these exponents cannot be completely independent of each other. The aim of this chapter is to employ scaling ideas to uncover relationships between them.

3.1 Homogeneity

The non-analytic structure of the Ginzburg-Landau model was found to be a coexistence line for \( t < 0 \) and \( h = 0 \) that terminates at the critical point \( h = t = 0 \). Thermodynamic quantities \( Q(t, h) \) in the vicinity of the critical point are characterised by various exponents. In particular, within the saddle-point approximation we found that the free energy density was given by

\[
f \equiv \frac{\beta F}{V} = \min_m \left[ \frac{1}{2} m^2 + um^4 - h \cdot m \right] \sim \begin{cases} -\frac{t^2}{u}, & h = 0, \ t < 0, \\ -\frac{h^{4/3}}{u^{1/2}}, & h \neq 0, \ t = 0. \end{cases}
\]  

(3.1)

In fact, the free energy can be described by a single homogeneous function\(^1\) in \( t \) and \( h \)

\[
f(t, h) = t^2 g_f(h/t^\Delta),
\]  

(3.2)

where \( \Delta \) is known as the “Gap exponent”. Comparison with Eq. (3.1) shows that, if we set \( \Delta = 3/2 \), the correct asymptotic behaviour of \( f \) is obtained,

\[
\lim_{x \to 0} g_f(x) \sim -\frac{1}{u}, \quad f(t, h = 0) \sim -\frac{t^2}{u},
\]

\[
\lim_{x \to \infty} g_f(x) \sim \frac{x^{4/3}}{u^{1/3}}, \quad f(t = 0, h) \sim t^2 \left( \frac{h}{t^{\Delta}} \right)^{4/3} \sim h^{4/3}.
\]

\(^1\)A function \( f(x) \) is said to be homogeneous of degree \( k \) if it satisfies the relation \( f(bx) = b^k f(x) \).
The assumption of homogeneity is that, on going beyond the saddle-point approximation, the singular form of the free energy (and of any other thermodynamic quantity) retains a homogeneous form

\[ f_{\text{sing}}(t, h) = t^{2-\alpha} g_f \left( \frac{h}{t^\Delta} \right), \]  

where the actual exponents \( \alpha \) and \( \Delta \) depend on the critical point being considered.

**Heat Capacity**: For example, the dependence on \( t \) is chosen to reproduce the heat capacity singularity at \( h = 0 \). The singular part of the energy is obtained from

\[ E_{\text{sing}} \sim \frac{\partial f}{\partial t} \sim (2 - \alpha)t^{1-\alpha} g_f(h/t^\Delta) - \Delta h t^{1-\alpha-\Delta} g'_f(h/t^\Delta) \equiv t^{1-\alpha} g_E(h/t^\Delta), \]

where the prime denotes the derivative of the function with respect to the argument. Thus, the derivative of one homogeneous function is another. Similarly, the second derivative takes the form

\[ C_{\text{sing}} \sim -\frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha} g_C(h/t^\Delta), \]

reproducing the scaling \( C_{\text{sing}} \sim t^{-\alpha} \) as \( h \to 0 \).

**Magnetisation**: Similarly the magnetisation is obtained from Eq. (3.3) using the expression

\[ m(t, h) \sim \frac{\partial f}{\partial h} \sim t^{2-\alpha-\Delta} g_m(h/t^\Delta). \]

In the limit \( x \to 0 \), \( g_m(x) \) is a constant, and \( m(t, h = 0) \sim t^{2-\alpha-\Delta} \) (i.e. \( \beta = 2 - \alpha - \Delta \)). On the other hand, if \( x \to \infty \), \( g_m(x) \sim x^p \), and \( m(t = 0, h) \sim t^{2-\alpha-\Delta}(h/t^\Delta)^p \). Since this limit is independent of \( t \), we must have \( p\Delta = 2 - \alpha - \Delta \). Hence \( m(t = 0, h) \sim h^{(2-\alpha-\Delta)/\Delta} \) (i.e. \( \delta = \Delta/(2 - \alpha - \Delta) = \Delta/\beta \)).

**Susceptibility**: Finally, calculating the susceptibility we obtain

\[ \chi(t, h) \sim \frac{\partial m}{\partial h} \sim t^{2-\alpha-2\Delta} g_\chi(h/t^\Delta) \Rightarrow \chi(t, h = 0) \sim t^{2-\alpha-2\Delta} \Rightarrow \gamma = 2\Delta - 2 + \alpha. \]

Thus the consequences of homogeneity are:

- The singular parts of all critical quantities, \( Q(t, h) \) are homogeneous, with the same exponents above and below the transition.
- Because of the interconnections via thermodynamic derivatives, the same gap exponent, \( \Delta \) occurs for all such quantities.
- All critical exponents can be obtained from only \textit{two} independent ones, e.g. \( \alpha, \Delta \).

\[ ^2 \text{It may appear that we have the freedom to postulate a more general form, } C_\pm = t^{-\alpha\pm} g_\pm(h/t^\Delta) \text{ with different functions for } t > 0 \text{ and } t < 0 \text{ that match at } t = 0. \text{ However, this can be ruled out by the condition that the free energy is analytic everywhere except on the coexistence line } h = 0 \text{ and } t < 0. \]
3.2. HYPERSCALING AND THE CORRELATION LENGTH

• As a result of above, one obtains a number of exponent identities:

\[ \alpha + 2\beta + \gamma = 2. \]  
(Rushbrooke’s Identity)

\[ \delta - 1 = \gamma / \beta. \]  
(Widom’s Identity)

These identities can be checked against the following table of critical exponents. The first three rows are based on a number of theoretical estimates in \( d = 3 \); the last row comes from an exact solution in \( d = 2 \). The exponent identities are approximately consistent with these values, as well as with all reliable experimental data.

<table>
<thead>
<tr>
<th>( d = 3 )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( \nu )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ising</td>
<td>0.12</td>
<td>0.31</td>
<td>1.25</td>
<td>5</td>
<td>0.64</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XY-spin</td>
<td>0.00</td>
<td>0.33</td>
<td>1.33</td>
<td>5</td>
<td>0.66</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heisenberg</td>
<td>-0.14</td>
<td>0.35</td>
<td>1.4</td>
<td>5</td>
<td>0.7</td>
<td>0.04</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( d = 2 )</th>
<th>( n = 1 )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( \nu )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ising</td>
<td>0</td>
<td>1/8</td>
<td>7/4</td>
<td>15</td>
<td>1</td>
<td>1/4</td>
<td></td>
</tr>
</tbody>
</table>

3.2 hyperscaling and the correlation length

The homogeneity assumption relates to the free energy and quantities derived from it. It says nothing about correlation functions. An important property of a critical point is the divergence of the correlation length, which is responsible for (and can be deduced from) the divergence of response functions. In order to obtain an identity involving the exponent \( \nu \) describing the divergence of the correlation length, we replace the homogeneity assumption for the free energy with the following two conditions:

1. The correlation length has a homogeneous form,

\[ \xi(t, h) \sim t^{-\nu} g_\xi \left( \frac{h}{t^\Delta} \right). \]

For \( t = 0 \), \( \xi \) diverges as \( h^{-\nu_h} \) with \( \nu_h = \nu / \Delta \).

2. Close to criticality, the correlation length \( \xi \) is the most important length scale, and is solely responsible for singular contributions to thermodynamic quantities.

The second condition determines the singular part of the free energy. Since \( \ln Z(t, h) \) is dimensionless and extensive (i.e. scales in proportion with the volume \( L^d \)), it must take the form

\[ \ln Z = \left( \frac{L}{\xi} \right)^d g_s + \left( \frac{L}{a} \right)^d g_a, \]
CHAPTER 3. THE SCALING HYPOTHESIS

where \( g_s \) and \( g_a \) are non-singular functions of dimensionless parameters (\( a \) is an appropriate microscopic length). The singular part of the free energy comes from the first term and behaves as

\[
f_{\text{sing.}}(t, h) \sim \frac{\ln Z}{L^d} \sim \xi^{-d} \sim t^{\nu} g_f(t/h^\Delta).
\]  

(3.4)

A simple interpretation of this result is obtained by dividing the system into units of the size of the correlation length (Fig. 3.2). Each unit is then regarded as an independent random variable, contributing a constant factor to the critical free energy. The number of units grows as \( (L/\xi)^d \).

The consequences of Eq. (3.4) are:

- Homogeneity of \( f_{\text{sing.}} \) emerges naturally.
- We obtain the additional exponent relation

\[
2 - \alpha = d\nu.
\]

(Josephson’s Identity)

Identities obtained from the generalised homogeneity assumption involve the space dimension \( d \), and are known as hyperscaling relations. The relation between \( \alpha \) and \( \nu \) is consistent with the exponents in the table above. However, it does not agree with the mean-field values, \( \alpha = 0 \) and \( \nu = 1/2 \), which are valid for \( d > 4 \). Any theory of critical behaviour must therefore account for the validity of this relation in low dimensions, and its breakdown in \( d > 4 \).

3.3 Correlation Functions and Self-Similarity

So far we have not accounted for the exponent \( \eta \) which describes the decay of correlation functions at criticality. Exactly at the critical point the correlation length is infinite, and there is no other length scale to cut-off the decay of correlation functions. Thus all
correlations decay as a power of the separation. As discussed in the previous chapter, the magnetisation falls off as

\[ G_c(x) \equiv \langle \mathbf{m}(x) \cdot \mathbf{m}(0) \rangle - \langle m^2 \rangle \sim \frac{1}{|x|^{d-2+\eta}}, \]

where \( \eta \) was deduced from the form factor.

Away from criticality, the power laws are cut-off for distances \( |x| \gg \xi \). As the response functions can be obtained from integrating the connected correlation functions, there are additional exponent identities such as Fisher’s identity

\[ \chi \sim \int d^d x \ G_c(x) \sim \int \xi \frac{d^d x}{|x|^{d-2+\eta}} \sim \xi^{2-\eta} \sim t^{-\nu(2-\eta)} \implies \gamma = (2 - \eta)\nu. \]

Therefore, two independent exponents are sufficient to describe all singular critical behaviour.

An important consequence of these scaling ideas is that the critical system has an additional dilation symmetry. Under a change of scale, the critical correlation functions behave as

\[ G_{\text{critical}}(\lambda x) = \lambda^p G_{\text{critical}}(x). \]

This implies a scale invariance or self-similarity: If a snapshot of the critical system is enlarged by a factor of \( \lambda \), apart from a change of contrast \( (\lambda^p) \), the resulting snapshot is statistically similar to the original. Such statistical self-similarity is the hallmark of fractal geometry. The Ginzburg-Landau functional was constructed on the basis of local symmetries such as rotational invariance. If we could add to the list of constraints the requirement of dilation symmetry, the resulting probability would indeed describe the critical point. Unfortunately, it is not in general possible to see directly how such a requirement constrains the effective Hamiltonian.\(^3\) We shall instead prescribe a less direct route by following the effects of the dilation operation on the effective energy; a procedure known as the renormalisation group.

\(^3\)One notable exception is in \( d = 2 \), where dilation symmetry implies conformal symmetry.
Chapter 4

Renormalisation Group

Previously, our analysis of the Ginzburg-Landau Hamiltonian revealed a formal breakdown of mean-field theory in dimensions below some upper critical dimension. Although the integrity of mean-field theory is sometimes extended by resolution limitations in experiment, the breakdown of mean-field theory is often associated with the appearance of qualitatively new critical behaviour. In the previous section, we saw that a simple scaling hypothesis can lead to useful insight into critical behaviour below the upper critical dimension. However, to complement the ideas of scaling, a formal theoretical approach to the analysis of the Ginzburg-Landau Hamiltonian is required. In this section we will introduce a general scheme which allows one to explore beyond the realms of mean-field theory. Yet the method, known as the Renormalisation Group, is not exact nor completely controlled. Instead, it should be regarded as largely conceptual — i.e. its application, which relies fundamentally only on scaling, can be tailored to the particular application at hand.

4.1 Conceptual Approach

The success of the scaling theory in correctly predicting various exponent identities strongly supports the contention that close to the critical point the correlation length $\xi$ is the only important length scale, and that the microscopic lengths are irrelevant. The critical behaviour is governed by fluctuations that are statistically self-similar up to the scale $\xi$. Can this self-similarity be used to develop a theory of critical phenomena below the upper critical dimension? Kadanoff\(^1\) suggested taking advantage of the self-similarity to gradually eliminate the correlated degrees of freedom at length scales $x \ll \xi$, until one

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\(^1\)Leo Kadanoff: recipient of the 1999 National Medal of Science and the 1998 Lars Onsager Prize “for his numerous and profound contributions to statistical physics, including the introduction of the concepts of universality and block spin scaling that are central to the modern understanding of the critical phenomena”.
is left with the relatively simple uncorrelated degrees of freedom at length scale $\xi$. This is achieved through a procedure known as the Renormalisation Group (RG), whose conceptual foundation is outlined below:

1. **Coarse-Grain**: The first step of the RG is to decrease the resolution by changing the minimum length scale from the microscopic scale $a$ to $ba$ where $b > 1$. This is achieved by integrating out fluctuations of the fields $m$ which occur on length scales finer than $ba$. The result is a renormalisation of the Hamiltonian $\beta H$ which leads to an effective Hamiltonian expressed in terms of a ‘coarse-grained’ magnetisation field

$$\bar{m}(x) = \frac{1}{(ba)^d} \int_{\text{Cell}} dy \, m(y),$$

where the integral runs over a cell of size $(ba)^d$ centred on $x$.

2. **Rescale**: Due to the change in resolution, the coarse-grained “picture” is grainier than the original. The original resolution $a$ can be restored by decreasing all length scales by a factor $b$, i.e. defining

$$x' = \frac{x}{b}.$$

Thus, at each position $x'$ we have defined an average moment $\bar{m}(x')$.

3. **Renormalise**: The relative size of the fluctuations of the rescaled magnetisation profile is in general different from the original, i.e. there is a change in contrast between the pictures. This can be remedied by introducing a factor $\zeta$, and defining a renormalised magnetisation

$$m'(x') = \frac{1}{\zeta} \bar{m}(x').$$

The choice of $\zeta$ will be discussed later.

By following these steps, for each configuration $m(x)$ one generates a renormalised configuration $m'(x')$. It can be regarded as a mapping of one set of random variables to another, and can be used to construct the probability distribution. Kadanoff’s insight was to realise that since, on length scales less than $\xi$, the renormalised configurations are statistically similar to the original ones, they must be distributed by a Hamiltonian that is also close to the original. In particular, if the original Hamiltonian $\beta H$ is at a critical point, $t = h = 0$, the new $\beta H'$ is also at criticality since no new length scale is generated in the renormalisation procedure, i.e. $t' = h' = 0$.

However, if the Hamiltonian is originally off criticality, then the renormalisation takes us further away from criticality because $\xi' = \xi/b$ is smaller. The next assumption is that since any transformation only involves changes at the shortest length scales it can not produce singularities. The renormalised parameters must be analytic functions, and hence expandable as

$$\begin{cases} t(b; t, h) = A(b) t + B(b) h + O(t^2, h^2, th), \\ h(b; t, h) = C(b) t + D(b) h + O(t^2, h^2, th). \end{cases}$$
However, the known behaviour at $t = h = 0$ rules out a constant term in the expansion, and to prevent a spontaneously broken symmetry we further require $C(b) = B(b) = 0$. Finally, commutativity $A(b_1 \times b_2) = A(b_1) \times A(b_2)$ implies $A(b) = b^{y_t}$ and $D(b) = b^{y_h}$. So, to lowest order

\[
\begin{aligned}
&\{ t(b) = b^{y_t} t, \\
&h(b) = b^{y_h} h, \\
\}
\end{aligned}
\]  

(4.1)

where $y_t, y_h > 0$ (to ensure that $\xi$ diminishes under the RG procedure). As a consequence:

1. **The free energy**: Since the statistical Boltzmann weight of the new configuration, $\exp[\beta H'[m']]$ is obtained by summing the weights $\exp[\beta H[m]]$ of old ones, the partition function is preserved

\[
Z = \int Dm \ e^{-\beta H[m]} = \int Dm' \ e^{-\beta H'[m']} = Z'.
\]

From this it follows that the free energies density takes the form

\[
f(t, h) = -\frac{\ln Z}{V} = -\frac{\ln Z'}{V' y_d} = b^{-d} f(t(b), h(b)) = b^{-d} f(b^{y_t} t, b^{y_h} h),
\]  

(4.2)

where we have assumed that the two free energies are obtained from the *same Hamiltonian* in which only the parameters $t$ and $h$ have changed according to
Eq. (4.1) describes a homogeneous function of \( t \) and \( h \). This is made apparent by choosing a rescaling factor \( b \) such that \( b^y t \) is a constant, say unity, i.e. \( b = t^{-1/y_t} \), and

\[
f(t, h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) = t^{d/y_t} g_f(h/t^{y_h/y_t}).
\]

We have thus recovered the scaling form of Eq. (3.2) and can identify the exponents

\[
2 - \alpha = d/y_t, \quad \Delta = y_h/y_t
\]

(4.3)

So if \( y_t \) and \( y_h \) are known we can generate all critical exponents.

2. **Correlation Length**: All length scales are reduced by a factor of \( b \) during the RG transformation. This is also true of the correlation length \( \xi' = \xi/b \) implying

\[
\xi(t, h) = b \xi(b^y t, b^{y_h} h) = t^{-1/y_t} \xi(1, h/t^{y_h/y_t}) = t^{-1/y_t} g_\xi(h/t^{y_h/y_t}).
\]

This identifies \( \nu = 1/y_t \) and using Eq. (4.3), the hyperscaling identity \( 2 - \alpha = d\nu \) is recovered.

3. **Magnetisation**: From the homogeneous form of the free energy we can obtain other bulk quantities such as magnetisation. Alternatively, from the RG results for \( Z \), \( V \), and \( h \) we conclude

\[
m(t, h) = \frac{1}{V} \frac{\partial \ln Z(t, h)}{\partial h} = \frac{1}{b^d V' b^{-y_h}} \frac{\partial \ln Z'(t', h')}{\partial h'} = b^{y_h-d} m(b^y t, b^{y_h} h)
\]

Choosing \( b = t^{-1/y_t} \), we find \( m(t, h) = t^{-(y_h-d)/y_t} g_m(h/t^{y_h/y_t}) \) which implies that \( \beta = (y_h - d)/y_t \) and \( \Delta = y_h/y_t \) as before.

It is therefore apparent that quite generally, a quantity \( X \) will have a homogeneous form

\[
X(t, h) = b^{y_X} X(b^{y_h} t, b^{y_h} h) = t^{-y_X/y_t} g_X(h/t^{y_h/y_t}).
\]

(4.4)

In general, for any conjugate pair of variables contributing a term \( \int dx \ F \cdot X \) to the Hamiltonian (e.g. \( \mathbf{m} \cdot \mathbf{h} \)), the **scaling dimensions** are related by \( y_X + y_F = d \).

### 4.2 Formal Approach

In the previous section we found that all critical properties can be abstracted from a scaling relation. Though conceptually appealing, it is not yet clear how such a procedure can be formally implemented. In particular, why should the form of the two Hamiltonians be identical, and why are the two parameters \( t \) and \( h \) sufficient to describe the transition? In this section we outline a more formal procedure for identifying the effects of the dilation operation on the Hamiltonian. The various steps of the program are as follows:
4.2. FORMAL APPROACH

1. Start with the most general Hamiltonian allowed by symmetry. For example, in the presence of rotational symmetry,
\[
\beta H[m] = \int dx \left[ \frac{t}{2} m^2 + um^4 + vm^6 + \cdots + \frac{K}{2} (\nabla m)^2 \right].
\]
(4.5)

2. Apply the three steps of the renormalisation in configuration space: (i) Coarse grain by \( b \); (ii) rescale, \( x' = x/b \); and (iii) renormalise, \( m' = m/\zeta \). This defines a change of variables
\[
m'(x') = \frac{1}{\zeta b^d} \int \text{Cell centred at } bx \, m(x).
\]
Given the Boltzmann weight \( \exp[-\beta H[m(x)]] \) of the original configurations, we can use the change of variables above to construct the corresponding weight \( \exp[-\beta H'[m'(x)]] \) of the new configurations. Naturally this is the most difficult step in the program.

3. Since rotational symmetry is preserved by the RG procedure, the rescaled Hamiltonian must also be described by a point in parameter space,
\[
\beta H'[m'] = \int dx' \left[ \frac{t'}{2} m'^2 + u'm'^4 + v'm'^6 + \cdots + \frac{K'}{2} (\nabla m')^2 \right].
\]
The renormalised coefficients are functions of the original ones, i.e. \( t' = t(b; t, u, \cdots) \); \( u' = u(b; t, u, \cdots) \), etc., defining a mapping \( S' \mapsto R_b S \) in parameter space. In general such a mapping is non-linear.

4. The operation \( R_b \) describes the effects of dilation on the Hamiltonian of the system. Hamiltonians that describe statistically self-similar configurations must thus correspond to fixed points \( S^* \) such that \( R_b S^* = S^* \). Since the correlation length, a function of Hamiltonian parameters, is reduced by \( b \) under the RG operation (i.e. \( \xi(S) = b \xi(R_b S) \)), the correlation length at a fixed point must be zero or infinity. Fixed points with \( \xi^* = 0 \) describe independent fluctuations at each point and correspond to complete disorder (infinite temperature), or complete order (zero temperature). Fixed points with \( \xi^* = \infty \) describe the critical point \( (T = T_c) \).

5. Eq. (4.1) represents a simplified case in which the parameter space is two-dimensional. The point \( t = h = 0 \) is a fixed point, and the lowest order terms in these equations describe the behaviour in the neighbourhood of the fixed point. In general, we can study the stability of a fixed point by linearising the recursion relations in its vicinity: under RG, a point \( S^* + \delta S \) is transformed to
\[
S_i^* + \delta S_i' = S_i^* + \sum_j [R_b]_{ij} \delta S_j + \cdots,
\]
(\( [R_b]_{ij} \equiv \frac{\partial S_i'}{\partial S_j} \))
Because of the semi (i.e. irreversible)-group property we have
\[
R_b R_{b'} \delta O_i = \lambda_i(b) \lambda_i(b') \delta O_i = R_{bb'} \delta O_i = \lambda_i(bb') \delta O_i,
\]
The vectors $O_i$ are called scaling directions associated with the fixed point $S^*$, and $y_i$ are the corresponding anomalous dimensions. Any Hamiltonian in the vicinity of the fixed point can be described by a set of parameters $S = S^* + \sum_i g_i O_i$. The renormalised Hamiltonian has the interaction parameters $S' = S^* + \sum_i g_i b^{y_i} O_i$.

If $y_i > 0$, $g_i$ increases under scaling, and $O_i$ is a relevant operator.

If $y_i < 0$, $g_i$ decreases under scaling, and $O_i$ is a irrelevant operator.

If $y_i = 0$, $O_i$ is a marginal operator, and higher order terms are necessary to track the behaviour.

The subspace spanned by the irrelevant directions is called the basin of attraction of the fixed point $S^*$. Since $\xi$ always decreases under RG ($\xi' = \xi/b$), and $\xi(S^*) = \infty$, $\xi$ is also infinite for any point on the basin of attraction of $S^*$. The surface defines the phase transition — it is equivalent to varying $\beta$ (i.e., the temperature) at different values of the parameters and eventually meeting the surface.

In fact for a general point in the vicinity of $S^*$, the correlation length satisfies the relation

$$\xi(g_1, g_2, \cdots) = b^{y_1} g_1, b^{y_2} g_2, \cdots).$$

For sufficiently large $b$ all the irrelevant operators scale to zero. The leading singularities of $\xi$ are then determined by the remaining set of relevant operators. In particular, if
the operators are indexed in order of decreasing dimensions, we can choose \( b \) such that \( b^n g_1 = 1 \). In this case Eq. (4.6) implies

\[
\xi(g_1, g_2, \cdots) = g_1^{-1/y_1} f(g_2/g_1^{y_2/y_1}, \cdots).
\]

We have thus obtained an exponent \( \nu_1 = 1/y_1 \) for the divergence of \( \xi \), and a generalised set of gap exponents \( \Delta_\alpha = y_\alpha/y_1 \) associated with \( g_\alpha \).

Let us imagine that the fixed point \( S^* \) describes the critical point of the magnet in Eq. (4.5) at zero magnetic field. As the temperature, or some other control parameter, is changed, the coefficients of the Hamiltonian are altered, and the point \( S \) follows a different trajectory in parameter space under renormalisation (see Fig. 4.2). Except for a single point (at the critical temperature) the magnet has a finite correlation length. This can be achieved if the experimental trajectory of the unrenormalised parameters \( S \) intersects the basin of attraction of \( S^* \) only at one point. To achieve this the basin must have co-dimension one, i.e. the fixed point \( S^* \) must have one and only one relevant operator.

This provides an explanation of universality in that the very many microscopic details of the system make up a huge space of irrelevant operators comprising the basin of attraction. In the presence of a magnetic field, two system parameters must be adjusted to reach the critical point, \( (T = T_c, h = 0) \). Thus the magnetic field corresponds to an additional relevant operator of \( S^* \). *In general, for fixed points describing second-order critical points, there are two relevant parameters: the temperature and the field conjugate to the order parameter* (for the magnet it is the magnetic field).

Although the formal procedure outlined in this section is quite rigorous, it suffers from some quite obvious shortcomings: how do we actually implement the RG transformations analytically? There are an infinite number of interactions allowed by symmetry, and hence the space of parameters of \( S \) is inconveniently large. How do we know *a priori* that there are fixed points for the RG transformation; that \( R_b \) can be linearised; that relevant operators are few; etc? The way forward was presented by Wilson who showed how these steps can be implemented (at least perturbatively) in the Ginzburg-Landau model.

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Kenneth G. Wilson, 1936–: Recipient of the 1982 Nobel Prize in Physics, awarded for “discoveries he made in understanding how bulk matter undergoes phase transition, i.e., sudden and profound structural changes resulting from variations in environmental conditions”. Wilson’s background ranges from elementary particle theory and condensed matter physics (critical phenomena and the Kondo problem) to quantum chemistry and computer science.
4.3 The Gaussian Model

In this section we will apply the RG approach to study the Gaussian theory obtained by retaining only the terms to quadratic order in the Ginzburg-Landau Hamiltonian,

$$Z = \int Dm(x) \exp \left\{ -\int d^d x \left[ \frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 - h \cdot m \right] \right\}, \quad (4.7)$$

where, as usual, $m$ represents an $n$-component vector field. The absence of a term at order $m^4$ makes the Hamiltonian meaningful only for $t \geq 0$. The singularity at $t = 0$ can be considered as representing the ordered side of the phase transition.

4.3.1 Exact Solution

Before turning to the RG analysis, let us first obtain the exact homogeneous form for the free energy density. Being of quadratic form, the Hamiltonian is diagonalised in Fourier space and generates the partition function

$$Z = \int Dm(q) e^{-\beta H[m]}, \quad \beta H[m] = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{2} \left( t + Kq^2 \right) |m(q)|^2 - h \cdot m(q = 0).$$

Performing the Gaussian integral, and neglecting the constant factor $(2\pi)^{nN/2}$ arising from the Gaussian functional integral, we obtain the free energy density,

$$f(t, h) = \frac{-\ln Z}{V} = \frac{n}{2} \int \frac{d\mathbf{q}}{(2\pi)^d} \ln(t + Kq^2) - \frac{h^2}{2t}.$$

Although the integral runs over the whole Fourier space $\mathbf{q}$, the important singular contributions originate from long wavelength modes (i.e. those around $\mathbf{q} = 0$). To study the non-analytic contributions to $f$, it is convenient to approximate the domain of integration by a “hypersphere” of radius $\Lambda \approx \pi/a$ where $a$ denotes the short-length scale cut-off. The functional form of the integral can be obtained on dimensional grounds by rescaling $\mathbf{q}$ by a factor $\sqrt{t/K}$. Neglecting the upper limit to the integral, and logarithmic factors, the free energy takes the scaling form

$$f_{\text{sing.}}(t, h) = t^{d/2} \left[ A + B \frac{h^2}{t^{1+d/2}} \right] \equiv t^{2-\alpha} g_f(h/t^\Delta),$$

$^3$Setting $m(x) = \int (d\mathbf{q}/(2\pi)^d) \ m(q) \ e^{i\mathbf{q} \cdot \mathbf{x}}, \ m(q) = \int d\mathbf{x} \ m(x) \ e^{-i\mathbf{q} \cdot \mathbf{x}},$

$$\int d\mathbf{x} \ m(x) \cdot m(x) = \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d\mathbf{q}'}{(2\pi)^d} \ m(q) \cdot m(q') \frac{1}{(2\pi)^d} \delta(q + q') \frac{d\mathbf{x} e^{i(q+q') \cdot \mathbf{x}}}{L^d},$$

$$= \int \frac{d\mathbf{q}}{(2\pi)^d} \ m(q) \cdot m(-q) = \int \frac{d\mathbf{q}}{(2\pi)^d} \ |m(q)|^2,$$

where we have used in the identity $m^*(q) = m(-q)$. 


4.3. THE GAUSSIAN MODEL

Figure 4.3: Diagram showing the shell in Fourier space that is integrated out in the renormalisation procedure.

where \( A \) and \( B \) represent dimensionless constants.

Thus, matching the points \((h = 0, t = 0^+)\) and \((h \to 0)\), the singular part of the free energy is described by the exponents

\[
\alpha_+ = 2 - d/2, \quad \Delta = (2 + d)/4.
\]

Since the ordered phase for \( t < 0 \) is not stable, the exponent \( \beta \) is undefined. The susceptibility, \( \chi \propto \partial^2 f/\partial h^2 \propto 1/t \), diverges with an exponent \( \gamma_+ = 1 \).

4.3.2 The Gaussian Model via RG

The RG of the Gaussian model is most conveniently performed in terms of the Fourier modes. The goal is to evaluate the partition function (4.7) indirectly via the three steps of the RG:

1. **Coarse-Grain:** The first step involves the elimination of fluctuations at scales \( a < |x| < ba \). In spirit, it is similar to removing Fourier modes with wavenumbers \( \Lambda/b < |q| < \Lambda \) (see Fig. 4.3.2). We thus separate the fields into slowly and rapidly varying functions, \( m(q) = m_>(q) + m_<(q) \), with

\[
m(q) = \begin{cases} 
m_<(q) & 0 < |q| < \Lambda/b, \\
m_>(q) & \Lambda/b < |q| < \Lambda.
\end{cases}
\]

The partition function can be re-expressed in the form

\[
Z = \int Dm_<(q) \int Dm_>(q) e^{-\beta H[m_>,m_<]}.
\]

Since the two sets of modes are decoupled in the Gaussian model, the integration is straightforward, and gives

\[
Z = Z_> \int Dm_>(q) \exp \left[ - \int_0^{\Lambda/b} dq \left( \frac{t + Kq^2}{2} \right) |m_<(q)|^2 + h \cdot m_<(0) \right],
\]
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where \( \mathcal{Z}_\gamma = \exp[-(nV/2) \int_{\lambda/b}^\Lambda (dq/(2\pi)^d) \ln(t + kq^2)] \).

2. **Rescale:** The partition function for the modes \( m_\omega(q) \) is similar to the original, except that the upper cut-off has decreased to \( \Lambda/b \), reflecting the coarse-graining in resolution. The rescaling, \( x' = x/b \) in real space, is equivalent to \( q' = bq \) in momentum space, and restores the cut-off to the original value.

3. **Renormalise:** The final step of the RG involves the renormalisation of magnetisation field, \( m'(x') = m_\omega(x')/\zeta \). Alternatively, we can renormalise the Fourier modes according to \( m'(q') = m_\omega(q')/z \), resulting in

\[
\mathcal{Z} = \mathcal{Z}_\gamma \int Dm'(q')e^{-\beta H'[m'(q')]},
\]

\[
\beta H' = \int_0^\Lambda \frac{dq'}{(2\pi)^d} b^{-d} z^2 \left( \frac{t + Kb^{-2}q^2}{2} \right) |m'(q')|^2 - zh \cdot m'(0).
\]

The constant factor change from the Jacobian can be neglected in favour of the singular contribution from the exponent.

This procedure has transformed from a set of parameters \( S = \{K, t, h\} \) to a new set

\[
S' = \begin{cases} 
K' = Kb^{-d-2}z^2, \\
t' = tb^{-d}z^2, \\
h' = hz.
\end{cases}
\]

(Note that in general, such transformations can and often will lead to the appearance of new terms absent in the original Hamiltonian.) The singular point \( t = h = 0 \) is mapped onto itself as expected. To make the fluctuations scale invariant at this point, we must ensure that the remaining parameter in the Hamiltonian, \( K \), stays fixed. This is achieved by the choice \( z = b^{1+d/2} \) which implies that

\[
\begin{cases} 
t' = b^2 t \\
h' = b^{1+d/2} h
\end{cases}
\]

\( y_t = 2, \quad y_h = 1 + d/2. \)

For the fixed point \( t = t', \) \( K \) becomes weaker and the spins become uncorrelated — the high temperature phase.

From these equations, we can predict the scaling of the Free energy

\[
f_{\text{sing}}(t, h) = b^{-d} f_{\text{sing}}(b^2 t, b^{1+d/2} h) \quad (b^2 t = 1)
\]

\[
= t^{d/2} g_f (h/t^{1/2+d/4}).
\]

This implies exponents: \( 2 - \alpha = d/2, \) \( \Delta = y_t/y_h = 1/2 + d/4, \) and \( \nu = 1/y_t = 1/2. \) Comparing with the results from the exact solution we can confirm the validity of the RG.
At the fixed point \((t = h = 0)\) the Hamiltonian must be scale invariant. This allows us to fix the value of the renormalisation parameter \(\zeta\). By dimensional analysis \(x = bx'\), \(m(x) = \zeta m'(x')\) and

\[
\beta H^* = \frac{K}{2} b^{d-2} \zeta^2 \int dx' (\nabla m')^2, \quad \zeta = b^{1-d/2}.
\]

Therefore, for small perturbations

\[
\beta H^* + u_p \int dx |m(x)|^p \rightarrow \beta H^* + u_p b^d \zeta^p \int dx' |m'(x')|^p.
\]

Thus, in general \(u_p \mapsto u'_p = b^d \zeta^p u_p\), where \(y_p = p - d(p/2 - 1)\), in agreement with our earlier findings that \(y_1 = y_h = 1 + d/2\) and \(y_2 = y_t = 2\). For the Ginzburg-Landau Hamiltonian, the quartic term scales with an exponent \(y_4 = 4 - d\) and is therefore relevant for \(d < 4\) and irrelevant for \(d > 4\). Sixth order perturbations scale with an exponent \(y_6 = 6 - 2d\) and is therefore irrelevant for \(d > 3\).

4.4 Wilson’s Perturbative Renormalisation Group

In this section we will assess the extent to which the higher order terms in the Ginzburg-Landau expansion can be treated as a perturbation of the Gaussian model. Our method will be to combine the momentum space RG with a perturbative treatment of the Hamiltonian.

Since the unperturbed part of the Hamiltonian is diagonal in Fourier space, it is convenient to switch to that representation and re-express

\[
\beta H[m] = \int dx \left[ \beta H_0 \left[ m \right] + \frac{1}{2} m^2 \right] + U
\]

as

\[
\beta H_0 = \frac{1}{(2\pi)^d} \int \frac{dq}{2} \frac{1}{2} (t + Kq^2) |m(q)|^2,
\]

\[
U = u \int \frac{dq_1}{(2\pi)^d} \int \frac{dq_2}{(2\pi)^d} \int \frac{dq_3}{(2\pi)^d} m(q_1) \cdot m(q_2) \cdot m(q_3) \cdot m(-q_1 - q_2 - q_3).
\]

To implement the perturbative RG we proceed, as before, in three steps

1. **Coarse-Grain**: Subdividing the fluctuations into two components \(m(q) = m_<(q) + m_>(q)\) the contribution to the unperturbed (Gaussian) part of the Hamiltonian is separable while the perturbation mixes the terms. Integrating, we obtain

\[
Z = Z_0 \int Dm_\le e^{-\beta H_0[m_\le]} \frac{1}{Z_0} \int Dm_\ge e^{-\beta H_0[m_\ge] - U[m_\le, m_\ge]} \langle e^{-U[m_\le, m_\ge]} \rangle_{m_\ge}
\]

\[
= Z_0^2 \int Dm_\le e^{-\beta H_0[m_\le]} + \ln \langle e^{-U[m_\le, m_\ge]} \rangle_{m_\ge}
\]
where $Z_0^\infty$ denotes the contribution to the Gaussian (unperturbed) partition function arising from $m_<$. 

In general, the renormalisation of the Hamiltonian would call for the expansion
\[
\ln \langle e^{-U} \rangle = -\langle U \rangle + \frac{1}{2} (\langle U^2 \rangle - \langle U \rangle^2) + \cdots + \frac{(-1)^\ell}{\ell!} \langle U^\ell \rangle_c + \cdots,
\]
where $\langle U^\ell \rangle_c$ denotes the $\ell$th cumulant. However, for simplicity, we will stop here at leading order in the perturbation from which we obtain
\[
\beta H [m_<] = \beta H_0 [m_<] - \ln [Z_0^\infty] + \langle U \rangle_{m_>} + O(a^2).
\]

Only terms which are of an even order in $m_>$ contribute to the average $\langle U \rangle_{m_>}$. In particular, we will require averages of the form
\[
C_1(\{q_1\}) = \langle m_< (q_1) \cdot m_< (q_2) \cdot m_< (q_3) \cdot m_< (q_4) \rangle_{m_>},
\]
\[
C_2(\{q_1, q_2\}) = \langle m_> (q_1) \cdot m_< (q_2) \cdot m_< (q_3) \cdot m_< (q_4) \rangle_{m_>},
\]
\[
C_3(\{q_1, q_2, q_3\}) = \langle m_> (q_1) \cdot m_< (q_2) \cdot m_> (q_3) \cdot m_< (q_4) \rangle_{m_>},
\]
\[
C_4(\{q_1, q_2, q_3, q_4\}) = \langle m_> (q_1) \cdot m_> (q_2) \cdot m_> (q_3) \cdot m_> (q_4) \rangle_{m_>}.
\]

$C_1$ simply generates $U[m_<]$ while $C_4$ gives some constant independent of $m_>$. The important contributions arise from $C_2$ and $C_3$ which can be represented diagrammatically as in Fig. 4.4.

For the unperturbed Hamiltonian the two-point expectation value is equal to
\[
\langle m_\alpha (q) m_\beta (q') \rangle_0 = \delta_{\alpha\beta} (2\pi)^d \delta^d (q + q') G_0 (q), \quad G_0 (q) = \frac{1}{t + K q^2},
\]
where the subscript zero indicates that the average is with respect to the unperturbed (Gaussian) Hamiltonian.\footnote{In general, the expectation value involving any product of $\bar{m}$’s can be obtained from the identity for Gaussian distributed random variables}

\[
\langle \exp \left[ \int dx \, a(x) \cdot m(x) \right] \rangle_0 = \exp \left[ \int dx \int dx' \frac{1}{2} a_\alpha (x) \langle m_\alpha (x) m_\beta (x') \rangle_0 a_\beta (x') \right]
\]
Dropping the irrelevant constant terms, $C_4$ and $\ln Z_0^>$, we find that no new relevant terms appear in the coarse-grained Hamiltonian $\beta H[m_\prec]$, and the coefficients $K$ and $u$ are unrenormalised, while

$$t \mapsto \tilde{t} = t + 4u(n + 2) \int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^d} G_0(\mathbf{q}),$$

the factor of $4(n + 2)$ arising from enumerating all permutations.

2. **Rescale**: As usual we set $q' = bq$.

3. **Renormalise**: Finally we set $m' = m_\prec(q')/z$ and obtain

$$\beta H'[m'] = \int_0^{\Lambda} \frac{dq'}{(2\pi)^d} b^{-d} z^2 \left( \frac{\tilde{t} + K b^{-2} q^d}{2} \right) |m'(q')|^2$$

$$+ uz^4b^{-3d} \int_0^{\Lambda} \frac{dq'_1}{(2\pi)^d} \int_0^{\Lambda} \frac{dq'_2}{(2\pi)^d} \int_0^{\Lambda} \frac{dq'_3}{(2\pi)^d} m'(q'_1) \cdot m'(q'_2) \cdot m'(q'_3) \cdot m'(-q'_1 - q'_2 - q'_3).$$

The renormalised Hamiltonian is defined by

$$t' = b^{-d} z^2 \tilde{t}, \quad K' = b^{-d-2} z^2 K, \quad u' = b^{-3d} z^4 u.$$

As in the Gaussian model, if we set $z = b^{1+d/2}$ such that $K' = K$, there is a fixed point at $t^* = u^* = 0$. The recursion relations for $t$ and $u$ in the vicinity of this point are given by

$$t' \equiv t(b) = b^2 \left[ t + 4u(n + 2) \right] \int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^d} G_0(\mathbf{q}),$$

$$u' \equiv u(b) = b^{4-d} u.$$

Expanding both sides in powers of $\{a(x)\}$ we obtain **Wick’s theorem**

$$\left\langle \prod_{i=1}^{\ell} m_\alpha(x_i) \right\rangle_0 = \begin{cases} 0 & \text{sum over all pairwise contractions} \\ \ell \text{ odd} \\ \ell \text{ even} \end{cases}$$

For example

$$\left\langle m_\alpha(x_i)m_\alpha(x_j)m_\alpha(x_k)m_\alpha(x_l) \right\rangle_0 = \left\langle m_\alpha(x_i)m_\alpha(x_j) \right\rangle_0 \left\langle m_\alpha(x_k)m_\alpha(x_l) \right\rangle_0$$

$$+ \left\langle m_\alpha(x_i)m_\alpha(x_k) \right\rangle_0 \left\langle m_\alpha(x_j)m_\alpha(x_l) \right\rangle_0 + \left\langle m_\alpha(x_k)m_\alpha(x_l) \right\rangle_0 \left\langle m_\alpha(x_i)m_\alpha(x_j) \right\rangle_0.$$ 

Moreover, in the presence of a perturbation $U$, the expectation value of any operator $O$ can be expressed using the identity

$$\left\langle O \right\rangle = \frac{\int Dm \, O \, e^{-\beta H}}{\int Dm \, e^{-\beta H}} = \frac{\int Dm \, [1 - U + U^2/2 - \cdots] e^{-\beta H_0}}{\int Dm \, [1 - U + U^2/2 - \cdots] e^{-\beta H_0}}$$

$$= \frac{Z_0 \left( O \right)_0 - \left( OU \right)_0 + \left( OU^2/2 \right)_0 - \cdots}{Z_0[1 - \left( U \right)_0 + \left( U^2/2 \right)_0 - \cdots]} \equiv \sum_n \frac{(-1)^n}{n!} \left( O^{(n)} \right)_0 = \left( O e^{-U} \right)_0,$$

where the different orders in the expansion define the connected average denoted by the superscript $c$. 

### 4.4. Wilson’s Perturbative Renormalisation Group
The recursion relation for $u$ at this order is identical to that obtained by dimensional analysis; but that of $t$ is modified. It is conventional to convert the above discrete recursion relations to continuous differential equations by setting $b = e^\ell$. For an infinitesimal $\delta \ell$, 

\[ t(b) \equiv t(1 + \delta \ell + \cdots) = t + \delta \ell \frac{dt}{d\ell} + O(\delta \ell^2), \]

\[ u(b) = u + \delta \ell \frac{du}{d\ell} + O(\delta \ell^2). \]

Expanding the recursion relations, we obtain \(^5\)

\[ \frac{dt}{d\ell} = 2t + \frac{4u(n + 2)K_d\Lambda^d}{t + K\Lambda^2}, \]

\[ \frac{du}{d\ell} = (4 - d)u, \]

where $K_d \equiv S_d/(2\pi)^d$. Integrated, the second equation gives $u(\ell) = u_0 e^{(4-d)\ell} = u_0 b^{4-d}$.

The recursion relations can be linearised in the vicinity of the fixed point $t^* = u^* = 0$ by setting $t = t^* + \delta t$ and $u = u^* + \delta u$, as

\[ \frac{d}{d\ell} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & 4(n + 2)K_d\Lambda^{d-2}/K \\ 0 & 4 - d \end{pmatrix} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix}. \]

In the differential form, the eigenvalues of the matrix that enter the recursion relations determine the relevance of the operators. Since the matrix above has zero elements on one side, its eigenvalues are the diagonal elements and, as in the Gaussian model, we can identify $y_t = 2$, and $y_u = 4 - d$. The results at this order are identical to those obtained from dimensional analysis of the Gaussian model. The only difference is in the eigen-directions. The exponent $y_t = 2$ is still associated with $u = 0$, while $y_u = 4 - d$ is actually associated with the direction $t = 4u(n + 2)K_d\Lambda^{d-2}/(2 - d)K$.

For $d > 4$ the Gaussian fixed point has only one unstable direction associated with $y_t$. It thus correctly describes the phase transition. For $d < 4$ it has two relevant directions and is unstable. Unfortunately, the recursion relations have no other fixed point at this order and it appears that we have learned little from the perturbative RG. However, since we are dealing with a perturbative series alternating in sign, we can anticipate that the recursion relations at the next order are modified according to

\[ \frac{dt}{d\ell} = 2t + \frac{4u(n + 2)K_d\Lambda^d}{t + K\Lambda^2} - Au^2, \]

\[ \frac{du}{d\ell} = (4 - d)u - Bu^2, \]

\(^5\)Here we have made use of the approximation

\[ \int_{\lambda/b}^{\Lambda} \frac{dq}{(2\pi)^d} G_0(q) \simeq \left( \Lambda - \frac{\Lambda}{b} \right)^{d-1} \frac{S_d}{(2\pi)^d K\Lambda^2 + t} \]

and set $\Lambda(1 - e^{-\delta \ell}) \simeq \Lambda \delta \ell + \cdots$. 

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**CHAPTER 4. RENORMALISATION GROUP**
4.5. \(^{†}\) The \(\epsilon\)-Expansion

with \(A\) and \(B\) both positive. There is now an additional fixed point at \(u^* = (4 - d)/B\) for \(d < 4\). For a systematic perturbation theory we need to keep the parameter \(u\) small. Thus the new fixed point can be explored systematically only for \(\epsilon = 4 - d\); we are led to consider an expansion in the dimension of space in the vicinity of \(d = 4\)! For a calculation valid at \(O(\epsilon)\) we have to keep track of terms of second order in the recursion relation for \(u\), but only first order in \(t\). It would thus be unnecessary to calculate the term \(A\) in the expression above.

4.5. \(^{†}\) The \(\epsilon\)-Expansion

INFO: It is left as an exercise (see problem set II) to show that expansion to second order ("two-loop") in \(u\) leads to the identity

\[
B = -\frac{4(n + 8)Kd\Lambda^d}{(t + K\Lambda^2)^2}.
\]

Thus, in addition to the Gaussian fixed point at \(u^* = t^* = 0\), there is now a non-trivial fixed point \((dt/d\ell = du/d\ell = 0)\) at

\[
\begin{align*}
  u^* &= \frac{(t^* + K\Lambda^2)^2}{4(n+8)Kd\Lambda^d} = \frac{K^2}{4(n+8)K^4} \epsilon + O(\epsilon^2), \\
  t^* &= \frac{-2u^*(n+2)Kd\Lambda^d}{t^* + K\Lambda^2} = \frac{-(n+2)}{2(n+8)} K\Lambda^2 \epsilon + O(\epsilon^2),
\end{align*}
\]

where only those terms at leading order in \(\epsilon = 4 - d\) have been retained.

Linearising the recursion relations in the vicinity of the new fixed point we obtain

\[
\frac{d}{dt} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 - \frac{n+2}{n+8} \epsilon & \cdots \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} \delta t \\ \delta u \end{pmatrix}.
\]
The first eigenvalue is positive controlling the instability of the fixed point

\[ y_t = 2 - \frac{n + 2}{n + 8} \epsilon + O(\epsilon^2) \]

while the second eigenvalue

\[ y_u = -\epsilon + O(\epsilon^2) \]

is negative for \( d < 4 \). The new fixed point thus has co-dimension one and can describe the phase transition in these dimensions. Although the position of the fixed point depends on microscopic parameters such as \( K \) and \( \Lambda \), the final eigenvalues are pure numbers that depend only on \( n \) and \( d = 4 - \epsilon \). These eigenvalues characterise the universality classes of rotational symmetry breaking in \( d < 4 \).

Continuing it is possible to obtain better estimates for critical exponents. However, even at second order, the \( \epsilon \)-expansion does not make numerically accurate predictions in physical dimensions. Why then should one bother with such calculations? Their great virtue is that they provide a relatively straightforward way of determining what types of universality classes exist. Although the numerical values of the critical exponents change considerably as one moves away from the upper critical dimension, the topology of the flow diagrams does not. Thus one can investigate which interactions will lead to new universality classes and which will not. It is in this sense that the \( \epsilon \)-expansion is largely responsible for our rather detailed understanding of critical phenomena.

The perturbative implementation of the RG procedure for the Ginzburg-Landau Hamiltonian was first performed by K. G. Wilson in the early 1970’s, while the \( \epsilon \)-expansion was developed jointly with M. E. Fisher.\(^6\)

Wilson was awarded the Nobel prize in 1982. Historical details can be found in his Nobel lecture reprinted in Rev. Mod. Phys. 55, 583 (1983). This concludes our investigation of the scaling theory and renormalisation group.
4.6 Problem Set II

4.6.1 Problems on Scaling and the Renormalisation Group

The problems below are designed to develop some of the ideas concerning scaling and the renormalisation group introduced in lectures. Note that the problems on the $\epsilon$-expansion (marked by a "†") invite you to go well beyond the material covered in lectures. It is designed to help you become familiar with the concept and application of the RG, and should NOT be regarded as typical of the type of problems that will be set in the examinations!

Although it was not discussed at length in lectures, the first (and in some sense the easiest) application of the Renormalisation Group was to a lattice spin Hamiltonian. The first problem in this section goes, step by step, through the RG transformation for the one-dimensional Ising model (where in fact the RG is exact). If you get lost refer to e.g. Chaikin and Lubensky, section 5.6 p 242.

1. The Migdal-Kadanoff Method: The partition function for the one-dimensional ferromagnetic Ising model with nearest neighbour interaction is given by

$$Z = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H[\sigma]}, \quad \beta H = -\sum_{\langle ij \rangle} \left[ J\sigma_i\sigma_j + \frac{h}{2} (\sigma_i + \sigma_j) + g \right],$$

where $\langle ij \rangle$ denotes the sum over neighbouring lattice sites. The Migdal-Kadanoff scheme involves an RG procedure which, by eliminating a certain fraction of the spins from the partition sum, reduces the number of degrees of freedom by a factor of $b$. Their removal induces an effective interaction of the remaining spins which renormalises the coefficients in the effective Hamiltonian. The precise choice of transformation is guided by the simplicity of the resulting RG. For $b = 2$ (known as decimation) a natural choice is to eliminate (say) the even numbered spins.

(a) By applying this procedure, show that the partition function is determined by a renormalised Hamiltonian involving spins at odd numbered sites $\sigma'_i$,

$$Z = \sum_{\{\sigma'_i = \pm 1\}} e^{-\beta H'[\sigma'_i]},$$

Michael E. Fisher: recipient of the 1995 Lars Onsager Prize “for his numerous and seminal contributions to statistical mechanics, including but not restricted to the theory of phase transitions and critical phenomena, scaling laws, critical exponents, finite size effects, and the application of the renormalisation group to many of the above problems”.

where the Hamiltonian $\beta H'$ has the same form as the original but with renormalised interactions determined by the equation
\[
\exp \left[ J' \sigma'_1 \sigma'_2 + \frac{h'}{2} (\sigma'_1 + \sigma'_2) + g' \right]
= \sum_{s = \pm 1} \exp \left[ J s (\sigma'_1 + \sigma'_2) + \frac{h}{2} (\sigma'_1 + \sigma'_2) + hs + 2g \right].
\]

(b) Substituting different values for $\sigma'_1$ and $\sigma'_2$ obtain the relationship between the renormalised coefficients and the original. Show that the recursion relations take the general form
\[
g' = 2g + \delta g(J, h), \\
h' = h + \delta h(J, h), \\
J' = J'(J, h).
\]

(c) For $h = 0$ check that no term $h'$ is generated from the renormalisation (as is clear from symmetry). In this case, show that
\[
J' = \frac{1}{2} \ln \cosh(2J).
\]

Show that this implies a stable (infinite $T$) fixed point at $J = 0$ and an unstable (zero $T$) fixed point at $J = \infty$. Any finite interaction renormalises to zero indicating that the one-dimensional chain is always disordered at sufficiently long length scales.

(d) Linearising (in the exponentials) the recursion relations around the unstable fixed point, show that
\[
e^{-J'} = \sqrt{2} e^{-J}, \quad h' = 2h.
\]

(e) Regarding $e^{-J}$ and $h$ as scaling fields, show that in the vicinity of the fixed point the correlation length satisfies the homogeneous form ($b = 2$)
\[
\xi(e^{-J}, h) = 2\xi(\sqrt{2} e^{-J}, 2h) = 2^f \xi(2^{f/2} e^{-J}, 2^f h).
\]

Note that choosing $2^{f/2} e^{-J} = 1$ we obtain the scaling form
\[
\xi(e^{-J}, h) = e^{2J} g_J(h e^{2J}).
\]

The correlation length diverges on approaching $T = 0$ for $h = 0$. However, its divergence is not a power law of temperature. Thus there is an ambiguity in identifying the exponent $\nu$ related to the choice of measure in the vicinity of $T = 0$ ($1/J$ or $e^{-J}$). The hyperscaling assumption states that the singular part of the free energy in $d$-dimensions is proportional to $\xi^{-d}$. Hence we expect
\[
f_{\text{sing.}}(J, h) \propto \xi^{-1} = e^{-2J} g_J(h e^{2J}).
\]
At zero field, the magnetisation is always zero, while the susceptibility behaves as
\[ \chi(J) \sim \frac{\partial^2 f}{\partial h^2}\bigg|_{h=0} \sim e^{2J}. \]

On approaching \( T = 0 \), the divergence of the susceptibility is proportional to that of the correlation length. Using the general form \( \langle \sigma_i \sigma_{i+x} \rangle \sim e^{-x/\xi} / x^{d-2+\eta} \) and \( \chi \sim \int dx \langle \sigma_0 \sigma_x \rangle_c \sim \xi^{2-\eta} \) we conclude \( \eta = 1 \).

[The results of the RG are confirmed by exact calculation using the so-called transfer matrix method.]

---

2. The Hamiltonian describing a nearly flat two-dimensional membrane embedded in three-dimensions is given approximately by
\[
\beta H = \frac{1}{2} \int d^2x \left[ r_0 (\nabla h)^2 + \kappa_0 (1 + (\nabla h)^2)^{-5/2} (\nabla^2 h)^2 \right],
\]
where \( h(x) \) denotes the height of the vertical displacement of the membrane at position \( x \), \( r_0 \) represents the bare interfacial tension, and \( \kappa_0 \) represents the bare bending modulus.

(a) In the harmonic approximation (i.e. neglecting terms of quartic order in \( h \) and higher), and taking \( r_0 > 0 \) show that the nature of the long-range fluctuations is determined solely by the interfacial tension. Estimate the long-distance behaviour of the autocorrelator \( \langle [h(x) - h(0)]^2 \rangle \).

(b) In the same approximation, obtain an estimate for the long-distance autocorrelator of the surface normal \( \langle [\nabla h(x) - \nabla h(0)]^2 \rangle \). Taking this result together with that found in part (a), what conclusions can be drawn from these results about the nature of long-range order in the system?

(c) Short-range fluctuations of the membrane give rise to a renormalisation of the bending modulus at longer length scales. Treating the quartic interaction as a perturbation (and neglecting higher order terms in the expansion), show that, up to a constant, a perturbative renormalisation group analysis leads to an effective Hamiltonian for the long-range fluctuations \( h_\prec \) of the form,
\[
\tilde{\beta} H[h_\prec] = \beta H[h_\prec] + \langle \beta H_I \rangle_{h_\prec} + \cdots
\]
where \( \beta H_I \) denotes the perturbation. Here \( h_\prec \) and \( h_\succ \) represent contributions to the field \( h \) involving Fourier components \( |q| < \Lambda e^{-\ell} \) and \( \Lambda e^{-\ell} \leq |q| < \Lambda \) respectively, where \( \Lambda \gg \sqrt{r_0/\kappa_0} \) denotes the inverse lattice cut-off.
(d) Applying this result, show that, to leading order, the differential recursion relation for the curvature modulus is given by
\[ \frac{d \kappa}{d \ell} = -\frac{5}{4\pi}. \]
Comment on the implications of this result.

3. The Lifshitz Point: (see Chaikin and Lubensky, p184) A number of materials, such as liquid crystals, are highly anisotropic and behave differently along directions parallel and perpendicular to some axis. An example is provided below. The \( d \) spatial dimensions are grouped into one parallel direction, \( x_\parallel \) and \( d-1 \) perpendicular directions, \( x_\perp \). Consider a one-component field \( m \) subject to the Hamiltonian
\[ \beta H = \beta H_0 + U, \]
\[ \beta H_0 = \int dx_\parallel \int dx_\perp \left[ \frac{K}{2} (\nabla_\parallel m)^2 + \frac{L}{2} (\nabla_\perp^2 m)^2 + \frac{t}{2} m^2 - hm \right], \]
\[ U = u \int dx_\parallel \int dx_\perp m^4. \]
A Hamiltonian of this kind is realised in the theory of fluctuations in stacked fluid membranes — the smectic liquid crystal. [Note that \( \beta H \) depends on the first gradient in the \( x_\parallel \) direction, and on the second gradient in the \( x_\perp \) directions.]
(a) Write \( \beta H_0 \) in terms of the Fourier transforms \( m(q_\parallel, q_\perp) \).
(b) Construct a renormalisation group transformation for \( \beta H_0 \) by rescaling distances such that \( q'_\parallel = bq_\parallel \), \( q'_\perp = cq_\perp \), and the field \( m' = m/z \).
(c) Choose \( c \) and \( z \) such that \( K' = K \) and \( L' = L \). At the resulting fixed point calculate the eigenvalues \( y_t \) and \( y_h \).
(d) Write down the relationship between the free energies \( f(t, h) \) and \( f(t', h') \) in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form
\[ f(t, h) = t^{2-\alpha} g_f(h/t^\Delta), \]
and identify the exponents \( \alpha \) and \( \Delta \).
(e) How does the unperturbed zero-field susceptibility \( \chi(t, 0) \) diverge as \( t \to 0 \)?
In the remainder of this problem set \( h = 0 \), and treat \( U \) as a perturbation
(f) In the unperturbed Hamiltonian calculate the expectation value \( \langle m(q)m(q') \rangle_0 \), and the corresponding susceptibility \( \chi(q) \), where \( q = (q_\parallel, q_\perp) \).
(g) Write the perturbation \( U \) in terms of the Fourier modes \( m(q) \).
(h) Obtain the expansion for \( \langle m(q)m(q') \rangle \) to first order in \( U \), and reduce the correction term to a product of two-point expectation values.

(i) Write down the expression for \( \chi(q) \) in the first order of perturbation theory, and identify the transition point \( t_c \) at first order in \( u \). [Do not evaluate the integral explicitly.]

(j) Using RG, or any other method, find the upper critical dimension \( d_u \) for validity of the Gaussian exponents.

The next problem concerns the \( \epsilon \)-expansion of the Ginzburg-Landau Hamiltonian to second order. Although outlined in the lectures, this problem leads you through a detailed investigation of the O(\( n \)) fixed point. In attacking this problem one may wish to consult a reference text such as Chaikin and Lubensky (p263).

4. Using Wilson’s perturbative renormalisation group, the aim of this problem is to obtain the second-order \( \epsilon = 4 - d \) expansion of the Ginzburg-Landau functional

\[
\beta H = \int dx \left[ \frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + u(m^2)^2 \right],
\]

where \( m \) denotes an \( n \)-component field.

(a) Treating the quartic interaction as a perturbation, show that an application of the momentum shell RG generates a Hamiltonian of the form

\[
\beta H[m_\prec] = \int_0^{\Lambda/b} (dq) \frac{G^{-1}(q)}{2} |m_\prec(q)|^2 - \ln \langle e^{-U} \rangle_{m_\prec}, \quad G^{-1}(q) = t + Kq^2,
\]

where we have used the notational shorthand \( (dq) \equiv dq/(2\pi)^d \).

(b) Expressing the interaction in terms of the Fourier modes of the Gaussian Hamiltonian, represent diagrammatically those contributions from the second order of the cumulant expansion. [Remember that the cumulant expansion involves only those diagrams which are connected.]

(c) Focusing only on those second order contributions that renormalise the quartic interaction, show that the renormalised coefficient \( u \) takes the form

\[
\tilde{u} = u - 4u^2(n + 8) \int_{\Lambda/b}^{\Lambda} (dq) G(q)^2.
\]

Comment on the nature of those additional terms generated at second-order.
(d) Applying the rescaling $q = q'/b$, performing the renormalisation $m_\prec = zm$, and arranging that $K' = K$, show that the differential recursion relations take the form ($b = e^\ell$)

$$
\frac{dt}{d\ell} = 2t + 4u(n+2)G(\Lambda)Kd\Lambda^d - u^2 A(q = 0),
$$

$$
\frac{du}{d\ell} = (4 - d)u - 4(n+8)u^2 G(\Lambda)^2 Kd\Lambda^d.
$$

(e) From this result, show that for $d < 4$ the Gaussian fixed point becomes unstable against a new fixed point (known as the $O(\epsilon)$ fixed point). [Remember to be consistent in keeping terms of definite order in $\epsilon$!] Linearising in the vicinity of the new fixed point, show that the scaling dimensions take the form

$$
y_t = 2 - \left(\frac{n+2}{n+8}\right) \epsilon + O(\epsilon^2), \quad y_u = -\epsilon + O(\epsilon^2).
$$

Sketch the RG flows for $d > 4$ and $d < 4$.

(f) Adding the magnetic field dependent part of the Hamiltonian, show that to leading order in $\epsilon$, the magnetic exponent $y_h$ is unchanged from the mean-field value.

(g) From the scaling relations for the free energy density and correlation length

$$
f(g_1 = \delta t, h) = b^{-d} f(b^{y_h} \delta t, b^{y_h} h),
$$

$$
\xi(\delta t, h) = b^{-1} \xi(b^{y_h} \delta t, b^{y_h} h).
$$

determine the critical exponents $\nu$, $\alpha$, $\beta$, and $\gamma$. [Recall: $\xi \sim (\delta t)^{-\nu}$, $C \sim (\delta t)^{-\alpha}$, $m \sim (\delta t)^{\beta}$, $\chi \sim (\delta t)^{-\gamma}$.]

The final problem in this set involves another investigation of an $\epsilon$-expansion this time applied to continuous spins near two-dimensions. In contrast to the $4 - \epsilon$ expansion of the Ginzburg-Landau Hamiltonian described above, a non-trivial fixed point emerges already at first order. The aim of this calculation is to study properties of the fixed point in the vicinity of two-dimensions. This calculation repeats steps first performed by Polyakov (Phys. Lett. 59B, 79 (1975)) in a seminal work on the properties of the non-linear $\sigma$-model. Once again, this calculation should be attempted with reference to a standard text such as Chaikin and Lubensky (p341).

5. †Continuous Spin Systems Near Two-Dimensions: The aim of this problem is to employ Wilson’s perturbative renormalisation group, to obtain the $\epsilon = d - 2$ expansion of the $n$-component non-linear $\sigma$-model

$$
Z = \int D\mathbf{S}(\mathbf{x}) \delta (S^2(\mathbf{x}) - 1) \exp \left[ -\frac{K}{2} \int d\mathbf{x} (\nabla \mathbf{S})^2 \right].
$$
In the vicinity of the transition temperature, it is convenient to expand the spin degrees of freedom around the (arbitrary) direction of spontaneous symmetry breaking, \( S_0(x) = (0, \cdots, 0, 1) \),

\[
S(x) = (\Pi_1(x), \cdots, \Pi_{n-1}(x), \sigma(x)) \equiv (\Pi(x), \sigma(x)),
\]

where \( \sigma(x) = (1 - \Pi^2)^{1/2} \).

(i) Substituting this expression, and expanding \( \sigma \) in powers of \( \Pi \), show that the Hamiltonian takes the form

\[
\beta H = \frac{K}{2} \int d\mathbf{x} \left[ (\nabla \Pi)^2 + \frac{1}{2} (\nabla \Pi^2)^2 + \cdots \right].
\]

(ii) Treating this expansion to quadratic order, show that the lower critical dimension is 2.

(iii) Taking \( \sigma > 0 \), and using the expression (true when \( \sigma > 0 \))

\[
\delta \left( \Pi^2 + \sigma^2 - 1 \right) = \frac{1}{2(1 - \Pi^2)^{1/2}} \delta \left( \sigma - (1 - \Pi^2)^{1/2} \right),
\]

show that the partition function can be written in the form

\[
Z = \int D\Pi(x) \exp \left[ -\rho \int d\mathbf{x} \ln(1 - \Pi^2) \right]
\times \exp \left\{ -\frac{K}{2} \int d\mathbf{x} \left[ (\nabla \Pi)^2 + (\nabla (1 - \Pi^2)^{1/2})^2 \right] \right\},
\]

where \( \rho \equiv (N/V) = \int_0^\Lambda (d\mathbf{q}) \) denotes the density of states.

(iv) Polyakov’s Perturbative Renormalisation Group: Expanding the Hamiltonian perturbatively in \( \Pi \), show that \( K\langle \Pi^2 \rangle \sim O(1) \), \( K(\nabla \Pi^2)^2 \sim O(K^{-1}) \), and \( \rho \Pi^2 \sim O(K^{-1}) \).

This suggests that we define

\[
\beta H_0 = \frac{K}{2} \int d\mathbf{x} (\nabla \Pi)^2,
\]

as the unperturbed Hamiltonian and treat

\[
U = \frac{K}{2} \int d\mathbf{x} (\Pi \cdot \nabla \Pi)^2 - \frac{\rho}{2} \int d\mathbf{x} \Pi^2,
\]

as a perturbation.

(v) Expand the interaction in terms of the Fourier modes and obtain an expression for the propagator \( \langle \Pi_\alpha(q_1) \Pi_\beta(q_2) \rangle_0 \). Sketch a diagrammatic representation of the components of the perturbation.
(vi) Perturbative Renormalisation Group: Applying the perturbative RG procedure, and integrating out the fast degrees of freedom, show that the partition function takes the form

\[ Z = \int D\Pi < e^{-\delta f_0^b - \beta H_0[\Pi <]} - \ln \langle e^{-U[\Pi <,\Pi >]} \rangle_0 >, \]

where \( \delta f_0^b \) represents some constant.

(vii) Expanding to first order, identify and obtain an expression for the two diagrams that contribute towards a renormalisation of the coupling constants. (Others either vanish or give a constant contribution.) [Note: the density of states is given by \( \rho = (N/V) = \int_0^\Lambda (d\mathbf{q}) = b^d \int_0^\Lambda (d\mathbf{q}). \)] As a result, show that the renormalised Hamiltonian takes the form

\[ -\beta H[\Pi <] = \delta f_0^b + \delta f_1^b - \frac{\tilde{K}}{2} \int_0^\Lambda (\nabla \Pi <)^2 + \frac{\rho}{2} b^{-d} \int_0^\Lambda d\mathbf{x} |\Pi <|^2 \]

\[ - \frac{K}{2} \int_0^\Lambda d\mathbf{x} (\Pi_{<\alpha} \nabla \Pi_{<\alpha})^2 + O(K^{-2}), \]

where \( \tilde{K} = K(1 + I_d(b)/K) \) and \( \delta f_0^b, \delta f_1^b \) are constants. Specify the function \( I_d(b) \).

(viii) Applying the rescaling \( x' = x/b \) and renormalising the spins,

\[ S' = \frac{S}{\zeta}, \quad \Pi < = \zeta \Pi', \]

obtain an expression for the renormalised coupling constant \( K' \).

To determine \( \zeta \), it is necessary to evaluate the average of the renormalised spin \( \langle S \rangle_0 = \langle (\Pi_{<1} + \Pi_{>1}, \cdots (1 - \Pi^2_{<} - \Pi^2_{>} 1/2))_0 \rangle. \) Expanding, we find

\[ \langle S \rangle_0^\gamma = (\Pi_{<1}, \cdots, 1 - \Pi^2_{<}/2 - \langle \Pi^2_{>} /2 \rangle) \approx (1 - \langle \Pi^2_{>} /2 \rangle) (\Pi_{<1}, \cdots, 1 - \Pi^2_{<}/2) = \zeta S' \]

From this expression, show that \( \zeta = 1 - (n - 1)I_d(b)/2K. \)

(ix) Using the expression for \( K' \) and \( \zeta \), show that the differential recursion relation takes the form

\[ \frac{dK}{d\ell} = (d-2)K - (n-2)K_d \Lambda^{d-2}, \]

where \( b = e^\ell \). Setting the temperature \( T = K^{-1} \), obtain the recursion relation \( dT/d\ell \) and confirm that the fixed point is given by

\[ T^* = \frac{d-2}{(n-2)K_d \Lambda^{d-2}} = \frac{2\pi \epsilon}{(n-2)} + O(\epsilon^2), \]

where \( d = 2 + \epsilon \). Sketch the RG flow diagram for \( d > 2, d = 2 \) and \( d < 2 \), for various values of \( n \).
(x) Linearising the RG flow in the vicinity of the fixed point, obtain the thermal exponent \( y_t \) to leading order in \( \epsilon \). Using this result, obtain the correlation length exponent \( \nu = 1/y_t \).

(xi) Adding a term \(- \int d\mathbf{x} \; \mathbf{h} \cdot \mathbf{S}\) show that the magnetic exponent takes the form

\[
y_h = 2 + \frac{n-3}{2(n-2)} \epsilon + O(\epsilon^2).
\]

(xii) Using an exponent identity, obtain the critical exponent \( \gamma \). Setting \( d = 3 \) and \( n = 3 \), how does this estimate compare to the best estimate of 1.38.
4.6.2 Answers

1. Applying the Migdal-Kadanoff procedure, we choose to trace out spins at odd numbered sites. [Note that in principle, we might have defined average variables \( \sigma'_i = (\sigma_i + \sigma_{i+1})/2 \). However, with this choice, the new variables are free to take values 0 and ±1 and do not, therefore, represent Ising spins.] Thus, setting \( \sigma'_i \equiv \sigma_{2i-1} \) and \( s_i \equiv \sigma_{2i} \), we have

\[
Z = \sum_{\{\sigma'_i \pm 1\}} \sum_{\{s'_i \pm 1\}} e^{\sum_{i}^{N/2}[B(\sigma'_i, s_i) + B(s_i, \sigma'_{i+1})]}
\]

\[
= \sum_{\{\sigma'_i\}} \prod_i \left\{ \sum_{s_i = \pm 1} e^{B(\sigma'_i, s_i) + B(s_i, \sigma'_{i+1})} \right\} = \sum_{\{\sigma'_i\}} \sum_{i}^{N/2} B'(\sigma'_i, \sigma'_{i+1}),
\]

where \( B(\sigma_1, \sigma_2) = g + h(\sigma_1 + \sigma_2)/2 + J\sigma_1\sigma_2 \) and similarly for the primed variables.

(b) Setting \( x = e^j, y = e^h, z = e^g \), using the identity,

\[
e^{B'(\sigma'_i, \sigma'_2)} = \sum_{s = \pm 1} e^{2g + (\sigma'_i + \sigma'_2)h/2 + hs(\sigma'_i + \sigma'_2)},
\]

and taking different permutations of \( \sigma'_i \), we obtain three independent equations

\[
\begin{align*}
    z'y'x' &= z^2 y(x^2 y + x^{-2} y^{-1}) \\
    z'y'x'^{-1} &= z^2 y^{-1}(x^2 y^{-1} + x^{-2} y) \\
    z'x'x' &= z^2 (y + y^{-1}).
\end{align*}
\]

Solving these equations, we obtain

\[
\begin{align*}
    z'^4 &= z^8 (x^2 y + x^2 y^{-1})(x^2 y^{-1} + x^{-2} y)(y + y^{-1})^2 \\
    y'^2 &= y^2 (x^2 y + x^2 y^{-1})/(x^2 y^{-1} + x^{-2} y) \\
    x'^4 &= (x^2 y + x^2 y^{-1})(x^2 y^{-1} + x^{-2} y)/(y + y^{-1})^2.
\end{align*}
\]

from which we find the recursion relations advertised in the text.

(c) From the form of the equations above, it is evident that \( h = 0 \) is a fixed subspace. In this case

\[
e^{4J''} = \left( e^{2J} + e^{-2J} \right)^2 / 4,
\]

from which we obtain the formula presented in the text. Expanding this equation, we obtain the asymptotics

\[
J'' = \begin{cases} 
    J^2 + O(J^4) & J \to 0, \\
    J - \frac{\ln 2}{2} + O(e^{-4J}) & J \to \infty.
\end{cases}
\]

from which we can deduce that \( J = 0 \) is a stable fixed point and \( J = \infty \) is an unstable fixed point. This implies that the one-dimensional Ising model has no phase transition.
(d) Linearising in the region around \( J = \infty, h = 0 \) we find
\[
\begin{align*}
x'^4 &\approx x^4/4 \\
y'^2 &\approx y^4 \rightarrow \begin{cases} 
  e^{-J'} = \sqrt{2} e^{-J} \\
  h' = 2h.
\end{cases}
\end{align*}
\]
(e) Notice that, after decimation, \( \xi' = \xi/2 \). Applying the decimation procedure \( \ell \) times, we obtain the scaling function advertised.

---

2. (a) In the harmonic approximation
\[
\langle h(q_1)h(q_2) \rangle = \frac{1}{r_0q^2 + \kappa_0 q^4} (2\pi)^2 \delta^2(q_1 + q_2).
\]
The corresponding autocorrelation function is given by
\[
\langle [h(x) - h(0)]^2 \rangle = \int \frac{d^2q}{(2\pi)^2} \frac{|1 - e^{iq\cdot x}|^2}{r_0q^2 + \kappa_0 q^4} \sim \frac{1}{\pi r_0} \ln (q_c|x|),
\]
where \( q_c \sim \sqrt{r_0/\kappa_0} \) represents the short-distance or ultraviolet cut-off of the integral. The divergence of this function at long distance implies that there is no long-range positional order.

(b) The fluctuation in the normals is given by
\[
\langle [\nabla h(x) - \nabla h(0)]^2 \rangle = \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{r_0q^2 + \kappa_0 q^4} \sim \text{const}.
\]
Since this result remains finite (independent of \( x \)) in the thermodynamic limit we can deduce that, while there is no long-range positional order, there is long-range orientational order of the membrane in three dimensions.

(c) The general formula for the perturbative RG is just bookwork. Simply writing the identity,
\[
Z = Z_0 \int Dh_1 e^{-\beta H_0[h]} \langle e^{-\beta H_1} \rangle_{h_1}
\]
and performing a cumulant expansion, one obtains the advertised formula.

(d) Evaluation of the perturbative correction leads to contributions of two non-trivial kinds. The first brings about a renormalisation of the interfacial tension while the second renormalises the bending modulus.
\[
\langle \beta H_1 \rangle_{h_1} \rightarrow \frac{5\kappa_0}{4} \int_{0}^{\Lambda} d^2q_1 d^2q_2 \int_{0}^{\Lambda} d^2q_3 d^2q_4 \\
\times q_1^2 q_2^2 q_3 \cdot q_4 h_{q_1}^h h_{q_2}^h \langle h_{q_3}^h h_{q_4}^h \rangle (2\pi)^2 \delta^2(q_1 + q_2 + q_3 + q_4).
\]
CHAPTER 4. RENORMALISATION GROUP

Substituting the form of the propagator, and making use of the approximation

\[
\frac{5}{4} \int_{\Lambda_{c-\epsilon}}^{\Lambda} \frac{d^2 q}{(2\pi)^2} \frac{\kappa_0 q^2}{r_0 q^2 + \kappa_0 q^4} \approx \frac{5}{4} \frac{1}{2\pi} \frac{\Lambda^2 (1 - e^{-\epsilon})}{r_0 \Lambda^2 + \kappa_0 \Lambda^4} \approx \frac{5\ell}{8\pi},
\]

we obtain the renormalisation in the text. Extra credit for providing a diagrammatic representation of the same.

In conclusion, we see that there is a renormalisation of the bending modulus to lower values at longer length scales. The eventual unphysical sign change of the modulus is a signature of the breakdown of the perturbative expansion of the Hamiltonian. It provides an estimate for the length over which the membrane remains approximately rigid (i.e. the persistence length).

3. This question presents a useful opportunity to review the main elements we have encountered in our review of statistical field theory.

(a) In Fourier representation the bare component of the Hamiltonian takes the diagonal form

\[
\beta H_0 = \frac{1}{2} \int (dq) G^{-1}(q)|m(q)|^2 - hm(q = 0),
\]

where the anisotropic propagator is given by

\[
G^{-1}(q) = t + Kq_\parallel^2 + Lq_\perp^4.
\]

(b) Course-Graining procedure: Separate the field \(m\) into components which are slowly and rapidly varying in space.

\[
m(q) = \begin{cases} 
m_<(q) & 0 < |q_\parallel| < \Lambda/b \text{ and } 0 < |q_\perp| < \Lambda/c, \\
m_>(q) & \Lambda/b < |q_\parallel| < \Lambda \text{ or } \Lambda/c < |q_\perp| < \Lambda. 
\end{cases}
\]

In this parameterisation, the bare Hamiltonian is separable. As such, an integration over the fast degrees of freedom can be performed explicitly.

\[
Z = Z_> \int Dm_< \exp \left[ -\frac{1}{2} \frac{\Lambda/b}{0} (dq_\parallel) \int_{0}^{\Lambda/c} (dq_\perp) G^{-1}(q)|m_<(q)|^2 + hm_< (0) \right],
\]

where the constant \(Z_>\) is obtained from performing the functional integral over the fast degrees of freedom. Applying the rescaling \(q_\parallel' = b q_\parallel\), and \(q_\perp' = c q_\perp\), the cut-off in the domain of momentum integration is restored. Finally, applying the renormalisation \(m'(q) = m<(q)/z\) to the Fourier components of the field, we obtain

\[
Z = Z_> \int Dm'(q') e^{-(\beta H')|m'(q')|},
\]
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where the renormalised Hamiltonian takes the form

\[(\beta H)' = \frac{1}{2} \int (dq) b^{-1} c^{-(d-1)} z^2 (t + K q^2 + L c^{-4} q^4) |m'(q')|^2 - z h m'(0).\]

From the result, we obtain the renormalisation

\[
\begin{align*}
  t' &= tb^{-1} c^{-(d-1)} z, \\
  K' &= Kb^{-3} c^{-(d-1)} z^2, \\
  L' &= Lb^{-4} c^{-(d+3)} z^2, \\
  h' &= hz.
\end{align*}
\]

(c) Choosing parameters \(c = b^{1/2}\) and \(z = b^{(d+5)/4}\) ensures that \(K' = K\) and \(L' = L\) and implies the scaling exponents \(y_t = 2\), \(y_h = (d + 5)/4\).

(d) From this result we obtain the renormalisation of the free energy density

\[
f_{\text{sing}}(t, h) = b^{-1} c^{-(d-1)} f_{\text{sing}}(t', h') = b^{-(d+1)/2} f_{\text{sing}}(b^2 t, b^{(d+5)/4} h).
\]

Setting \(b^2 t = 1\), we can identify the exponents \(2 - \alpha = (d + 1)/4\) and \(\Delta = y_h/y_t = (d + 5)/8\). To determine the exponent \(\nu\) it is necessary to introduce correlation lengths

\[
\begin{align*}
  \xi_\parallel &= b^{-1} \xi_\parallel (b^2 t, b^{(d+5)/4} h) = t^{-1/2} g_{\xi_\parallel} (h/t^{(d+5)/8}), \\
  \xi_\perp &= c^{-1} \xi_\perp (b^2 t, b^{(d+5)/4} h) = t^{-1/4} g_{\xi_\perp} (h/t^{(d+5)/8}),
\end{align*}
\]

from which we obtain \(\nu_\parallel = 1/2\) and \(\nu_\perp = 1/4\).

(e) From the scaling function, the variation of the order parameter at the transition can be obtained \((m = \partial f/\partial h)\). Taking \(\chi = \partial m/\partial h\), we obtain the exponent identity \(\gamma = 2\Delta - 2 + \alpha\). In this case, we obtain \(\gamma = 1\).

(f) At \(h = 0\) the correlation function in the Gaussian approximation is given by

\[\langle m(q) m(q') \rangle_0 = G(q) (2\pi)^d \delta^d (q + q').\]

The scattering amplitude, or susceptibility, \(S(q) \equiv \chi(q) = G(q)\).

(g) In Fourier representation

\[U = u \int \prod_i (dq_i) m(q_i)m(q_2)m(q_3)m(q_4)(2\pi)^d \delta^d(q_1 + q_2 + q_3 + q_4).\]

(h) In the first order of perturbation theory in \(U\), the expectation value of an operator \(A\) is given by

\[\langle A \rangle = \langle A \rangle_0 + \langle A \rangle_0 \langle U \rangle_0 - \langle AU \rangle_0 + \cdots.\]
At first order, symmetry limits contributions to the connected correlation function to only one type from which we obtain

\[
\langle AU \rangle_0 - \langle A \rangle_0 \langle U \rangle_0 = u \int \prod_i (d\mathbf{q}_i)(2\pi)^d \delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) \times \frac{4 \times 3 (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) G(\mathbf{q}) G(\mathbf{q}') G(\mathbf{q}_3) G(\mathbf{q}_4)}{(m(\mathbf{q}) m(\mathbf{q}') m(\mathbf{q}_1) m(\mathbf{q}_2) m(\mathbf{q}_3) m(\mathbf{q}_4))_c} \\
= 12u (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}') G^2(\mathbf{q}) \int (d\mathbf{q}') G(\mathbf{q}')
\]

where the factor of 12 accounts for the degeneracy. Thus, to leading order, the susceptibility takes the form

\[
\chi(\mathbf{q}) = \langle |m(\mathbf{q})|^2 \rangle = G(\mathbf{q}) \left[ 1 - 12uG(\mathbf{q}) \int (d\mathbf{q}') G(\mathbf{q}') + \cdots \right] + O(u^2).
\]

Substituting for \(G(\mathbf{q})\), and setting \(\mathbf{q} = 0\), we obtain the uniform inverse susceptibility

\[
\chi^{-1} \simeq t + 12u \int (d\mathbf{q}') G(\mathbf{q}') + \cdots
\]

from which we obtain the shift of the transition temperature (i.e. by finding the value of \(t\) for which \(\chi^{-1} = 0\)).

(j) To determine the upper critical dimension we can apply the momentum shell RG. Under the renormalisation procedure, it is straightforward to show that the coupling constant of the interaction becomes

\[
u' = ub^{-3}c^{-3(d-1)}z^4.
\]

Using the results from above, this implies \(u' = b^{(7-d)/2}u\). Thus, for \(d > 7\) the interaction constant renormalises down. Thus the upper critical dimension for this model is 7.

4. \(\epsilon\)-Expansion of the Ginzburg-Landau Hamiltonian:

(i) Following on from the analysis presented in the lectures, applying the renormalisation group procedure, and treating the quartic interaction as a perturbation, the renormalised Hamiltonian takes the form advertised. In the first order of the cumulant expansion, an analysis of the renormalisation was performed in the lectures. Here we will focus on the second-order, or “two-loop” expansion \(\langle U^2 \rangle_c\). Terms generated at this order are represented diagrammatically in Fig. 4.7. Terms contributing at second order in the fields lead to an additional renormalisation of
4.6. PROBLEM SET II

Figure 4.7: Diagrammatic representation of terms generated by the second order expansion. The dotted line represents the quartic vertex, while dashed and solid lines represent fields \( m_\leq \) and \( m_\geq \) respectively. I.e. all dashed legs must be connected after averaging. (Those diagrams related by symmetry are not drawn.)

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

const.

Figure 4.8: Typical diagram contributing at quartic order.

Such terms are irrelevant in the \( 4 - \epsilon \) expansion. Similarly, terms generated at sixth order in the fields can be neglected as they represent irrelevant operators. Instead, we focus on those terms contributing to the renormalisation of the quartic interaction,

\[
U = u \int_0^\Lambda (dq_1) \cdots (dq_4)(2\pi)^d \delta^d(q_1 + q_2 + q_3 + q_4) \\
\times m(q_1) \cdot m(q_2) m(q_3) \cdot m(q_4).
\]

(ii) A representation of the terms generated at second order in the expansion is shown in Fig. 4.7.
(iii) Focusing on the generic term shown in Fig. 4.8, we obtain

\[
8n^2 \frac{u^2}{2!} \int_{\Lambda/b}^\Lambda (dk_1) \cdots (dk_4) (2\pi)^d \delta^d(q_1 + q_2 + k_1 + k_2) \times (2\pi)^d \delta^d(q_3 + q_4 + k_3 + k_4) \cdot m_<(q_1) \cdot m_<(q_2) \cdot m_<(q_3) \cdot m_<(q_4) \\
\times (2\pi)^d \delta^d(k_1 + k_3) G(k_1) (2\pi)^d \delta^d(k_2 + k_4) G(k_2) \delta_{\alpha\beta} \\
= 4nu^2 (2\pi)^d \delta^d(q_1 + q_2 + q_3 + q_4) m_<(q_1) \cdot m_<(q_2) \cdot m_<(q_3) \cdot m_<(q_4) \\
\times \int_{\Lambda/b}^\Lambda (dk_1) G(k_1) G(k_1 + q_1 + q_2),
\]

where we must apply the constraint \( \Lambda/b < |q_1 + q_2 + k_1| < \Lambda \). An expansion of the \( k_1 \) integral in the vicinity of small \( q_1 + q_2 \) yields

\[
\int_{\Lambda/b}^\Lambda (dk_1) \left( \frac{1}{t + Kk_1^2} \right) \left[ 1 - \frac{2Kk_1 \cdot (q_1 + q_2)}{t + Kk_1^2} + \cdots \right]
\]

However, since terms of higher order in \( q_1 + q_2 \) generate terms such as \( m^2(\nabla m)^2 \), they can be neglected as irrelevant perturbations.

Other terms which renormalise \( u \) can be treated within the same approximation and together generate the renormalised Hamiltonian

\[
\beta H[m_<] = \int_{0}^{\Lambda/b} (dq) \left( \frac{\tilde{t} + \tilde{K}q^2}{2} \right) |m_<(q)|^2 \\
+ \tilde{u} \int_{0}^{\Lambda/b} (dq_1) \cdots (dq_4) (2\pi)^d \delta^d(q_1 + q_2 + q_3 + q_4) \cdot m_<(q_1) \cdot m_<(q_2) \cdot m_<(q_3) \cdot m_<(q_4)
\]

where, denoting by \( A \) the (physically irrelevant) \( q \)-dependent second order contribution to the renormalisation of \( t \) and \( K \), the renormalised parameters take the form

\[
\tilde{t} = t + 4u(n + 2) \int_{\Lambda}^{\Lambda/b} (dk) G(k) - u^2 A(t, K, \Lambda, q = 0), \\
\tilde{K} = K - u^2 \nabla^2 q A|_{q=0}, \\
\tilde{u} = u - 4u^2 (n + 8) \int_{\Lambda/b}^\Lambda (dk) G(k)^2.
\]

(iv) Applying the rescaling \( q = q'/b \) and performing the renormalisation \( m_<= z m \), we obtain

\[
t' = b^{-d} z^2 \tilde{t}, \quad K' = b^{-(d+2)} z^2 \tilde{K}, \quad u' = b^{-3d} z^4 \tilde{u}.
\]

Choosing \( z \) such that \( K' = K \), we find

\[
z^2 = \frac{b^{d+2}}{1 - (u^2/K) \nabla^2 q A|_{q=0}} = b^{d+2} (1 + O(u^2)).
\]
However, since $u^2 \sim O(\epsilon^2)$, the correction can be neglected and $z = b^{1+d/2+O(\epsilon^2)}$. As a result, setting $b = e^\ell$, we obtain the differential recursion relations

$$\frac{dt}{d\ell} = 2t + 4u(n + 2)G(\Lambda)K_d\Lambda^d - u^2 A(q = 0),$$

$$\frac{du}{d\ell} = (4 - d)u - 4(n + 8)u^2 G(\Lambda)K_d\Lambda^d.$$  

(v) Setting $\epsilon = 4 - d$, we can identify two fixed points: the Gaussian fixed point $t^* = u^* = 0$ and the new $O(n)$ fixed point

$$u^* = \frac{(t^* + K\Lambda^2)^2}{4(n + 8)K_d\Lambda^d}(4 - d) = \frac{K^2}{4(n + 8)K_4}\epsilon + O(\epsilon^2),$$

$$t^* = -\frac{2(n + 2)K_d\Lambda^d}{(t^* + K\Lambda^2)^2}u^* = -\frac{n + 2}{2(n + 8)}K^2\epsilon + O(\epsilon^2).$$

Linearising in the vicinity of the fixed point, we find

$$\frac{d}{d\ell} \left( \frac{\delta t}{\delta u} \right) = \left( \begin{array}{cc} 2 - \epsilon(n + 2)/(n + 8) + O(\epsilon^2) & O(\epsilon^0) \\ O(\epsilon^0) & -\epsilon + O(\epsilon^2) \end{array} \right) \left( \frac{\delta t}{\delta u} \right),$$

from which we obtain the exponents

$$y_t = 2 - \left( \frac{n + 2}{n + 8} \right) \epsilon + O(\epsilon^2), \quad y_u = -\epsilon.$$  

So, as expected, $y_t$ and $y_u$ are independent of $K$, $\Lambda$, etc. and are therefore universal. A sketch of the RG flow is shown in the lecture notes.

(vi) To determine the exponent $y_h$ we must examine what happens to the magnetic field dependent component of the Hamiltonian under RG. Setting

$$\beta H = (\beta H)^* - \int dx h \cdot m,$$

and applying the RG procedure, we find that $h' = zh$. Substituting from above, we find $y_h = 1 + d/2 + O(\epsilon^2)$.

(vii) Altogether we find

$$(\beta H)' = (\beta H)^* + g_t b^{\mu} \theta_t + g_u b^{\mu} \theta_u + g_h b^{\mu} \theta_h,$$

where $\theta_\alpha$ denote the eigen-directions, and $g_\alpha$ the coupling constants. According to the $\epsilon$-expansion around the $O(n)$ fixed point the perturbation $\theta_u$ is irrelevant in $d < 4$. Under rescaling, the free energy density and correlation length take the scaling form shown in the text.
Applying the exponent identities, we thus obtain

\[ \xi \sim (\delta t)^{-\nu}, \quad \nu = \frac{1}{y_t} = \frac{1}{2} \left[ 1 + \frac{(n+2)}{2(n+8)} \epsilon + O(\epsilon^2) \right], \]

\[ C \sim (\delta t)^{-\alpha}, \quad \alpha = 2 - \frac{d}{y_t} = \frac{(4-n)}{2(n+8)} \epsilon + O(\epsilon^2), \]

\[ m \sim (\delta t)^{\beta}, \quad \beta = \frac{d - y_h}{y_t} = \frac{1}{2} - \frac{3}{2(n+8)} \epsilon + O(\epsilon^2), \]

\[ \chi \sim (\delta t)^{-\gamma}, \quad \gamma = \frac{2y_h - d}{y_t} = 1 + \frac{(n+2)}{2(n+8)} \epsilon + O(\epsilon^2). \]

5. Continuous Spin Systems Near Two-Dimensions:

(i) Substituting the parameterisation into the Hamiltonian

\[ \beta H = \frac{K}{2} \int dx \left[ (\nabla \Pi)^2 + (\nabla \sigma)^2 \right], \]

and expanding \( \sigma \) we obtain the expression shown.

(ii) Treating the expansion to quadratic order, an estimate of the typical fluctuations gives

\[ \langle |\Pi(0)|^2 \rangle = \int (dq) \langle |\Pi(q)|^2 \rangle = \int_{1/L}^{1/a} (dq) \frac{n-1}{Kq^2} \]

\[ = \frac{n-1}{K} \frac{S_d}{(2\pi)^d} \frac{a^{2-d} - L^{2-d}}{d-2} \rightarrow_{L \to \infty} \begin{cases} T & d > 2, \\ \infty & d \leq 2. \end{cases} \]

This confirms that the lower critical dimension is \( d_c = 2 \). Close to two-dimensions it is possible to study the critical behaviour in the vicinity of the transition using a perturbative renormalisation group.

(iii) After representing the prefactor the \( \delta \)-function as an exponent, and integrating over \( \sigma \), the expression given in text is evident.

(iv) Polyakov’s Perturbative Renormalisation Group: Treating all but the quadratic component of the Hamiltonian perturbatively, the scaling described in the text is again evident.

(v) Expanding the perturbation in terms of Fourier modes we obtain

\[ U = -\frac{K}{2} \int (dq_1)(dq_2)(dq_3)q_1 \cdot q_3 \Pi_\alpha(q_1) \Pi_\alpha(q_2) \Pi_\beta(q_3) \Pi_\beta(-q_1 - q_2 - q_3) \]

\[ -\frac{\rho}{2} \int (dq)|\Pi(q)|^2; \]
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Figure 4.9: Diagrammatic representation of vertices. Note that the slash indicates those legs carrying the momenta.

Figure 4.10: Diagrams contributing to the first order of perturbation theory.

and

$$\langle \Pi_\alpha(q_1)\Pi_\beta(q_2) \rangle_0 = \frac{1}{Kq_1^2} \delta_{\alpha\beta}(2\pi)^d \delta^d(q_1 + q_2).$$

The perturbation can be represented diagrammatically as in Fig. 4.9 where the slashes denote those fields that carry the momenta.

(vi) **Perturbative Renormalisation Group**: To apply the perturbative RG, we separate the fields into components,

$$\Pi(q) = \begin{cases} 
\Pi_<(q) & 0 < |q| < \Lambda/b, \\
\Pi_>(q) & \Lambda/b < |q| < \Lambda.
\end{cases}$$

The partition function can then be expressed as

$$Z = \int D\Pi_< D\Pi_> e^{-\beta H_0[\Pi_<] - \beta H_0[\Pi_> - U[\Pi_<, \Pi_>]}.$$

from which we obtain the expression shown in the text. (Here $\delta f^0_b$ denotes an irrelevant constant arising from the bare integration.) The renormalisation of the Hamiltonian can be obtained from the cumulant expansion

$$-\ln \langle e^{-U} \rangle = \langle U \rangle - \frac{1}{2} \langle U^2 \rangle^0 + O(U^3).$$

(vii) Treating the expansion to first order (fig. 4.10), only two terms give a non-trivial contribution. The others vanish or give a constant. As a result we obtain for each diagram

$$D_1 = -K \int_0^{\Lambda/b} (dq)q^2|\Pi_<|^2(q) \frac{I_d(b)}{K}.$$
where
\[ I_d(b) = \int_{\Lambda/b}^\Lambda (dk) \frac{1}{k^2} = K_d \int_{\Lambda/b}^\Lambda dkk^{d-3}. \]
Similarly, using the formula for the density of states, we obtain
\[ D_2 = -\frac{1}{2} \int_0^{\Lambda/b} (dq)|\Pi_< (q)|^2 \rho(1-b^{-d}). \]
Collecting the contributions from these diagrams, we obtain the expression given in the text.

(viii) Applying the rescaling and renormalising the spins, we obtain (up to a constant)
\[ -\beta H' = -\frac{K}{2} b^{d-2} \xi^2 \int_0^\Lambda dx' (\nabla \Pi')^2 + \frac{\rho}{2} \xi^2 \int_0^\Lambda dx' |\Pi'|^2 \\
- \frac{K}{2} b^{d-2} \xi^4 \int_0^\Lambda dx' (\Pi'_\alpha \nabla' \Pi'_{\alpha'})^2 + O(K^{-2}), \]
where
\[ K' = b^{d-2} \xi^2 K \left( 1 + \frac{I_d(b)}{K} \right) \]
From the definition of \( \xi \) presented in the question, we find
\[ \xi = 1 - \frac{1}{2} (n-1) \int_{\Lambda/b}^\Lambda (dk) \frac{1}{Kk^2} = 1 - \frac{n-1}{2K} I_d(b), \]
and
\[ K' = K b^{d-2} \left[ 1 - \frac{n-2}{K} I_d(b) \right]. \]
(ix) Setting \( b = e^{\delta \ell} = 1 + \delta \ell \), we obtain the differential recursion relations
\[ K + \delta \ell \frac{dK}{d\ell} = K \left( 1 + (d-2)\delta \ell \right) \left( 1 - \frac{n-2}{K} K_d \Lambda^{d-2} \delta \ell \right), \]
\[ \frac{dK}{d\ell} = (d-2)K - (n-2)K_d \Lambda^{d-2}. \]
Assigning the temperature \( T = K^{-1} \), we obtain
\[ \frac{dT}{d\ell} = -\frac{1}{K^2} \frac{dK}{d\ell} = -(d-2)T + (n-2)K_d \Lambda^{d-2} T^2 + O(T^3). \]
The corresponding RG flows are shown in fig. 4.11. Solving this equation for the fixed point \( dT/d\ell = 0 \), we obtain \( T^* \).

(x) Linearising the RG flow in the vicinity of the fixed point we obtain
\[ \frac{d\delta T}{d\ell} = (-\epsilon + 2(n-2)K_d \Lambda^{d-2} T^*) \delta T = \epsilon \delta T. \]
Integrating, we obtain $\delta T(\epsilon) = b^\prime \delta T(0)$. From this we obtain $y_\epsilon = \epsilon + O(\epsilon^2)$ from which we deduce that the thermal exponents are independent of $n$ at this order. In $d = 3$, with $n = 3$ this predicts

$$\nu = \frac{1}{y_\epsilon} = \frac{1}{\epsilon} = 1.$$ 

This compares with the best estimate from numerics of $\nu = 0.7$ (and an estimate of $\nu = 0.67$ from the $4 - \epsilon$ expansion).

(xi) Incorporating a magnetic field dependence into the Hamiltonian, we find that, under RG,

$$-b^d \xi \int dx h \cdot S'$$

Thus, using $h' = b^d \xi h = b^{y_\epsilon} h$, we find

$$b^{y_\epsilon} = b^d \left[1 - \frac{n - 1}{2K^*} I_d(b)\right],$$

This implies the differential recursion relation

$$\frac{dh}{d\ell} = h \left[d - \frac{n - 1}{2(n - 2)} \epsilon\right]$$

from which we obtain the magnetic exponent shown in the text.

(xii) Using the exponent identities, we obtain

$$\gamma = \nu (2y_\epsilon - d) = \frac{1}{\epsilon} \left(2 - \frac{\epsilon}{n - 2}\right).$$

This predicts a value $\gamma = 1$ for $d = 3$ and $n = 3$. Finally, using the relation $\gamma = \nu (2 - \eta)$ we obtain $\eta = \epsilon/(n - 2)$. 
Chapter 5
Topological Phase Transitions

Previously, we have seen that the breaking of a continuous symmetry is accompanied by the appearance of massless Goldstone modes. Fluctuations of the latter lead to the destruction of long-range order at any finite temperature in dimensions \( d \leq 2 \) — the Mermin-Wagner theorem. However, our perturbative analysis revealed only a power-law decay of spatial correlations in precisely two-dimensions — “quasi long-range order”. Such cases admit the existence of a new type of continuous phase transition driven by the proliferation of topological defects. The aim of this section is to discuss the phenomenology of this type of transition which lies outside the usual classification scheme.

In classifying states of condensed matter, we usually consider two extremes: on the one hand there are crystalline solids in which atoms form a perfectly periodic array that extends to infinity in all directions. Such phases are said to possess long-range order (LRO). On the other hand there are fluids or glasses, in which the atoms are completely disordered and the system is both orientationally and positionally isotropic — that is the materials look the same when viewed from any direction. However, an intermediate state of matter is possible. In such a state the atoms are distributed at random, as in a fluid or glass, but the system is orientationally anisotropic on a macroscopic scale, as in a crystalline solid. This means that some properties of the fluid are different in different directions. Order of this sort is known as bond-orientational order.

This type of quasi long-range order is manifest in properties of superfluid and superconducting films (i.e. two-dimensions) and in the crystallisation properties of fluid membranes. As we have seen, according to the Mermin-Wagner theorem, fluctuations of a two-component or complex order parameter destroy LRO at all finite temperatures. However, at temperatures below \( T_c \), quasi-LRO is maintained. The nature of this topological phase transition was first resolved by Berezinskii (Sov. Phys. JETP 32, 493, (1971)) and later generalised to encompass a whole class of systems by Kosterlitz and Thouless\(^1\) (J. Phys. C 5, L124 (1972); 6, 1181 (1973)). These include the melting of a two-dimensional crystal, with dislocations taking the place of vortices (Halperin and Nelson, Phys. Rev. Lett. 41, 121 (1978)).

\(^1\)
In this chapter, we will exploit a magnetic analogy to explore this unconventional type of phase transition which is driven by the condensation of topological defects known as vortices. Note that this type of phase transition is qualitatively quite different from those we have met previously.

5.1 Continuous Spins Near Two-Dimensions

Suppose unit $n$-component spins $S_i = (s_{i1}, s_{i2}, \cdots s_{in})$ ($S_i^2 = 1$) which occupy the sites $i$ of a lattice interact ferromagnetically with their neighbours.

$$-\beta H = K \sum_{\langle ij \rangle} S_i \cdot S_j = -\frac{K}{2} \sum_{\langle ij \rangle} [(S_i - S_j)^2 - 2].$$

At zero temperature the ground state configuration is ferromagnetic with all spins aligned along some direction (say $S_i = \hat{e}_n \equiv (0, 0, \cdots, 1)$). At low temperatures statistical fluctuations involve only low energy long wavelength modes which can be treated within a continuum approximation. Accordingly the Hamiltonian can be replaced by

$$-\beta H[S] = -\beta E_0 - \frac{K}{2} \int dx \left( \nabla S \right)^2,$$

where the discrete lattice index $i$ has been replaced by a continuous vector $x \in \mathbb{R}^d$. The corresponding partition function is given by the so-called non-linear $\sigma$-model,

$$Z = \int D\mathbf{S}(x) \delta(S^2 - 1) e^{-\beta H[S]}.$$

Here we have used the notation $\delta(S^2 - 1)$ to represent a “functional $\delta$-function” — i.e. at all spatial coordinates, $S(x)^2 = 1$.

Fluctuations transverse to the ground state spin orientation $\hat{e}_n$ are described by $n - 1$ Goldstone modes. Adopting the parameterisation $S(x) = (\pi_1(x), \cdots \pi_{n-1}(x), (1 - \pi^2)^{1/2}) \equiv (\pi, (1 - \pi^2)^{1/2})$, and expanding to quadratic order in $\pi$ we obtain the following

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John Michael Kosterlitz and David James Thouless: co-recipients of the 2000 Lars Onsager Prize “for the introduction of the theory of topological phase transitions, as well as their subsequent quantitative predictions by means of early and ingenious applications of the renormalization group, and advancing the understanding of electron localization and the behavior of spin glasses”.

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expression for the average transverse fluctuation (cf. section 2.4)

\[ \langle |\pi(x)|^2 \rangle = \int \frac{d^d q}{(2\pi)^d} \langle |\pi(q)|^2 \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{n - 1}{K q^2} = \frac{n - 1}{K} \frac{S_d}{(2\pi)^d} a^{2-d} - L^{2-d} \quad L \to \infty \quad (n - 1) K_d \begin{cases} a^{2-d} \propto T & d > 2, \\ L^{2-d} \to \infty & d \leq 2. \end{cases} \]

This result suggests that in more than two dimensions we can always find a temperature where the magnitude of the fluctuations is small while in dimensions of two or less fluctuations always destroy long-range order. This is in accord with the Mermin-Wagner theorem discussed in section 2.4 which predicted the absence of long-range order in \( d \leq 2 \). Arguing that this result implies a critical temperature \( T_c \sim O(d - 2) \), Polyakov (Phys. Lett. 59B, 79 (1975)) developed a perturbative RG expansion close to two-dimensions.\(^2\)

The excitation of Goldstone modes therefore rules out spontaneous order in two-dimensional models with a continuous symmetry. An RG analysis of the non-linear \( \sigma \)-model indeed confirms that the transition temperature of \( n \)-component spins vanishes as \( T^* = 2\pi\epsilon/(n - 2) \) for \( \epsilon = (d - 2) \to 0 \). However, the RG also indicates that the behaviour for \( n = 2 \) is in some sense marginal.

The first indication of unusual behaviour in the two-dimensional XY-model \( (n = 2) \) appeared in an analysis of the high temperature series expansion by Stanley and Kaplan (1971). The series appeared to indicate a divergence of susceptibility at a finite temperature, seemingly in contradiction with the absence of symmetry breaking. It was this contradiction that led Wigner to explore the possibility of a phase transition without symmetry breaking. It is the study of this novel and important type of phase transition to which we now turn.

5.2 Topological Defects in the XY-Model

We begin our analysis with a study of the asymptotic behaviour of the partition function at high and low temperatures using a series expansion.

5.2.1 High Temperature Series

In two-dimensions it is convenient to parameterise the spins by their angle with respect to the direction of one of the ground state configurations \( S = (\cos \theta, \sin \theta) \). The spin Hamiltonian can then be presented in the form

\[ -\beta H = K \sum_{(ij)} \cos(\theta_i - \theta_j). \]

\(^2\)Polyakov’s work provided one of the milestones in the study of critical phenomena. The \( \epsilon = d - 2 \) expansion employed in the perturbative RG approach set the framework for numerous subsequent investigations. A description of the RG calculation can be found in Chaikin and Lubensky and is assigned as a question in the problem set 2.
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At high temperatures the partition function can be expanded in powers of $K$

$$Z = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} e^{-\beta H} = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \prod_{(ij)} \left[ 1 + K \cos(\theta_i - \theta_j) + O(K^2) \right].$$

Each term in the product can be represented by a “bond” that connects neighbouring sites $i$ and $j$. To the lowest order in $K$, each bond on the lattice contributes either a factor of one, or $K \cos(\theta_i - \theta_j)$. But, since $\int_0^{2\pi} (d\theta_1/2\pi) \cos(\theta_1 - \theta_2) = 0$ any graph with a single bond emanating from a site vanishes. On the other hand, a site at which two-bonds meet yields a factor $\int_0^{2\pi} (d\theta_2/2\pi) \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) = \cos(\theta_1 - \theta_3)/2$. The first non-vanishing contributions to the partition function arise from closed loop configurations that encircle one plaquette.

The high temperature expansion can be used to estimate the spin-spin correlation function $\langle S_0 \cdot S_x \rangle = \langle \cos(\theta_x - \theta_0) \rangle$. To leading order, only those graphs which join sites 0 and $r$ will survive and give a contribution

$$\langle S_0 \cdot S_x \rangle \sim \left( K \right)^{|r|} \sim \exp \left[ -|x|/\xi \right],$$

where $\xi^{-1} = \ln(2/K)$. This result implies an exponential decay of the spin-spin correlation function in the disordered phase.

5.2.2 Low Temperature Series

At low temperature the cost of small fluctuations around the ground state is obtained within a quadratic expansion which yields the Hamiltonian corresponding to Eq. (2.9)

$$-\beta H = \frac{K}{2} \int d\mathbf{x} (\nabla \theta)^2,$$

in the continuum limit. According to the standard rules of Gaussian integration

$$\langle S(0) \cdot S(\mathbf{x}) \rangle = \Re \left\langle e^{i(\theta(0) - \theta(\mathbf{x}))} \right\rangle = \Re \left[ e^{-\langle (\theta(0) - \theta(\mathbf{x}))^2 \rangle / 2} \right].$$

In section 2.4 we saw that in two-dimensions Gaussian fluctuations grow logarithmically $\langle (\theta(0) - \theta(\mathbf{x}))^2 \rangle / 2 = \ln(|\mathbf{x}|/a)/2\pi K$, where $a$ denotes a short distance cut-off (i.e. lattice spacing). Therefore, at low temperatures the spin-spin correlation function decays algebraically as opposed to exponential.

$$\langle S(0) \cdot S(\mathbf{x}) \rangle \sim \left( \frac{a}{|\mathbf{x}|} \right)^{1/2\pi K}.$$
The distinction between the nature of the asymptotic decays allows for the possibility of a finite temperature phase transition. However, the arguments so far put forward are not specific to the XY-model. Any continuous spin model will exhibit exponential decay of correlations at high temperature, and a power law decay in a low temperature Gaussian approximation. Strictly speaking, to show that Gaussian behaviour persists to low temperatures we must prove that it is not modified by the additional terms in the gradient expansion. Quartic terms, such as \( \int d^d x (\nabla \theta)^4 \), generate interactions between the Goldstone modes. The relevance of these interactions can be discussed within the framework of perturbative RG (see Problem Set 2). The conclusion is that the zero temperature fixed point in \( d = 2 \) is unstable for all \( n > 2 \) but apparently stable for \( n = 2 \). (There is only one branch of Goldstone modes for \( n = 2 \). It is the interactions between different branches of these modes for \( n > 2 \) that leads to instability towards high temperature behaviour.) The low temperature phase of the XY-model is said to possess quasi-long range order, as opposed to true long range order that accompanies finite magnetisation.

What is the mechanism for the disordering of the quasi-long range ordered phase? Since the RG suggests that higher order terms in the gradient expansion are not relevant it is necessary to search for other relevant operators.

### 5.3 Vortices

The gradient expansion describes the energy cost of small deformations around the ground state and applies to configurations that can be continuously deformed to the uniformly ordered state. Berezinskii, and later Kosterlitz and Thouless, suggested that the disordering is caused by topological defects that cannot be obtained from such deformations.

Since the angle describing the orientation of a spin is defined up to an integer multiple of \( 2\pi \), it is possible to construct spin configurations in which the traversal of a closed path will see the angle rotate by \( 2\pi n \). The integer \( n \) is the topological charge enclosed by the path. The discrete nature of the charge makes it impossible to find a continuous deformation which returns the state to the uniformly ordered configuration in which the charge is zero. (More generally, topological defects arise in any model with a compact group describing the order parameter — e.g. a ‘skyrmion in an \( O(3) \)’ or three-component spin Heisenberg Ferromagnet, or a dislocation in a crystal.)

The elementary defect, or vortex, has a unit charge. In completing a circle centred on the defect the orientation of the spin changes by \( \pm 2\pi \) (see Fig. 5.1). If the radius \( r \) of the circle is sufficiently large, the variations in angle will be small and the lattice structure can be ignored. By symmetry \( \nabla \theta \) has uniform magnitude and points along the azimuthal direction. The magnitude of the distortion is obtained by integrating around a path that encloses the defect,

\[
\oint \nabla \theta \cdot d\ell = 2\pi n \implies \nabla \theta = \frac{n}{r} \hat{e}_r \times \hat{e}_z,
\]
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Figure 5.1: Spin configurations of the two-dimensional XY-model showing vortices of charge ±1.

where \( \hat{e}_r \) and \( \hat{e}_z \) are unit vectors respectively in the plane and perpendicular to it. This (continuum) approximation fails close to the centre (core) of the vortex, where the lattice structure is important.

The energy cost from a single vortex of charge \( n \) has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius \( a \) to distinguish the two, i.e.

\[
\beta E_n = \beta E_n^0(a) + \frac{K}{2} \int_a \left( \nabla \theta \right)^2 = \beta E_n^0(a) + \pi K n^2 \ln \left( \frac{L}{a} \right).
\]

The dominant part of the energy comes from the region outside the core and diverges logarithmically with the system size \( L \).\(^3\) The large energy cost associated with the defects prevents their spontaneous formation close to zero temperature. The partition function for a configuration with a single vortex of charge \( n \) is

\[
Z_1(n) \approx \left( \frac{L}{a} \right)^2 \exp \left[ -\beta E_n^0(a) - \pi K n^2 \ln \left( \frac{L}{a} \right) \right],
\]  

(5.1)

where the factor of \( (L/a)^2 \) results from the configurational entropy of possible vortex locations in an area of size \( L^2 \). The entropy and energy of a vortex both grow as \( \ln L \), and the free energy is dominated by one or the other. At low temperatures, large \( K \), energy dominates and \( Z_1 \), a measure of the weight of configurations with a single vortex, vanishes. At high enough temperatures, \( K < K_n = 2/(\pi n^2) \), the entropy contribution is large enough to favour spontaneous formation of vortices. On increasing temperature, the first vortices that appear correspond to \( n = \pm 1 \) at \( K_c = 2/\pi \). Beyond this point many vortices appear and Eq. (5.1) is no longer applicable.

\(^3\)Notice that if the spin degrees of freedom have three components or more the energy cost of a defect is only finite.
In fact this estimate of $K_c$ represents only a lower bound for the stability of the system towards the condensation of topological defects. This is because pairs (dipoles) of defects may appear at larger couplings. Consider a pair of charges $\pm 1$ separated by a distance $d$. Distortions far from the core $|r| \gg d$ can be obtained by superposing those of the individual vortices (see fig. 5.2)

\[
\nabla \theta = \nabla \theta_+ + \nabla \theta_- \approx 2d \cdot \nabla \left( \frac{\hat{e}_r \times \hat{e}_z}{|r|} \right),
\]

which decays as $d/|r|^2$. Integrating this distortion leads to a finite energy, and hence dipoles appear with the appropriate Boltzmann weight at any temperature. The low temperature phase should therefore be visualised as a gas of tightly bound dipoles (see fig. 5.3), their density and size increasing with temperature. The high temperature phase constitutes a plasma of unbound vortices. A theory of the Berezinskii-Kosterlitz-Thouless transition based on an RG description can be found in Chaikin and Lubensky.