Creating novel quantum phases by artificial magnetic fields

Gunnar Möller
Cavendish Laboratory, University of Cambridge

Theory of Condensed Matter Group

TCM

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Trinity Hall Cambridge
The Leverhulme Trust
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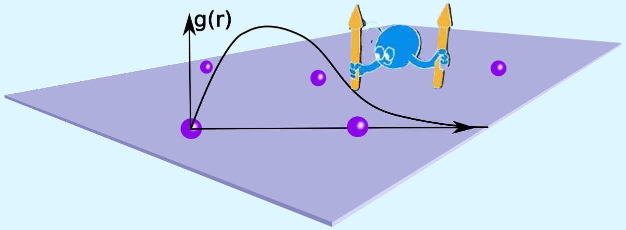
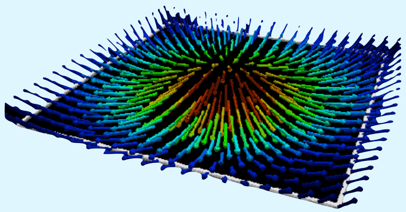
Topics for Conversation

Topological Phases of Matter

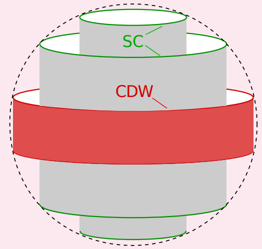


e.g., paired quantum Hall states



topological quasiparticle excitations

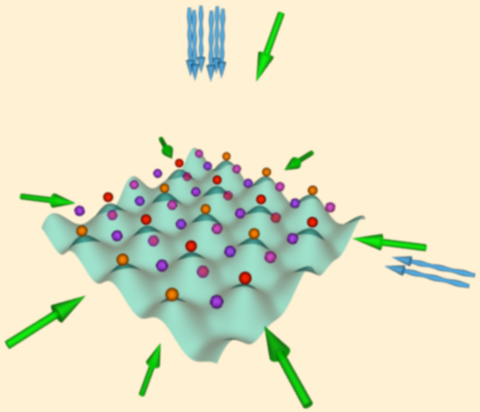
Novel Phases



e.g.,'high field' supersolid phase in rotating Fermi gases

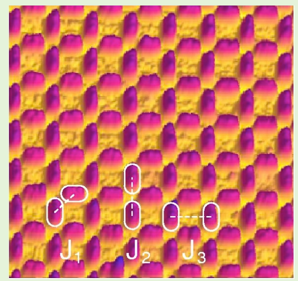
Today: novel systems for stabilising quantum Hall physics

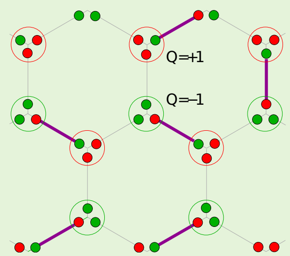
Cold Atomic Gases



many-body physics in artificial gauge fields

Nano-Magnetism





artificial frustrated compounds

Outline

- A brief introduction to the quantum Hall effect
- Topological aspects of Landau levels and how to simulate magnetic fields

- Adiabatic connection of fractionalized phases in topologically non-trivial Chern bands and FQH states

- Novel types of quantum liquids bands Chern-# $C > 1$
  - composite fermion approach for bosons in flux lattices
  - flat band projection in the Hofstadter butterfly

Quantum Hall Effect: Phenomenology

‣ a macroscopic quantum phenomenon observed in magnetoresistance measurements

Where?
‣ in semiconductor hetero-structures with clean two-dimensional electron gases

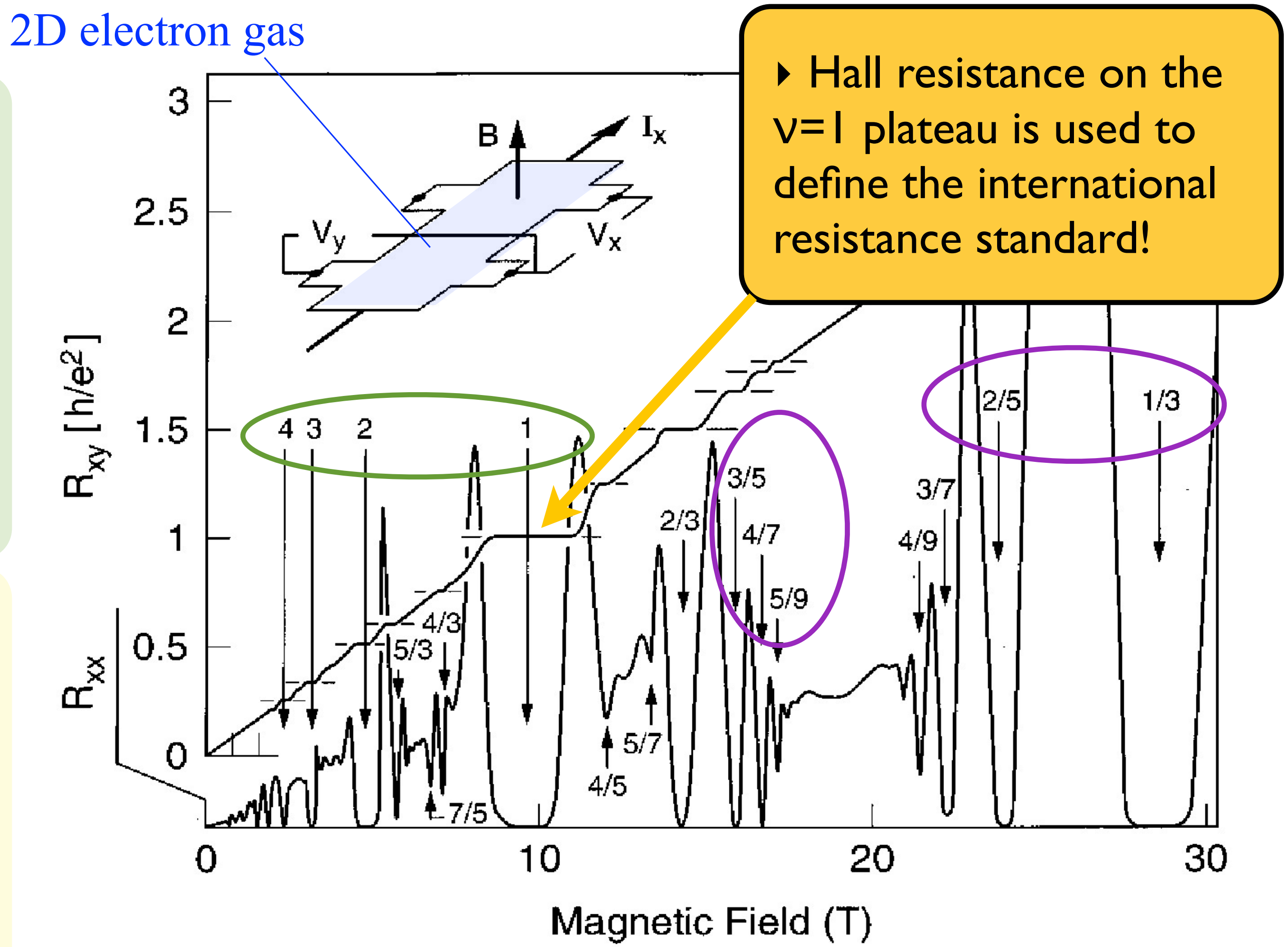
‣ at low temperatures (~0.1K) and in strong magnetic fields

$$k_B T \ll \hbar\omega_c = \hbar eB/m_e$$

What?
‣ plateaus in Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

‣ simultaneously: (near) zero longitudinal resistance



‣ Quantum number ν, observed to take integer or simple fractional values

Integer Quantum Hall Effect

- single-particle eigenstates (=bands) in a homogeneous magnetic field: degenerate Landau levels with spacing $\hbar\omega_c$

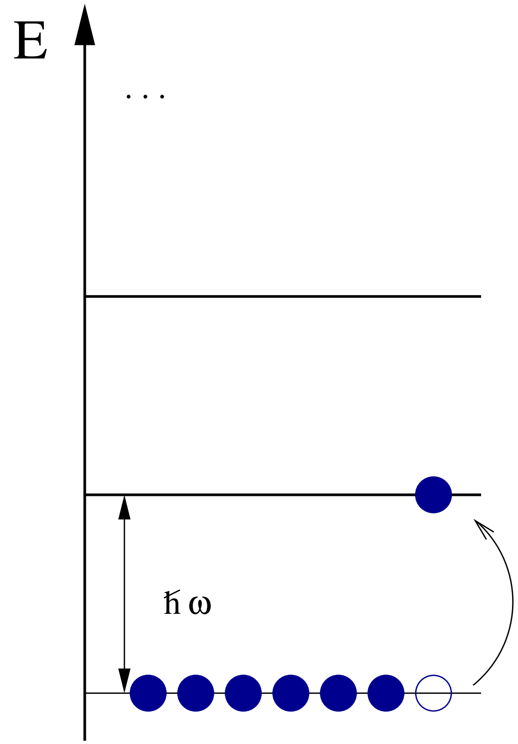
$$\mathcal{H} = \frac{(\vec{p} + e\vec{A})^2}{2m}$$

$$\vec{A} = Bx\vec{e}_y$$

$$\omega_c = eB/m_e \text{ cyclotron frequency}$$

- degeneracy per surface area: $d_{LL} = eB/h$

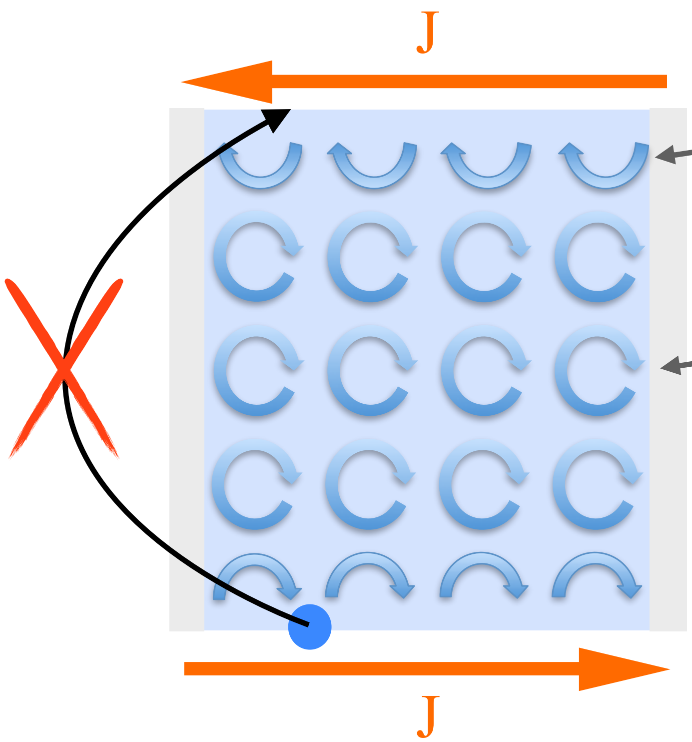
- fill a number of bands = integer filling factor $\nu = n/d_{LL}$



> $\Rightarrow$ large gap Δ for single particle excitations:
>
> naively, we should have a band insulator

- There must be something special about Landau-levels!

Semiclassical picture: skipping orbits



at edge of sample, 'skipping orbits' contribute a uni-directional current

cyclotron motion produces no net current in bulk of sample

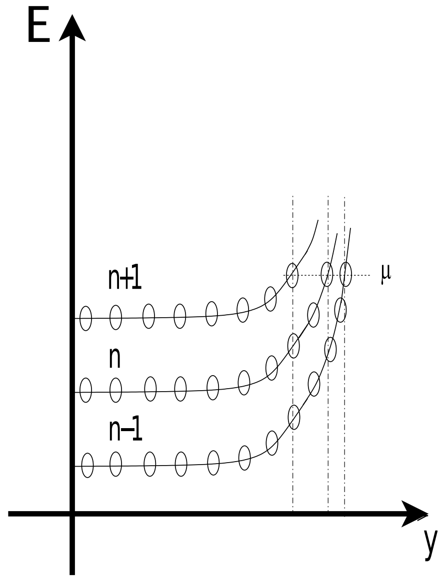
picture for quantum transport: absence of backscattering

⇒ dissipationless current

⇒ no voltage drop along lead!

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

several LLs filled:

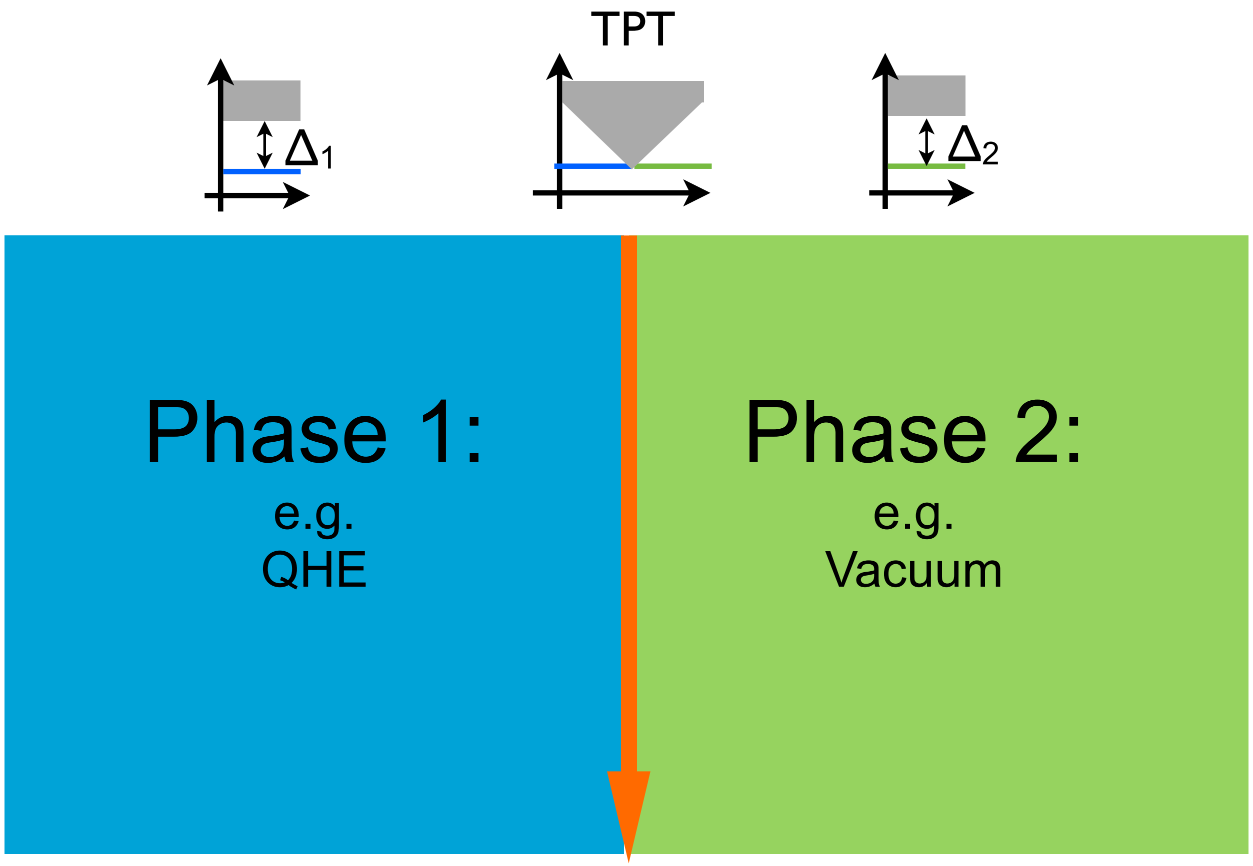


low-energy or 'gapless' excitations present near boundary

Edge States & Topological Order

Quantum Hall plateaus have a property called *topological order*

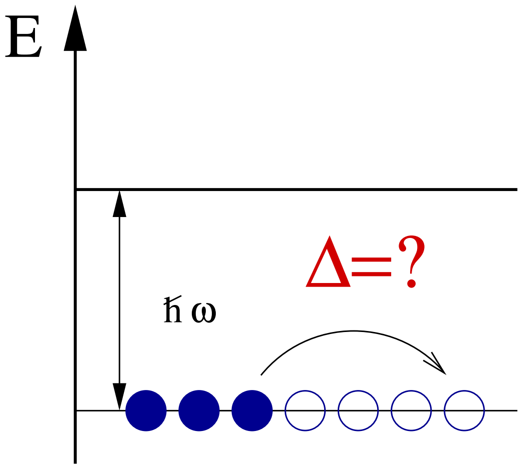
Edge states must occur where the topological order changes in space (gap must close)

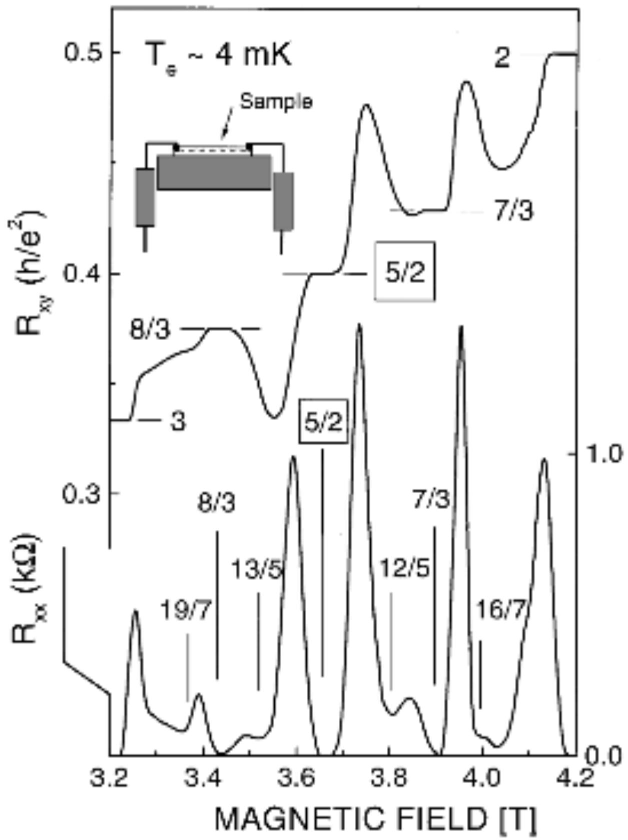


Can formulate *topological invariants* to characterise topological order: will see example, later

Fractional Quantum Hall Effect (FQHE)

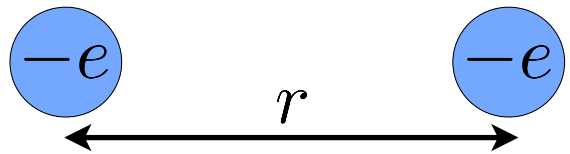
- plateaus seen also for non-integer ν
- not filled bands - but similar phenomenology as integer filling:





- nature of interactions determines how the system behaves:

$$\mathcal{H} = \underbrace{\sum_i \cancel{\frac{1}{2m}(\vec{p}_i - e\vec{A}_i)^2}}_{\text{within Landau-level: Kinetic Energy=constant}} + \underbrace{\sum_{i<j} V(|\vec{r}_i - \vec{r}_j|)}_{\text{interactions determine quantum state}}$$



$$V(\mathbf{r}) = \frac{e^2}{4\pi\epsilon_0|r|}$$

⇒ FQHE is an inherently many-body phenomenon

- each Hall plateau represents a kind of topological order

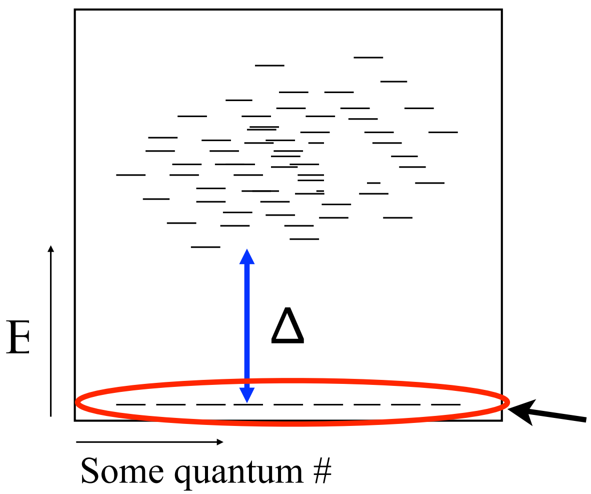
Why is the fractional quantum Hall effect important?

source of very unusual physics, for example:

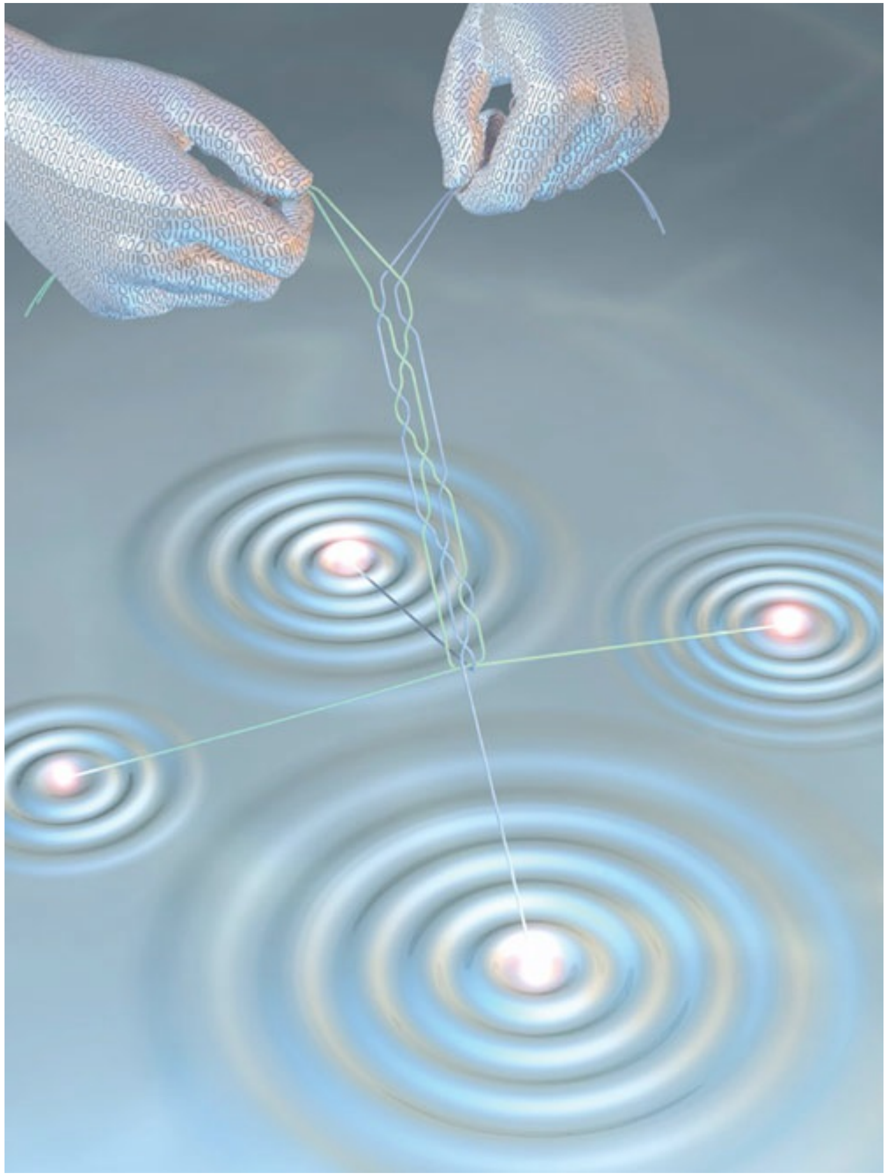
- quasi-particles with fractional electronic charge

$$e.g.,\ q = e/3$$

- manipulations of quasiparticles could provide the basis for a quantum computer that is protected from errors!

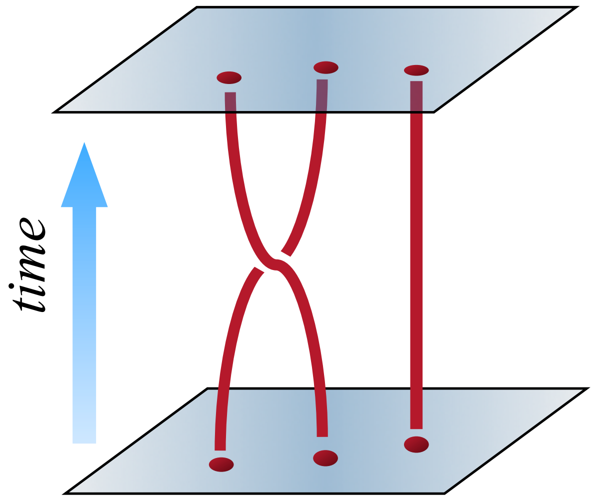


topological degeneracy of low-lying states



- quantum operations by braiding quasiparticles

Fractional statistics - Anyons and Non-Abelions



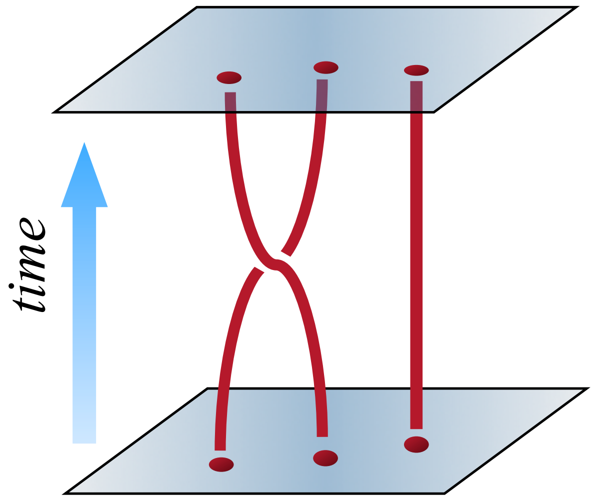
$$|\psi_f\rangle = e^{i\phi}|\psi_i\rangle$$

almost 'any' phase

$$|\psi_i\rangle$$

Abelian Anyons

Fractional statistics - Anyons and Non-Abelions



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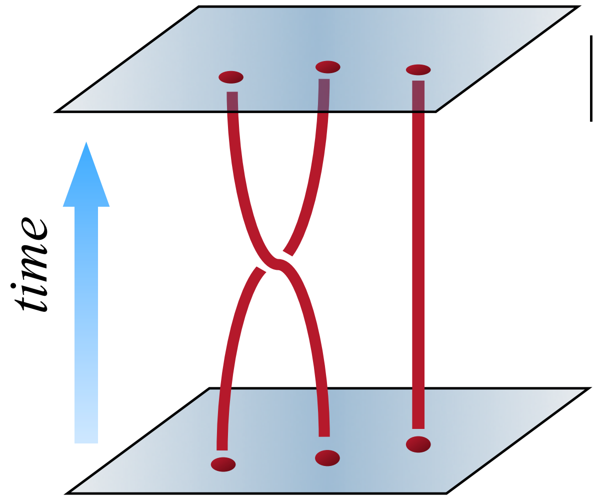


$$|\psi_f\rangle = \begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

Non-Abelian Anyons

non-Abelian representation of the braid group

$$|\psi_i\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

orthogonal degenerate states

Quantum Hall effect without magnetic fields

The fractional quantum Hall effect is observed under extreme conditions

- strong magnetic fields of several Tesla
- very low temperatures
- clean / high mobility semiconductor samples

Opportunities for creating novel types of quantum Hall systems

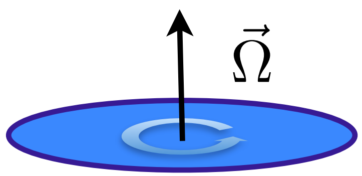
1. Cold Atomic Gases
   - both bosons and fermions
   - highly tuneable: density, interactions, tunnelling strengths, (effective) mass, ...
   - different types of experimental probes: local density, velocity distribution, correlations

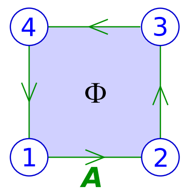
2. Novel classes of materials
   - strained graphene
   - materials with strong spin orbit coupling, such as topological insulators

Strategies for simulating magnetic fields

- Simulate a physical effect that a magnetic field B exerts particle of charge q

Signature		Simulated by
Lorentz Force	$F_L = q\,\vec{v} \times \vec{B}$	Coriolis Force in Rotating System
Aharonov-Bohm Effect	$\Psi \propto \exp\left\{ i\frac{q}{\hbar}\int \vec{A}\cdot d\vec{\ell} \right\}$	Complex Hopping Amplitudes A in Optical Lattices



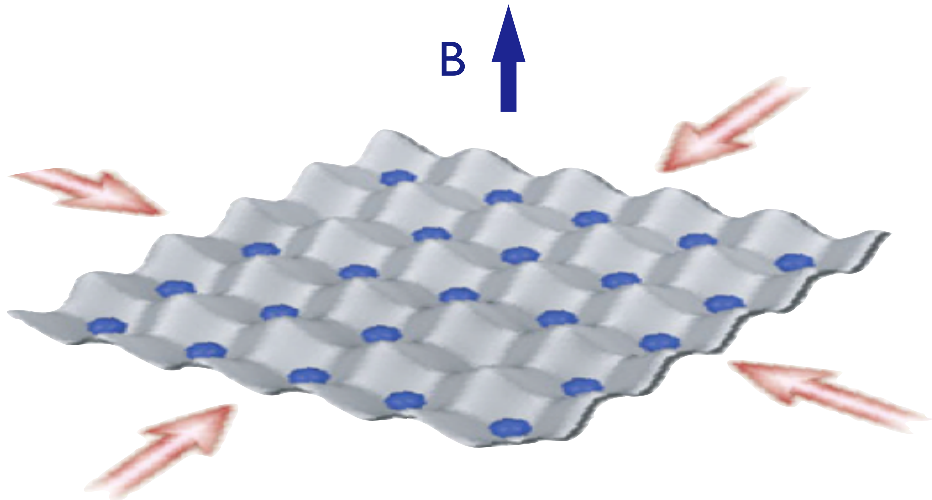


$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$

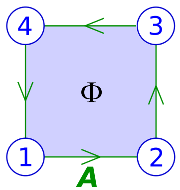
Motivation: Physical realizations in Cold Atoms I

experimental realisation: cold atoms with optical lattice + Raman lasers

$\Rightarrow$ possibility to simulate Aharonov-Bohm effect of magnetic field by imprinting phases for hopping via Raman transitions



$$\sum_{\square} A_{\alpha\beta} = 2\pi n_{\mathrm{V}}$$



$\Rightarrow$ Bose-Hubbard with a magnetic field ($\rightarrow$ Lorentz force)

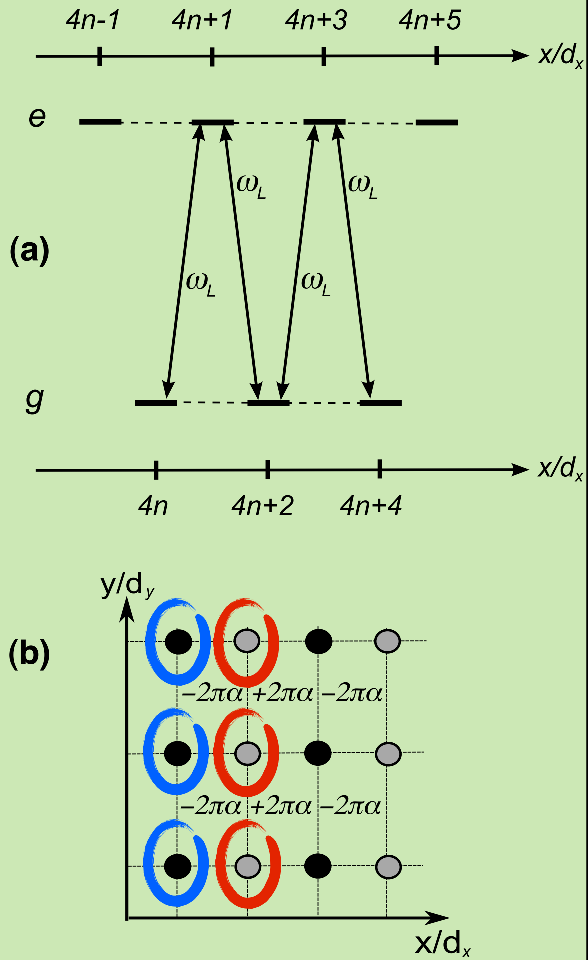
$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha(\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

particle density n vortex/flux density n_{V} interaction U/J

J. Dalibard, et al. Rev. Mod. Phys. 83, 1523 (2011)

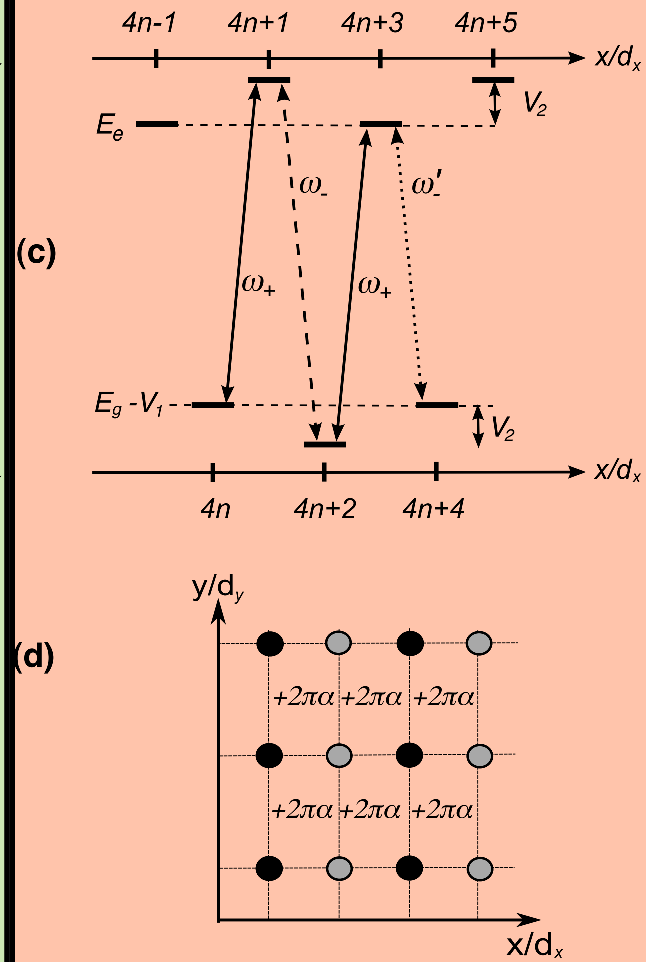
Realization I: Optical Lattices with Complex Hopping

Staggered Flux



Realized by group of I. Bloch:
Phys. Rev. Lett. **107**, 255301 (2011).

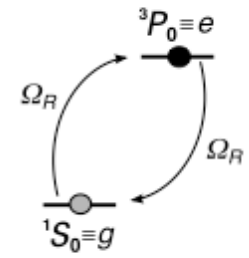
'Rectified' Flux

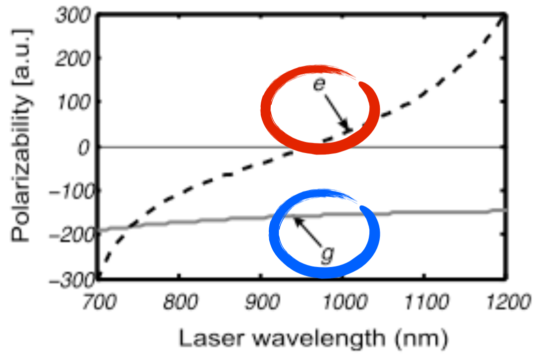


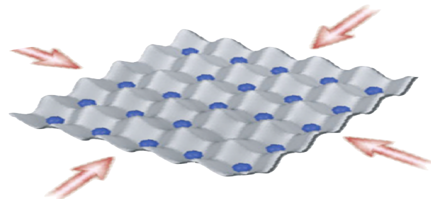
Can tune flux density!

Experimental realization outstanding

'anti-magic' optical lattice for Yb





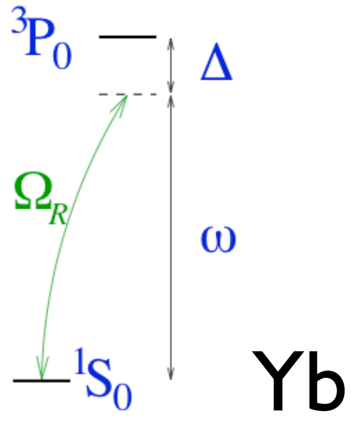


Gerbier & Dalibard, NJP 2010

Realization II: Berry phases of optically dressed states

J. Dalibard, F. Gerbier, G. Juzeliunas, P. Öhberg, RMP 2011

spacially varying optical coupling of *N* internal states (consider N=2)



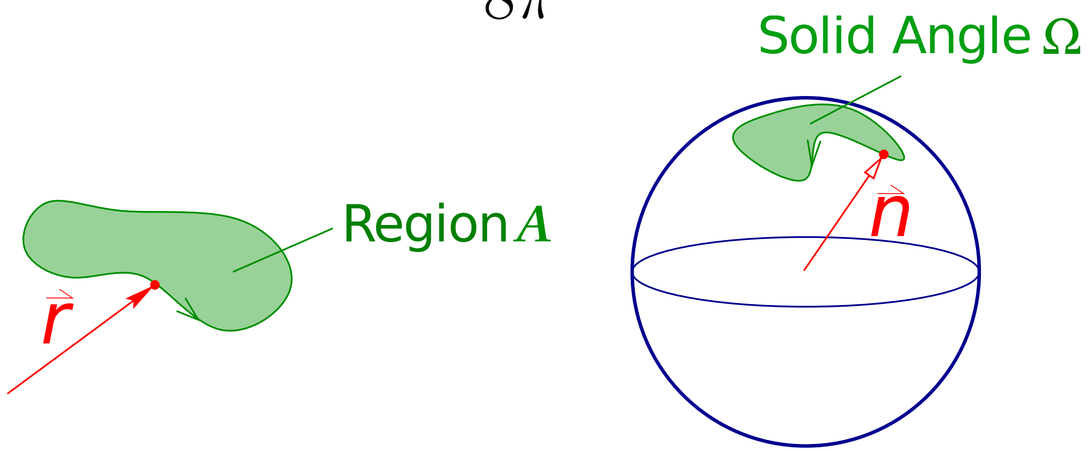
$$\mathcal{H} = \frac{\mathbf{p}^2}{2m}\hat{1} + \frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega_R(\mathbf{r}) \\ \Omega_R(\mathbf{r}) & \Delta \end{pmatrix}$$

local spectrum $E_n(\boldsymbol{r})$ and dressed states: $|\Psi_{\mathbf{r}}\rangle$

adiabatic motion of atoms *in space* in the optical potential generates a Berry phase

for *N*=2, consider Bloch sphere of unit vector $\vec{n} = \langle\Psi_{\mathbf{r}}|\hat{\vec{\sigma}}|\Psi_{\mathbf{r}}\rangle$

Berry flux generated: $n_\phi = \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{\mu\nu}n_i\partial_\mu n_j\partial_\nu n_k$



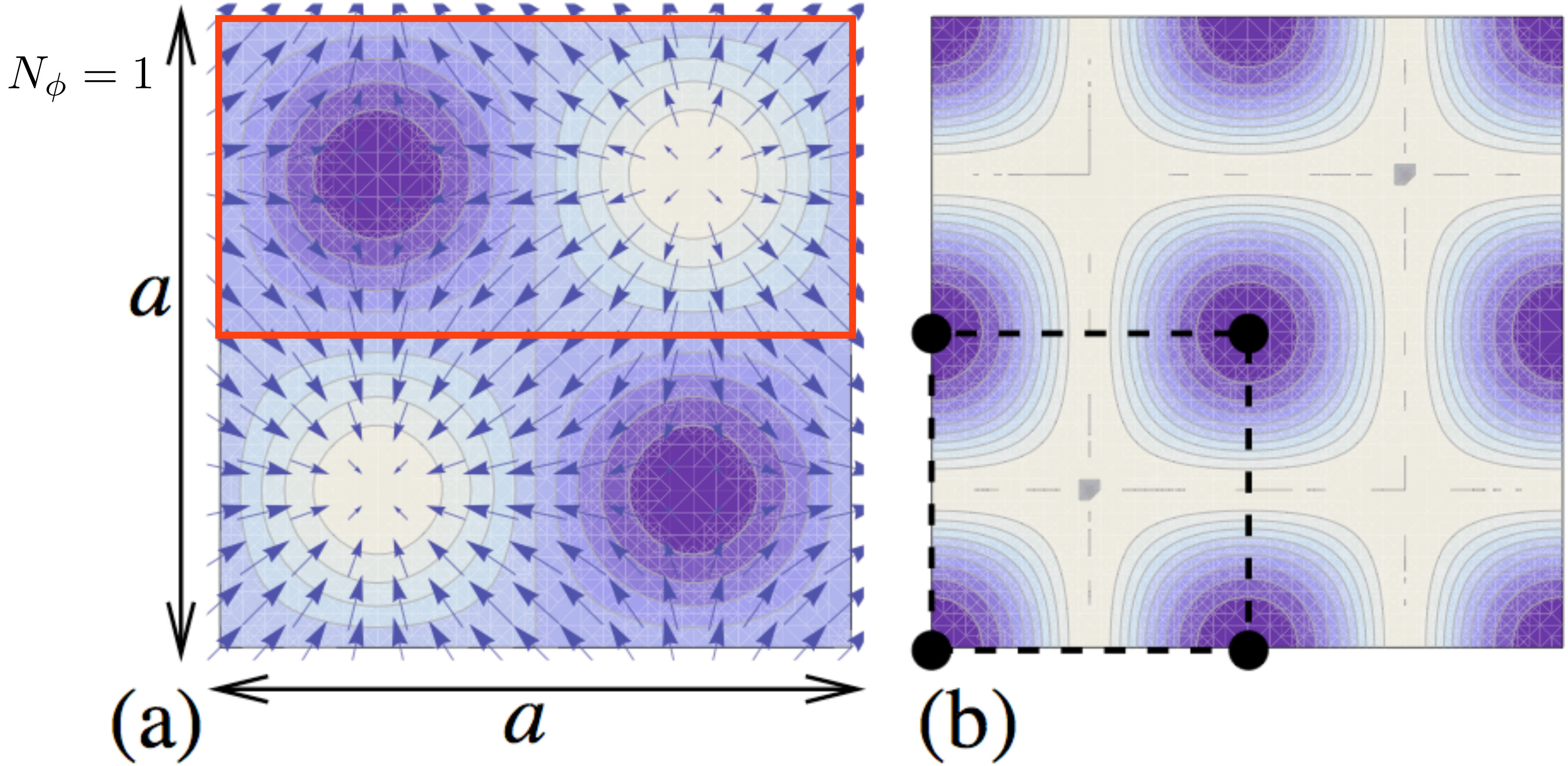
Total flux quanta
= # times Bloch
vector wraps sphere

$$\int_A n_\phi\, d^2r = \frac{\Omega}{4\pi}$$

Realization II: Optical Flux Lattices

Nigel Cooper, PRL (2011); N. Cooper & J. Dalibard, EPL (2011)

periodic optical Raman potentials are conveniently located by standing wave Lasers yielding an optical flux lattice of high flux density, here $n_\phi=1/2a^2$ (fixed)



(a) Local direction of unit Bloch vector $\vec{n}$ (b) Local density of Berry Flux n_ϕ

Strategies for simulating magnetic fields

- Simulate a physical effect that a magnetic field B exerts particle of charge q

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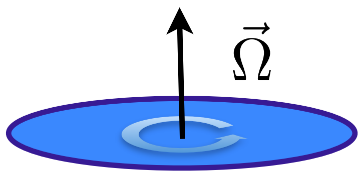
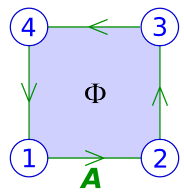
Aharonov-Bohm Effect	$\Psi \propto \exp\left\{ i\frac{q}{\hbar}\int \vec{A}\cdot d\vec{\ell} \right\}$	Complex Hopping Amplitudes A in Optical Lattices
Berry Curvature of Landau levels		?



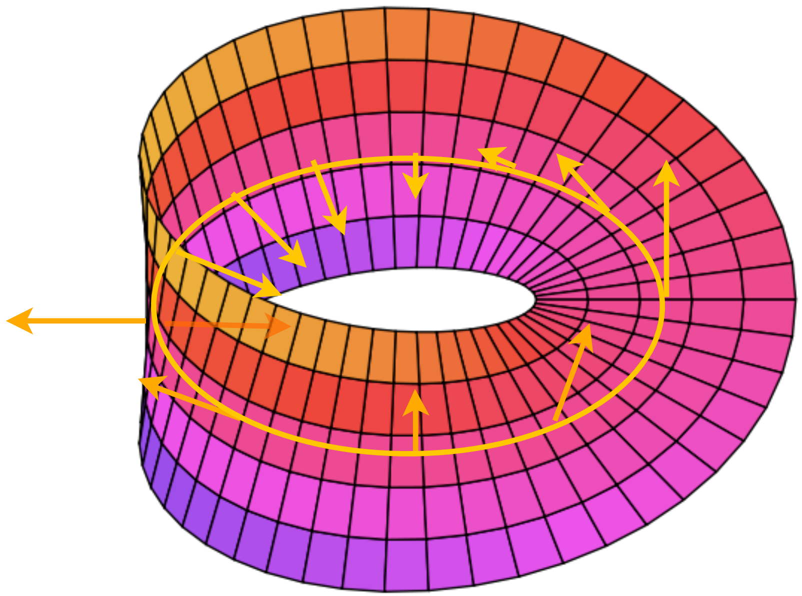


$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$

Landau-levels as a topological band-structure

- Can we see in which way Landau-levels are special, just by looking at the wavefunctions?

Start with an analogy:



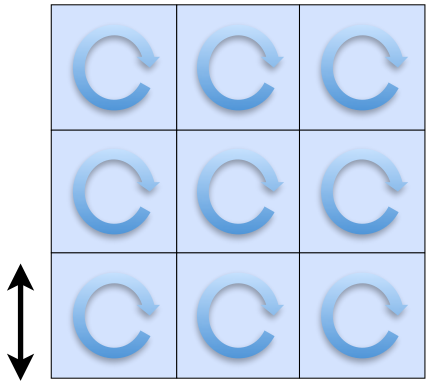
Recipe for calculating the twist in this Möbius band:

- choose a closed path around the surface
- construct normal vector to the surface at points along the curve
- add up the twist angle while moving along this contour

Calculating the 'twist' in wave functions I

1. How to think of the surface / manifold on which Landau level wavefunctions φ live?

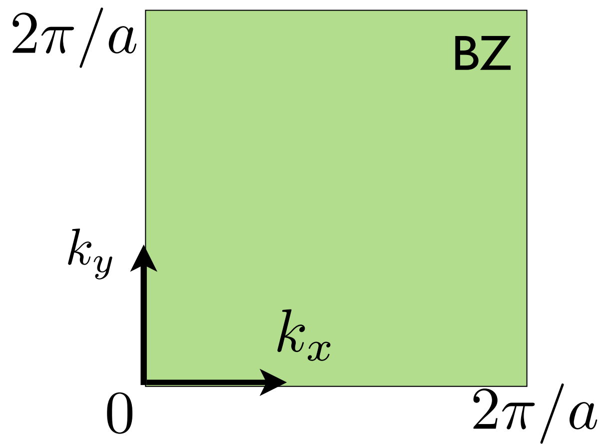
choose to make wavefunctions periodic (by superposition):



$$\varphi(\vec{r}+\vec{a}) \propto \varphi(\vec{r})$$

periodic repetitions of a unit cell (UC)

'crystal momentum' ***k*** parametrizes states in Brillouin zone (BZ)



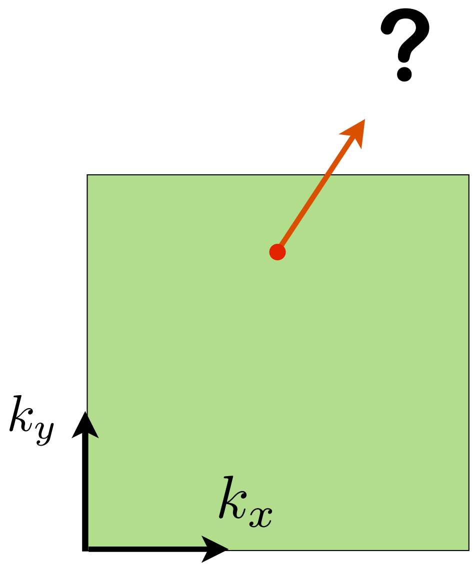
$$\varphi_{\vec{k}}(\vec{r}) = u_{\vec{k}}(\vec{r})e^{i\vec{k}\vec{r}}$$ (Bloch's theorem)

notation for strictly periodic part

$$u_{\vec{k}}(\vec{r}+\vec{a}) = u_{\vec{k}}(\vec{r})$$

Calculating the 'twist' in wave functions II

2. What is the equivalent of the normal vector?



The wavefunction itself is a vector!

lives in a function space with a norm = Hilbert space

$$\varphi_{\vec{k}}(\vec{r}) = u_{\vec{k}}(\vec{r})e^{i\vec{k}\vec{r}}$$

having chosen a point $\boldsymbol{k}$, only u provides non-trivial information and use the ket representation

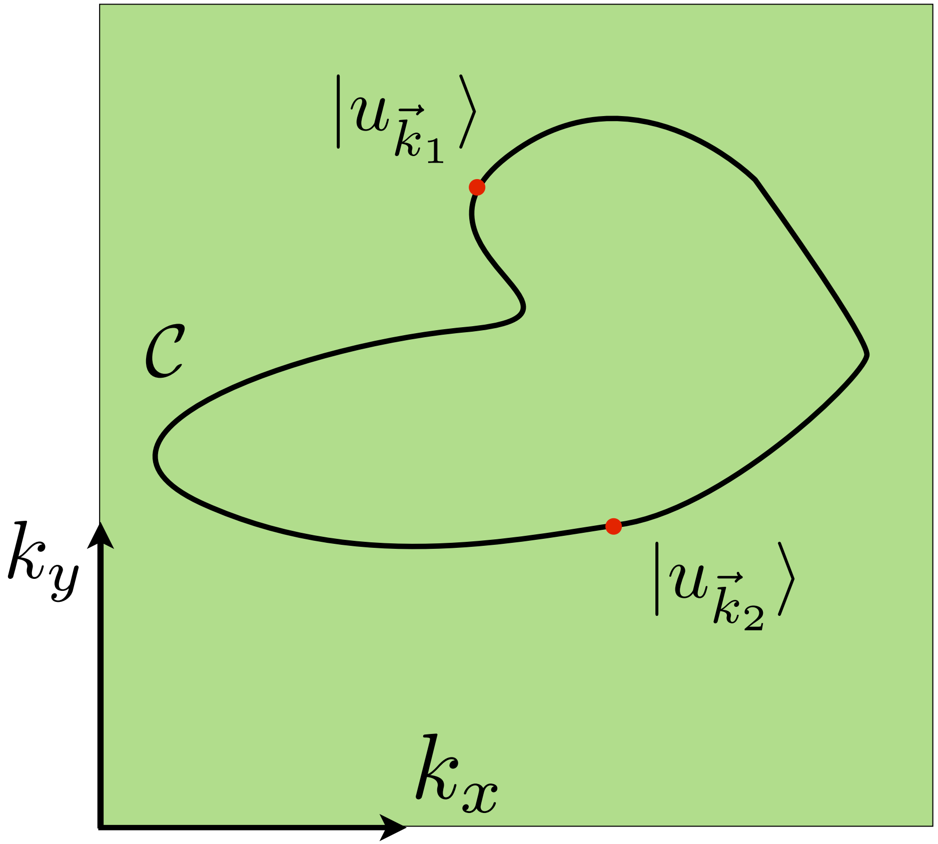
$$|u_{\vec{k}}\rangle$$

u's are eigenstates of a transformed Hamiltonian:

$$\tilde{\mathcal{H}} = e^{-i\vec{k}\vec{r}}\mathcal{H}e^{i\vec{k}\vec{r}} \quad \Rightarrow \quad \tilde{\mathcal{H}}|u_{\vec{k}}\rangle = \epsilon_k|u_{\vec{k}}\rangle$$

Calculating the ‘twist’ in wave functions III

3. What is the equivalent of the twist angle?



direction of vector fixed at any point ...

$$|u_{\vec{k}}\rangle$$

... up to an overall phase

$$e^{i\gamma}|u_{\vec{k}}\rangle$$

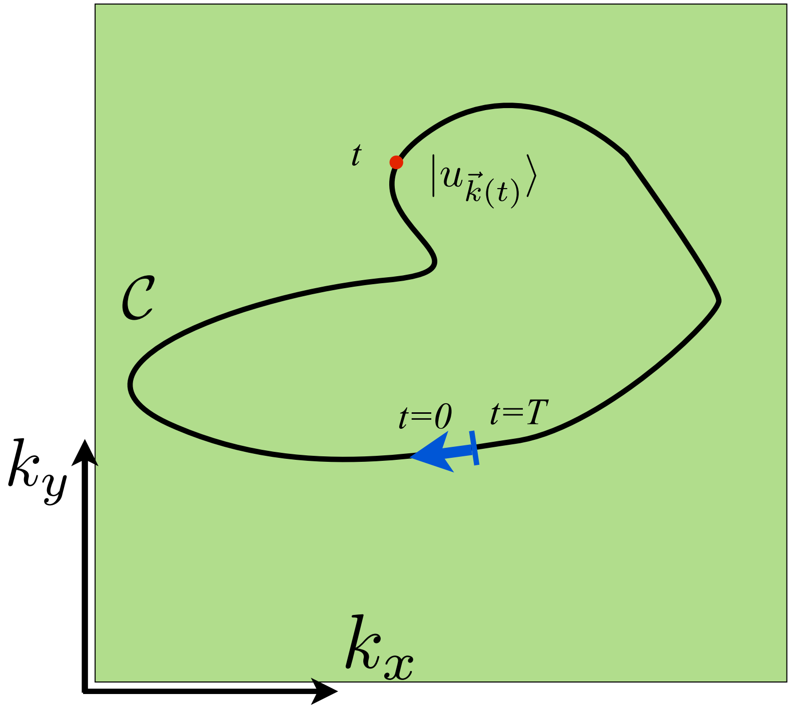
$\Rightarrow$ need to keep track of the phase!

- use scalar product to compare vectors

Calculating the Berry phase

Michael Berry (1984)

Calculate how wavefunction evolves while moving adiabatically through curve $C : \mathbf{k}(t),\ t=0\ldots T$



Local basis $\quad \tilde{\mathcal{H}}|u_{\vec{k}}\rangle = \epsilon_{\vec{k}}|u_{\vec{k}}\rangle$

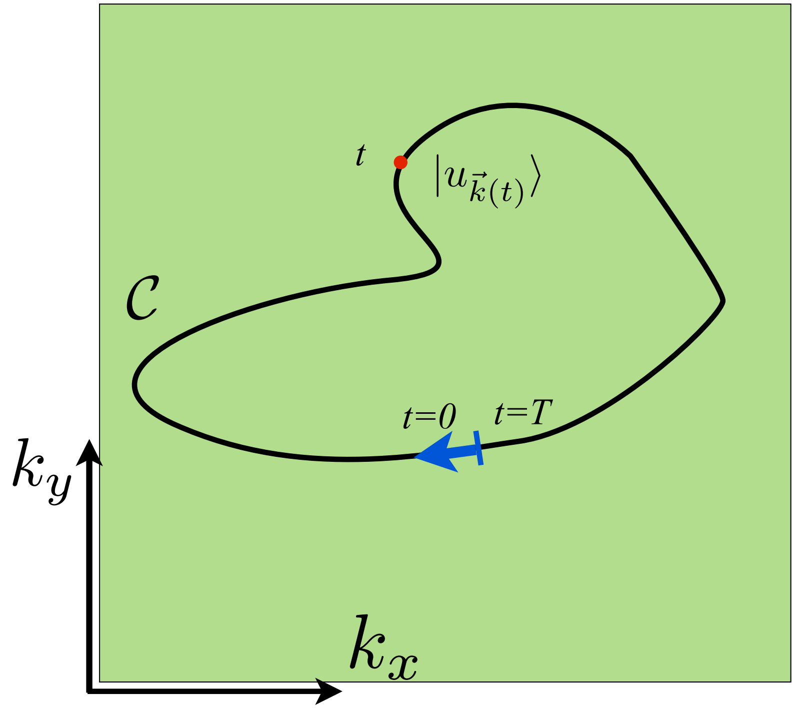
Phase evolution has two components:

$$|U(t)\rangle = \underbrace{\exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon_{\vec{k}(t')}dt'\right\}}_{\text{dynamical time evolution}} \underbrace{\exp\left\{i\gamma(t)\right\}}_{\text{'twist'}} |u_{\vec{k}(t)}\rangle$$

Calculating the Berry phase

Michael Berry (1984)

Calculate how wavefunction evolves while moving adiabatically through curve C : $\mathbf{k}(t)$, $t=0...T$



Local basis $\tilde{\mathcal{H}}|u_{\vec{k}}\rangle = \epsilon_{\vec{k}}|u_{\vec{k}}\rangle$

Phase evolution has two components:

$$|U(t)\rangle = \exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon_{\vec{k}(t')}dt'\right\}\exp\{i\gamma(t)\}\,|u_{\vec{k}(t)}\rangle$$

Substitute into Schrödinger equation:

$$\tilde{\mathcal{H}}|U(t)\rangle = i\hbar\frac{d}{dt}|U(t)\rangle$$

take

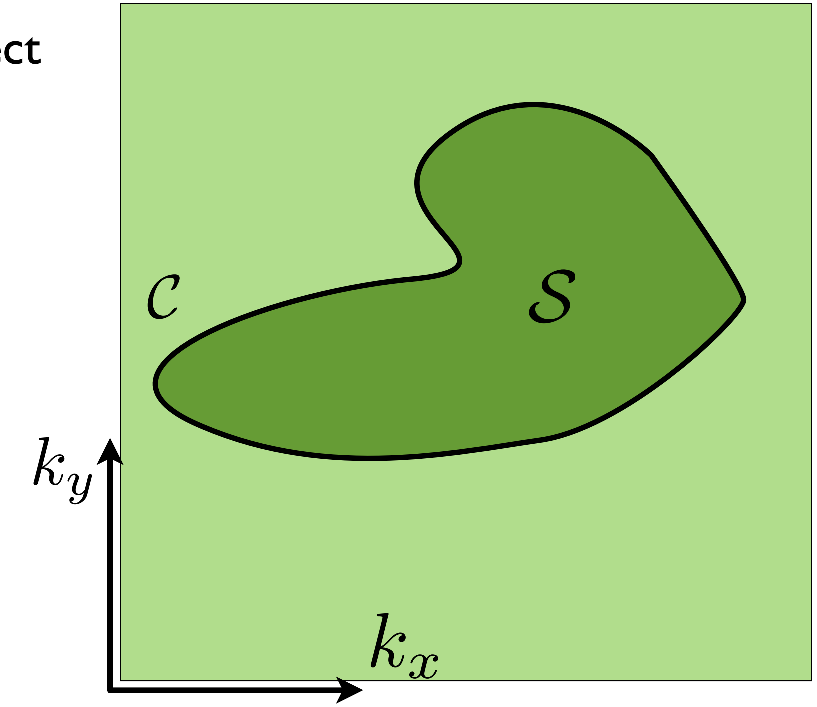
$$\int_0^T dt\langle U(t)|\;\Big|\quad \cancel{\epsilon_{\vec{k}(t)}|U(t)\rangle} = i\hbar\left(\cancel{-\frac{i}{\hbar}\epsilon_{\vec{k}(t)}} + i\frac{d}{dt}\gamma(t) + \frac{d\vec{k}}{dt}\frac{d}{d\vec{k}}\right)|U(t)\rangle$$

Berry phase: $$\gamma(\mathcal{C}) = i\int_{\mathcal{C}}\langle u_{\vec{k}}|\frac{d}{d\vec{k}}|u_{\vec{k}}\rangle d\vec{k}$$ purely geometrical!

Berry curvature and Chern number

Geometrical phase analogous to Aharonov-Bohm effect

$$\gamma(\mathcal{C}) = i \int_{\mathcal{C}} \langle u_{\vec{k}} | \frac{d}{d\vec{k}} | u_{\vec{k}} \rangle d\vec{k} \equiv \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k}) d\vec{k}$$



Effective 'vector potential' called *Berry connection*

$$\vec{\mathcal{A}}(\vec{k}) = i \int_{\mathrm{UC}} u_{\vec{k}}(\vec{r})^* \vec{\nabla}_k u_{\vec{k}}(\vec{r})\, d^2 r$$

Using Stokes' theorem:

$$\gamma(\mathcal{C}) = \int_{\mathcal{C}} \vec{\mathcal{A}}(\vec{k}) d\vec{k} = \int_{\mathcal{S}} \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k}) d\vec{\sigma}$$

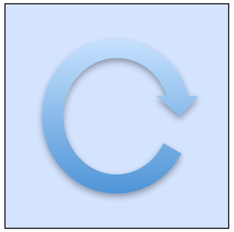
$\mathcal{C} = \partial\mathcal{S}$

Berry curvature: $\vec{\mathcal{B}} = \vec{\nabla}_k \times \vec{\mathcal{A}}(\vec{k})$ is a property of the band eigenfunctions, only!

Chern number: $C = \frac{1}{2\pi} \int_{BZ} d^2\mathbf{k}\, \mathcal{B}(\mathbf{k})$ takes only integer values!

Evaluating: Berry curvature for Landau-Levels

Choosing the Landau-gauge, i.e. vector potential $\vec{A} = Bx\vec{e}_y$



$$2\pi\ell_0^2 = \frac{h}{eB}$$ spatial extent of of one LL state

Berry connection $\vec{\mathcal{A}}(\vec{k}) = -k_y \ell_0^2 \vec{e}_x$ (this is gauge dependent)

Berry curvature: $\vec{\mathcal{B}} = \ell_0^2 \vec{e}_z$ constant curvature - reflects constant magnetic field

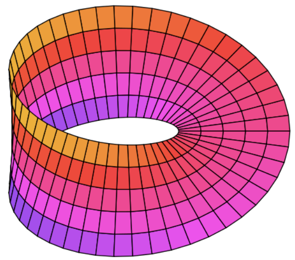
Chern number: $C = 1$

$\Downarrow$

Directly related to quantised Hall conductance:

$$\sigma_{xy} = \frac{e^2}{h} C$$

Like twist in Möbius strip:
Chern number does not vary under small perturbations:
'*Topological invariant*'



Emulating the effect of magnetic fields

Non-zero Berry curvature is not related specifically to magnetic fields only:

$$\vec{\mathcal{A}}(\vec{k}) = i \int_{\mathrm{UC}} u_{\vec{k}}(\vec{r})^* \vec{\nabla}_k u_{\vec{k}}(\vec{r})\, d^2 r$$

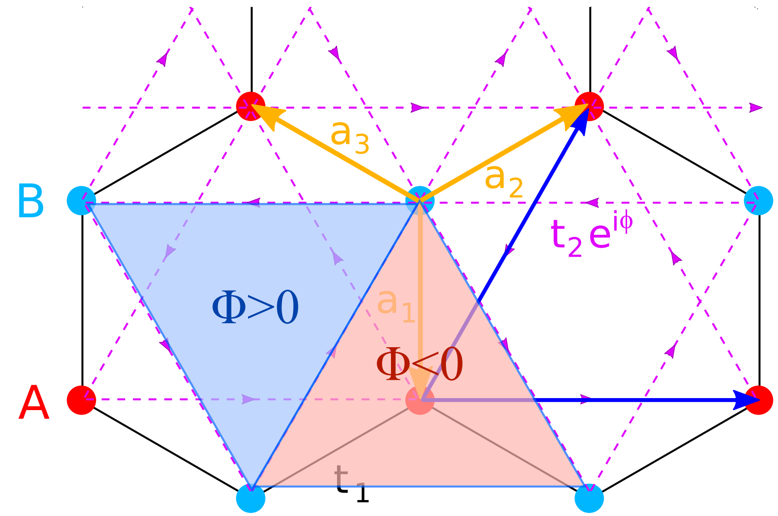
spatial dependency = physical implementation integrated out!

- Other systems with Chern number $C=1$ can give rise to a quantized Hall effect

F. D. M. Haldane (1988)

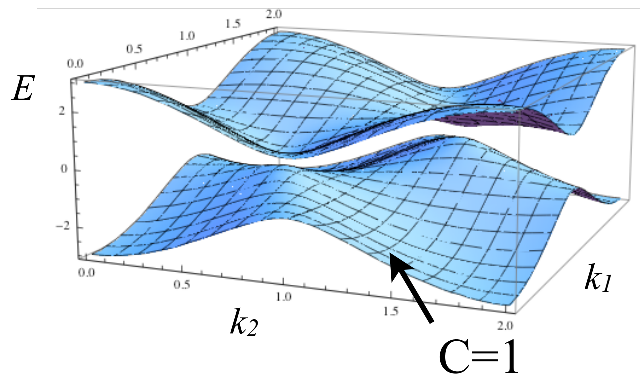
Haldane's Model

F. D. M. Haldane (1988)



Energy:
minimal dispersion

$\phi = \frac{\pi}{2},\ t_2 = 0.1 t_1$



$$\mathcal{H} = -\,t_1 \sum_{\langle \mathbf{r}\mathbf{r}'\rangle} \left(\hat{a}^\dagger_{\mathbf{r}}\hat{a}_{\mathbf{r}} + h.c.\right) - t_2 \sum_{\langle\langle \mathbf{r}\mathbf{r}'\rangle\rangle} \left(\hat{a}^\dagger_{\mathbf{r}}\hat{a}_{\mathbf{r}} e^{i\phi_{\mathbf{r}\mathbf{r}'}} + h.c.\right)$$

- Realization of such models might be possible thanks to spin-orbit coupling

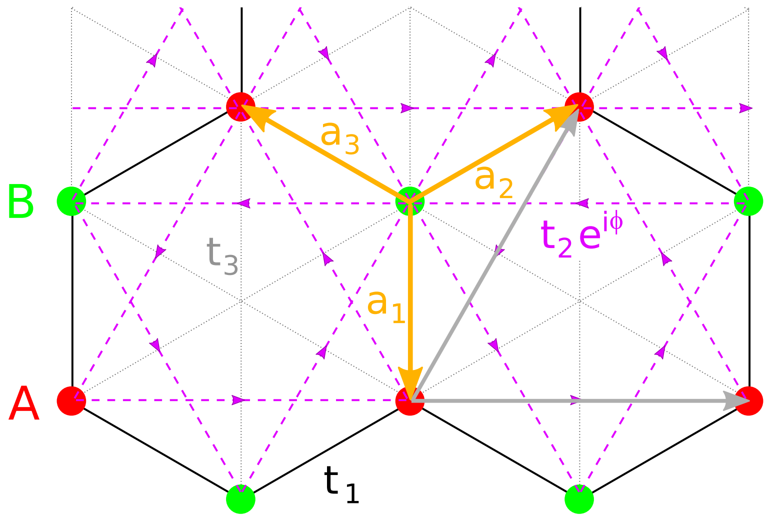
Kane (2005)

- Proposal: for *flat* bands with $|C|>0$, there may even be a fractional quantum Hall effect!

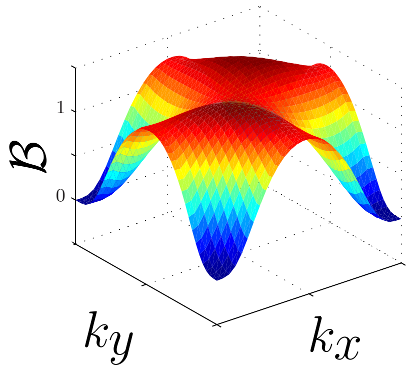
Tang et al. + Neupert et al. + Sun et al. in Phys. Rev. Lett. (2011)

Haldane's Model for strong correlations

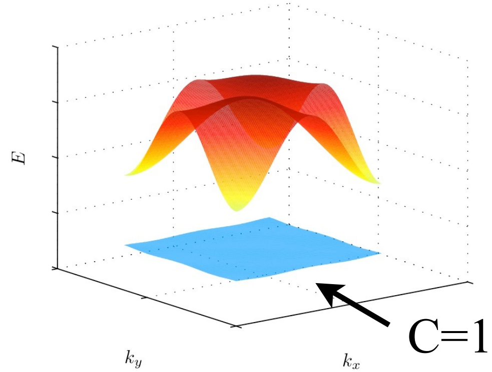
F. D. M. Haldane (1988)



Berry curvature: inhomogenous



Energy: minimal dispersion



$$\mathcal{H} = -\,t_1 \sum_{\langle \mathbf{r}\mathbf{r}'\rangle} \left(\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} + h.c.\right) - t_2 \sum_{\langle\langle \mathbf{r}\mathbf{r}'\rangle\rangle} \left(\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} e^{i\phi_{\mathbf{r}\mathbf{r}'}} + h.c.\right)$$

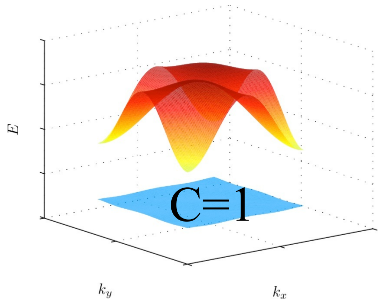
$$-\,t_3 \sum_{\langle\langle\langle \mathbf{r}\mathbf{r}'\rangle\rangle\rangle} \left(\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} + h.c.\right)$$

- tight binding model on hexagonal lattice
- no average magnetic flux (but time-reversal symmetry is broken)
- with fine-tuned hopping parameters: obtain flat lower band

$$t_1 = 1,\; t_2 = 0.60,\; t_3 = -0.58 \text{ and } \phi = 0.4\pi$$

Q: Do flat Chern bands yield FQHE physics?

Q: Particles in a nearly flat band with Chern #1 are similar to electrons in a Landau-level, but do repulsive interactions really induce the equivalent of fractional quantum Hall states?



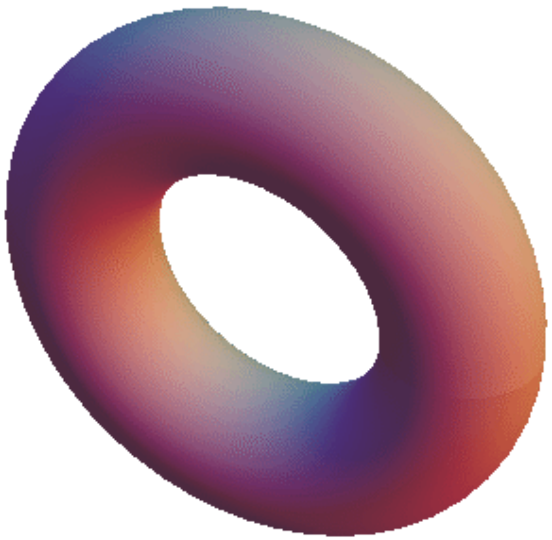
+ Interactions = FQHE ?

- some evidence for such states, which we call "Fractional Chern Insulators" (FCI):
  - existence of a gap & groundstate degeneracy [D. Sheng (2011)]
  - Finite size scaling of gap [Regnault & Bernevig (2011)]
  - count of quasiparticle excitations matches FQHE states (e.g. Laughlin state)

‣ Want to find a new technique that can be used to make robust conclusions about the nature of phases realised in Chern bands

Strategy

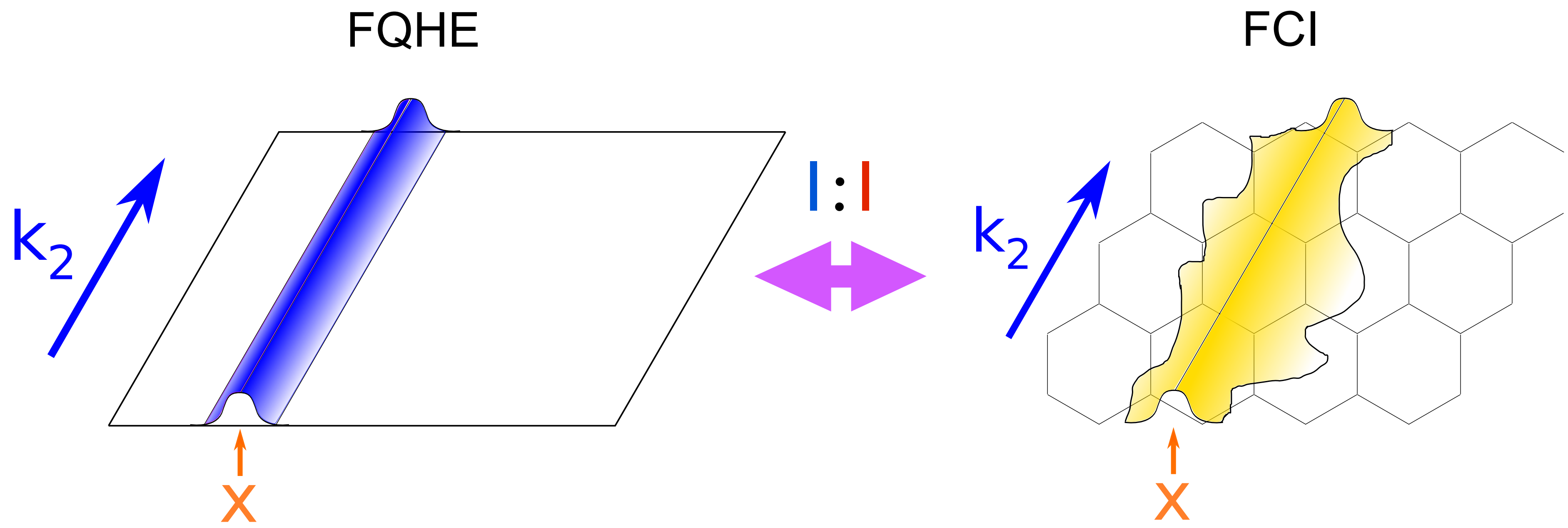
- Just like the topology of a geometrical object:



- Topological order is invariant under continuous / adiabatic deformations!

- My approach: Devise a method to deform the wavefunction of a fractional quantum Hall state into a fractional Chern insulator without closing the gap.

Mapping from FQHE to FCI: Single Particle Orbitals



- Proposal by X.-L. Qi [PRL '11]: Get FCI Wavefunctions by mapping single particle orbitals
- Idea: use Wannier states which are localized in the x-direction
- keep translational invariance in y (cannot create fully localized Wannier state if C>0!)

$$|W(x,k_y)\rangle = \sum_{k_x} f_{k_x}^{(x,k_y)} |k_x, k_y\rangle$$

- Qi's Proposition: using a mapping between the LLL eigenstates (QHE) and localized Wannier states (FCI), we can establish an exact mapping between their many-particle wavefunctions

Wannier states in Chern bands

- construction of a Wannier state at fixed k_y in gauge with $\mathcal{A}_y = 0$

$$|W(x,k_y)\rangle = \frac{\chi(k_y)}{\sqrt{L_x}} \sum_{k_x} e^{-i\int_0^{k_x} \mathcal{A}_x(p_x,k_y)dp_x} \times e^{ik_x\frac{\theta(k_y)}{2\pi}} \times e^{-ik_x x}|k_x,k_y\rangle$$

'Parallel transport' of phase — 'Polarization' — Fourier transform

Berry connection indicates change of phase due to displacement in BZ

ensures periodicity of WF in $x \to x + \mathrm{L_x}$

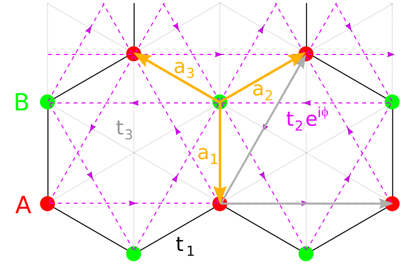
- or, more simply we can think of the Wannier states as the eigenstates of the position operator

$$\hat{X}^{cg} = \lim_{q_x\to 0} \frac{1}{i}\frac{\partial}{\partial q_x}\bar{\rho}_{q_x} \quad \Rightarrow \quad \hat{X}^{cg}|W(x,k_y)\rangle = [x - \theta(k_y)/2\pi]|W(x,k_y)\rangle$$

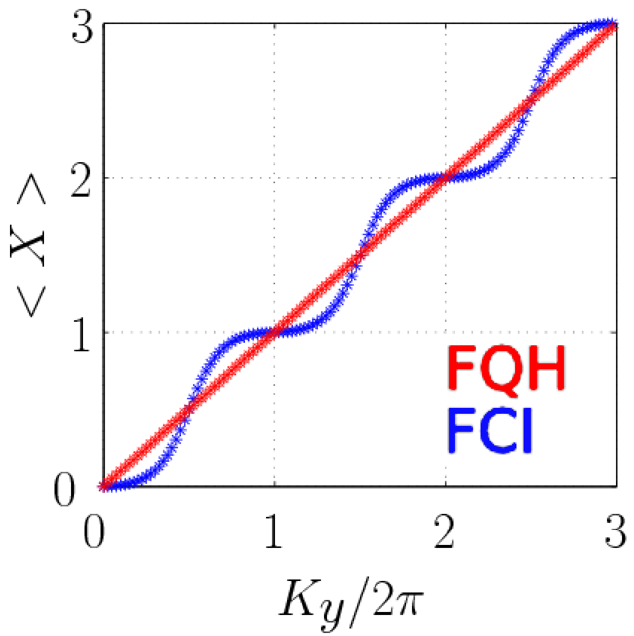
- role of polarization: displacement of centre of mass of the Wannier state

$$\theta(k_y) = \int_0^{2\pi} \mathcal{A}_x(p_x,k_y)dp_x$$

Associating Quantum Numbers



- The Wannier basis for the Chern band produces a sequence of single-particle states with monotonously increasing position
- Similar to the linearly increasing position of the Landau-level basis



$$k_y = 2\pi n_y / L_y$$

$$\boxed{K_y} = k_y + 2\pi x = \boxed{2\pi j / L_y}$$

$$j = n_y + L_y x = 0, 1, ..., N_\phi - 1$$

- Formulate degenerate perturbation theory in this basis for a fractionally filled Chern band

$$\hat{\mathcal{H}} = K.E. + \sum_{i<j} \hat{V}(\vec{r}_i - \vec{r}_j) \qquad \mathcal{H} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & \cdots \\ V_{21} & V_{22} & V_{23} & \cdots \\ V_{31} & V_{32} & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

many-body basis

with $\quad V_{ij} = \langle \alpha_j | \hat{V} | \alpha_i \rangle$

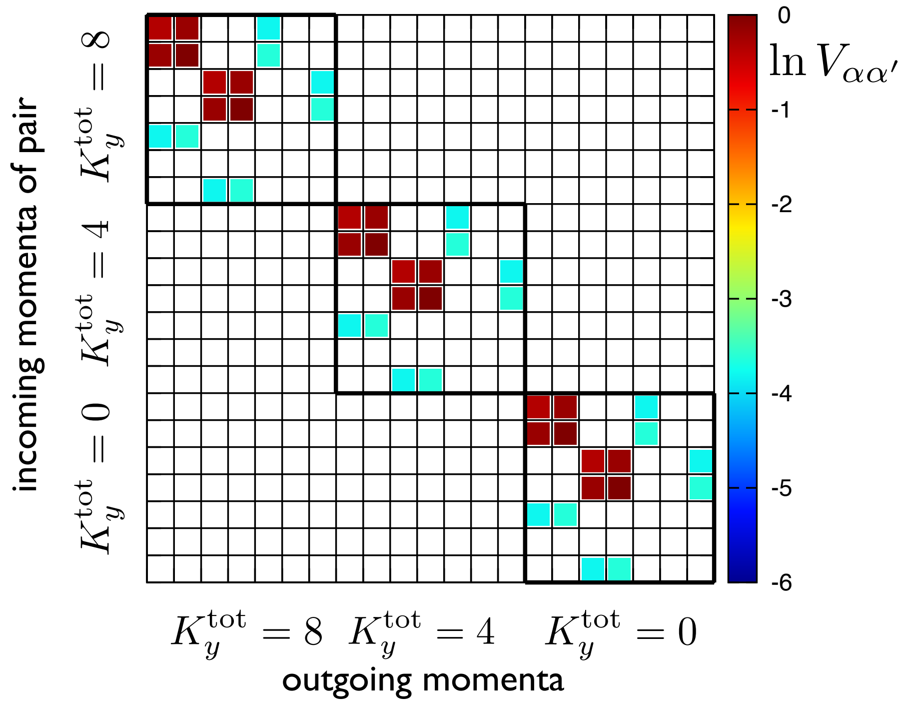
Visualising contact interactions for bosons

Th. Scaffidi & GM, Phys. Rev. Lett. (2012)

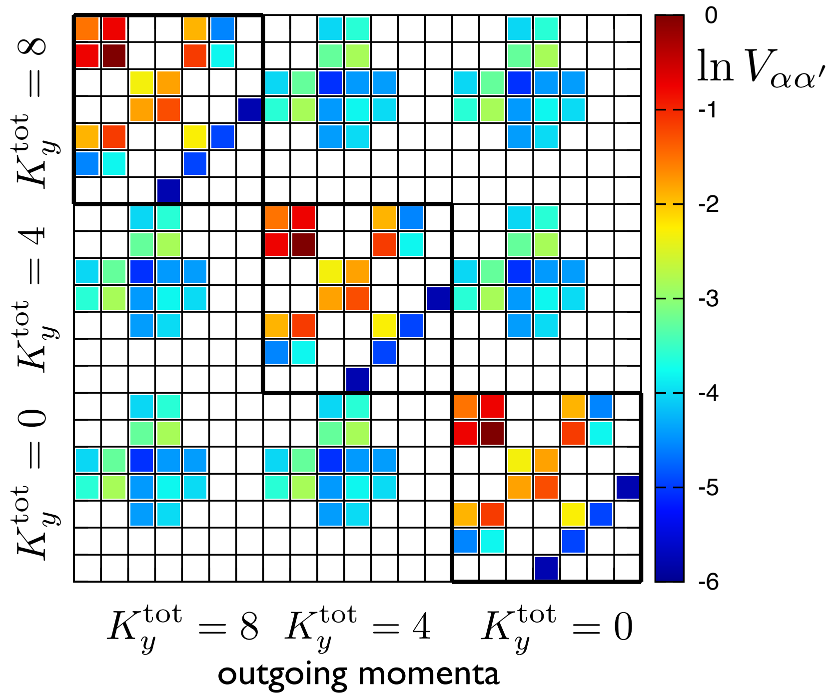
- Magnitude of two-body matrix elements for delta interactions in the Haldane model $\mathcal{H} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & \cdots \\ V_{21} & V_{22} & V_{23} & \cdots \\ V_{31} & V_{32} & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix}$

$$V(\vec{r}_i - \vec{r}_j) \propto \delta(\vec{r}_i - \vec{r}_j)$$

FQHE



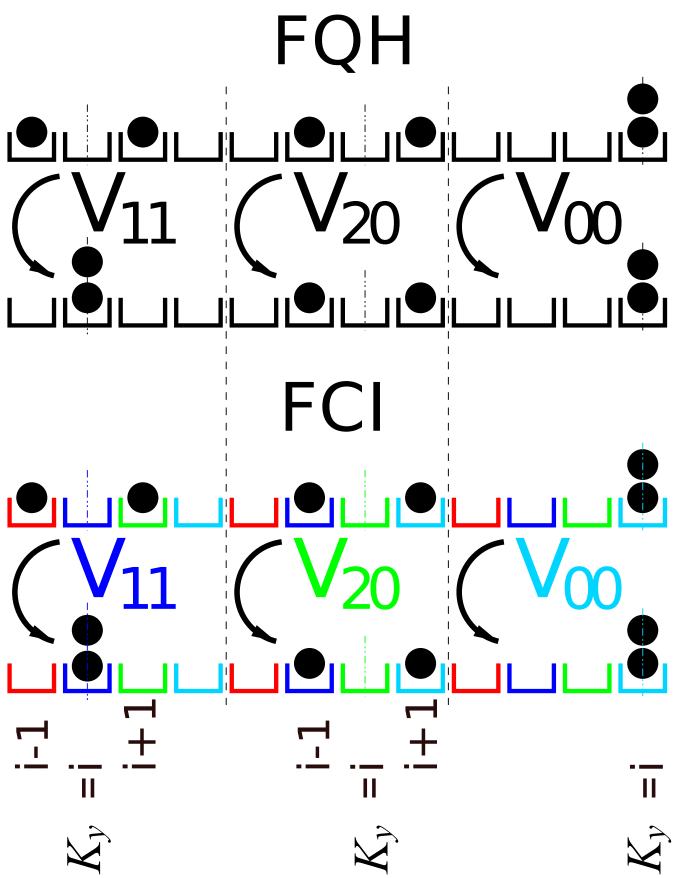
FCI

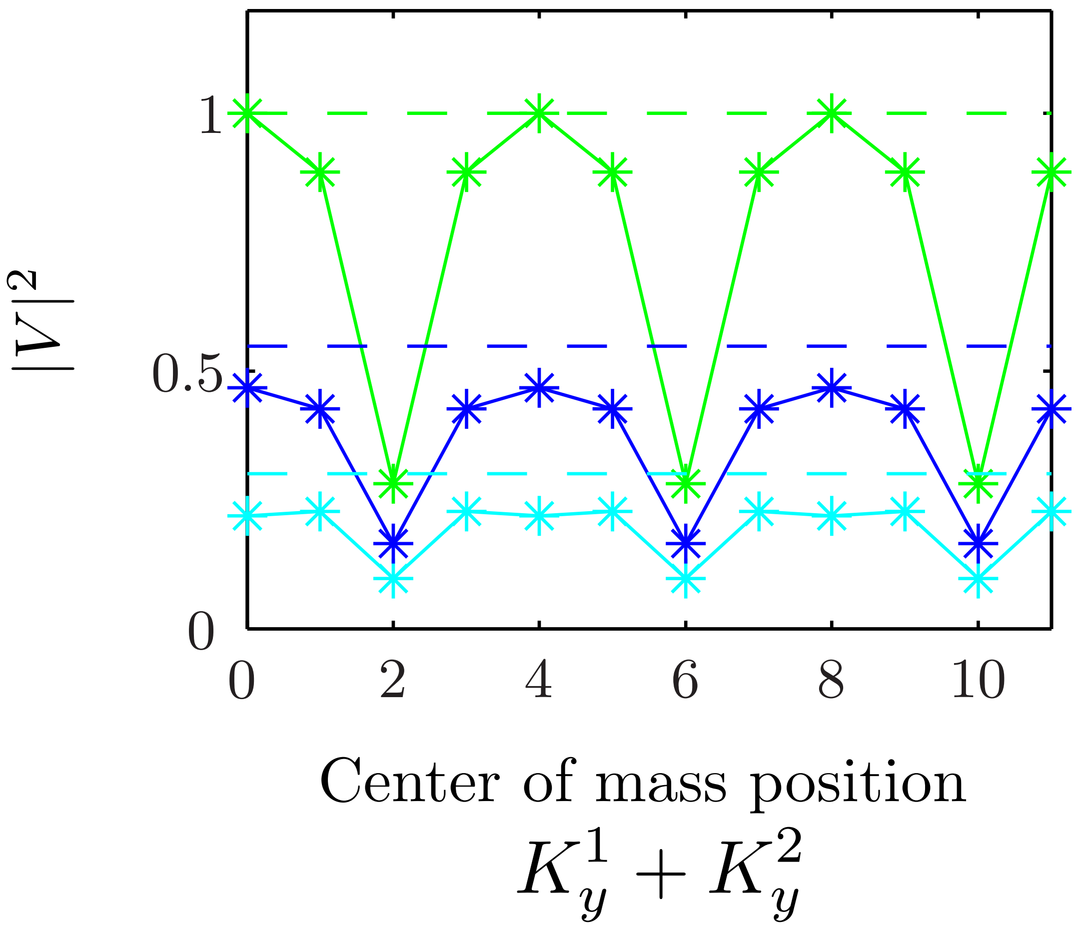


- System shown: two-body interactions for $L_x \times L_y = 3 \times 4$
- Matrix elements differ in magnitude, but overall similarities are present
- Different block-structure due to non-conservation of linearised momentum K_y

Reduced translational invariance in K_y

- A closer look at some short range hopping processes





- for FCI: hopping amplitudes depend on position of centre of mass / K_y

Interpolating in the Wannier basis

- Can write both states in single Hilbert space with the same overall structure (indexed by K_y) and study the low-lying spectrum numerically (exact diagonalization)
- Can study adiabatic deformations from the FQHE to a fractionally filled Chern band

$$\mathcal{H}(x) = \frac{\Delta_{\mathrm{FCI}}}{\Delta_{\mathrm{FQHE}}}(1-x)\mathcal{H}^{\mathrm{FQHE}} + x\mathcal{H}^{\mathrm{FCI}}$$

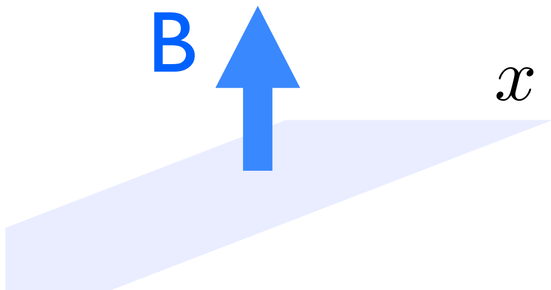
- Here: look at half-filled band for bosons

FQHE of Bosons at $\nu = 1/2$

Half filled band of the (flattened) Haldane-model

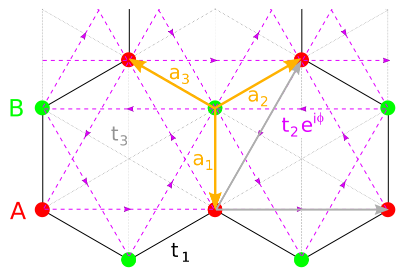
Laughlin state

Same topological phase?



$x = 0$

$x = 1$

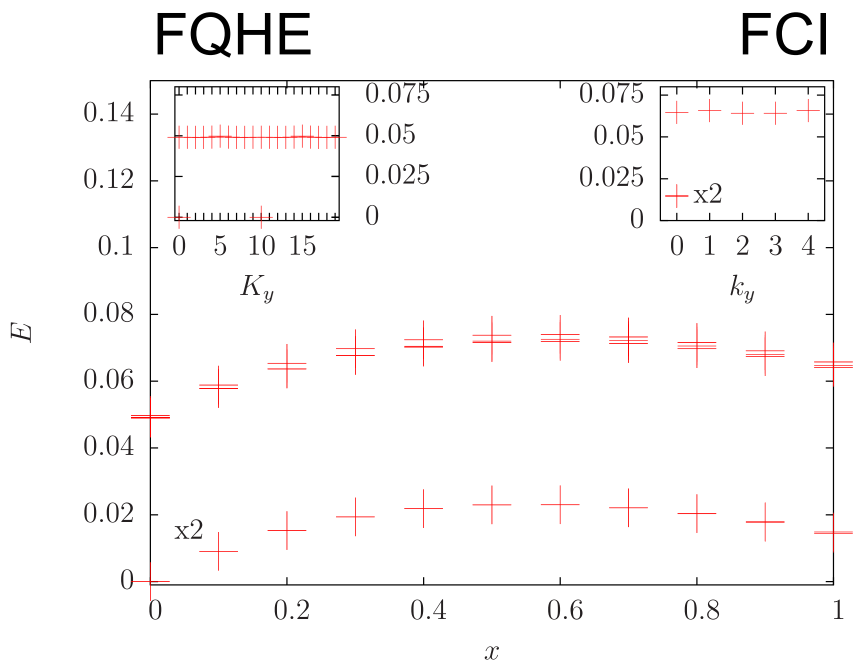


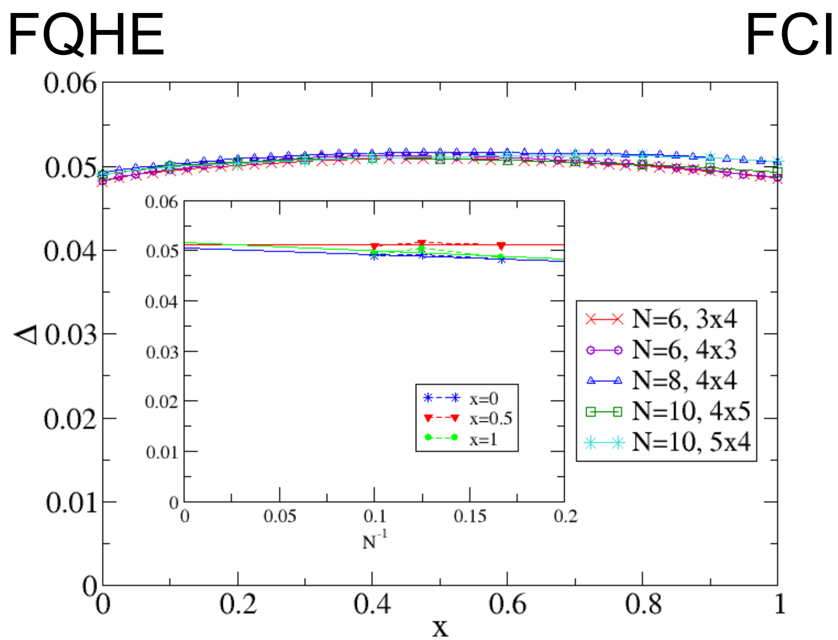
Th. Scaffidi & GM, Phys. Rev. Lett. (2012)

Adiabatic continuation in the Wannier basis

- Spectrum for N=10
(Hilbert space of dimension d=5x10^6):

- Gap for different system sizes & aspect ratios:





- The gap remains open for all x!

- We confirm the Laughlin state is adiabatically connected to the groundstate of the half-filled topological flat band of the Haldane model

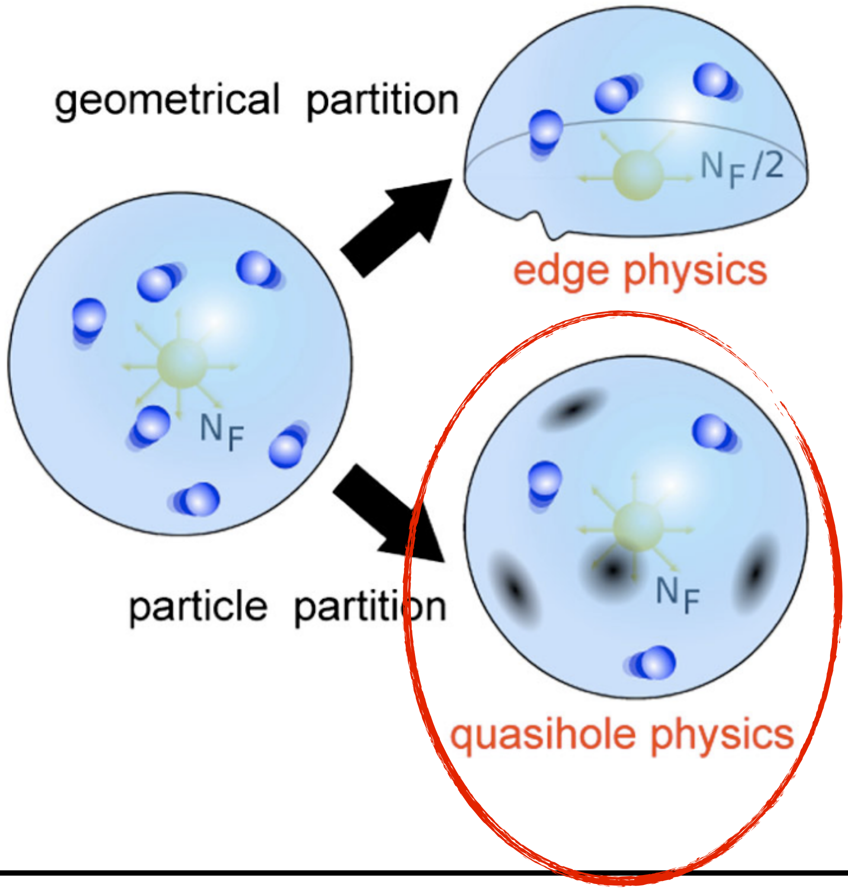
‣ General strategy with possibility to test & predict topological order in the thermodynamic limit

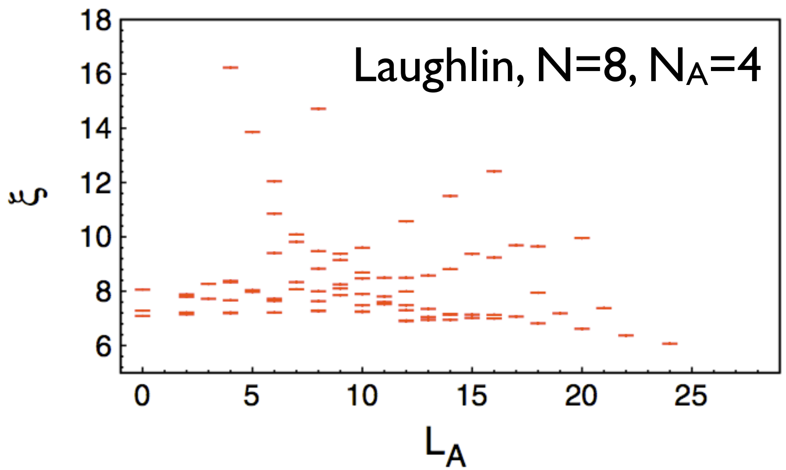
Th. Scaffidi & GM, Phys. Rev. Lett. (2012)

Entanglement spectra and quasiparticle excitations

• Entanglement spectrum: arises from Schmidt decomposition of ground state into two groups A, B
=> Schmidt eigenvalues ξ plotted over quantum numbers for symmetries within each block

$$|\Psi\rangle = \sum_{\varpi} \sum_{i} e^{-\xi_{\varpi,i}/2} |\Psi^{A}_{\varpi,i}\rangle \otimes |\Psi^{B}_{\varpi,i}\rangle$$





Dominant (universal) eigenvalues of PES yield count of excited states - and their wavefunctions - from groundstate wavefunction only!

credit: Sterdyniak et al. PRL 2011

FCI: Adiabatic continuation of the entanglement spectrum

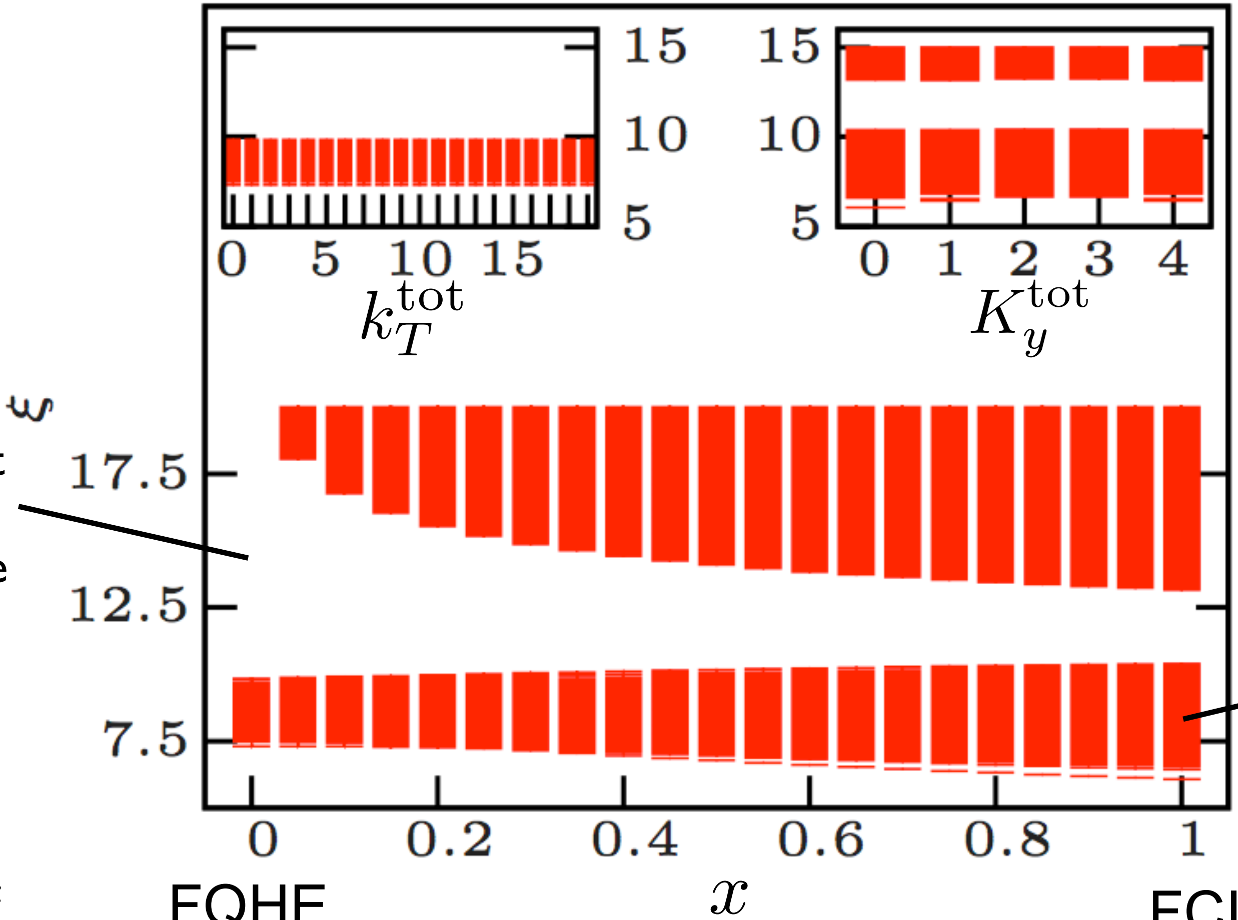
Total #eigenvalues below entanglement gap
= 4x(201 + 200 + 200 + 200 + 200)

Total #eigenvalues below entanglement gap
= 804 + 800 + 800 + 800 + 800

$N = 10$

$N_A = 5$

'Infinite' entanglement gap for pure Laughlin state

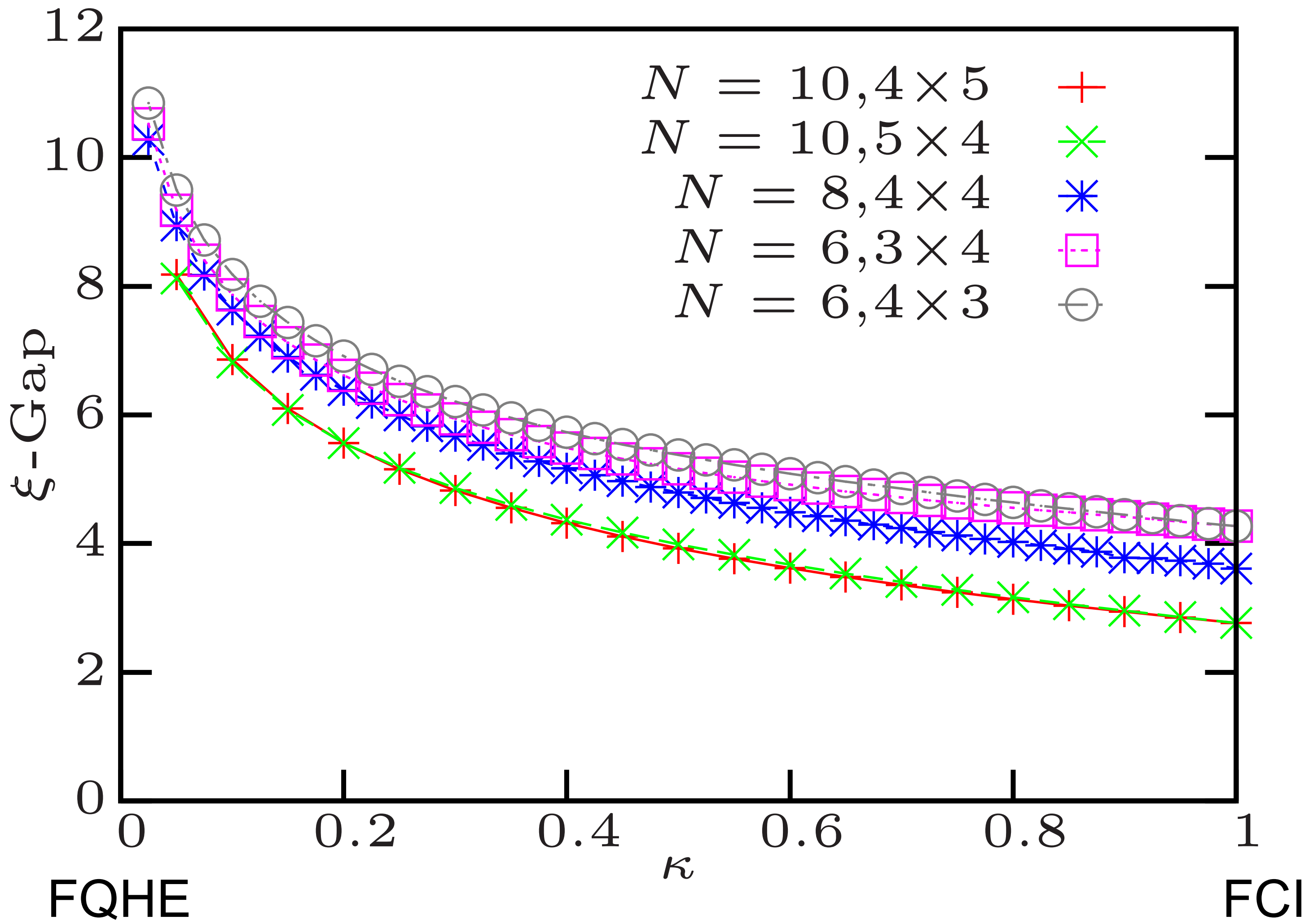


Same number of states for all x

Dictionary: FQHE FCI

$$|\Psi\rangle = \sum_{\varpi} \sum_{i} e^{-\xi_{\varpi,i}/2} |\Psi^A_{\varpi,i}\rangle \otimes |\Psi^B_{\varpi,i}\rangle$$

Finite size behaviour of entanglement gap



- The entanglement gap remains open for all values of the interpolation parameter k
- Finite size scaling behaviour encouraging, but analytic dependency on system size unknown

Summary for far

Analytic continuation between FQHE and FCI:

- provides a formal proof of identical topological order between two phases

- may allows robust conclusions about the thermodynamic limit from finite size data

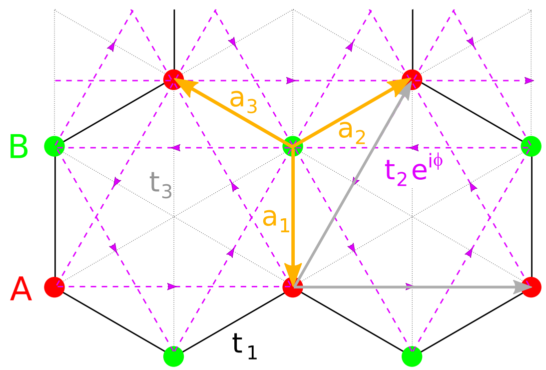
- Successfully applied to a range of systems:
  - Haldane model: Laughlin state (Scaffidi & Möller, PRL 2012)
  - Kagomé lattice model: Laughlin & Moore-Read states (Liu & Bergholtz, PRB 2013)

- Possibility to simulate quantum Hall physics using spin-orbit coupling in solid state materials confirmed.

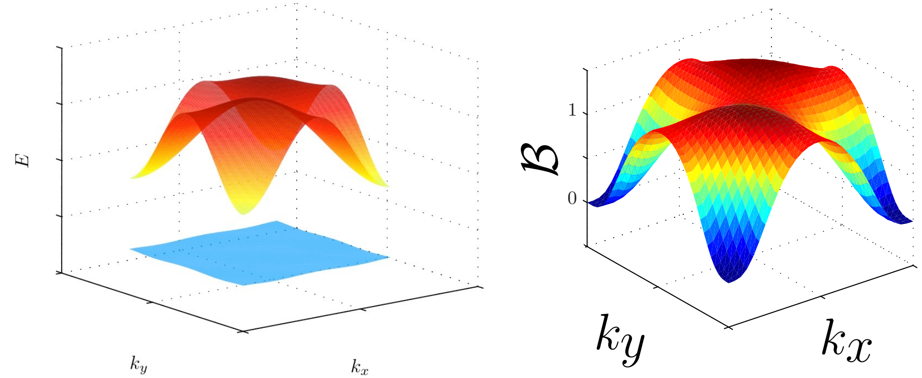
A side note: Flux Lattices vs Chern Bands

CBs = tight binding model in *real* space with topological flat bands

real space

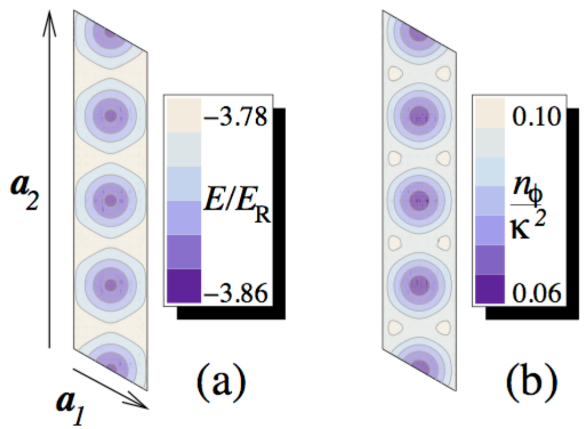


reciprocal space

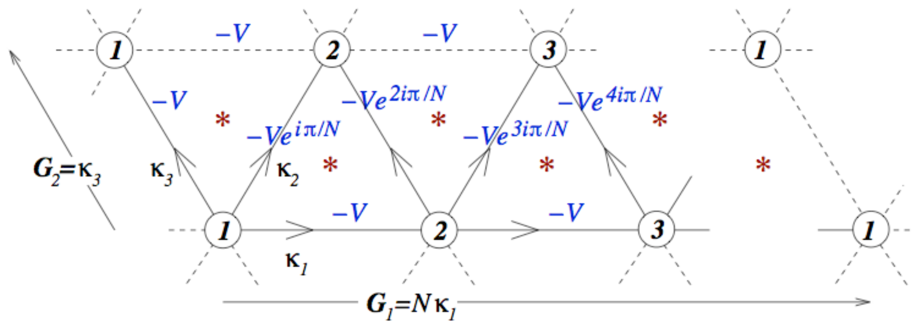


Flux Lattice = tight binding model in *reciprocal* space with topological flat bands

N. R. Cooper, R. Moessner, PRL 109, 215302 (2012)



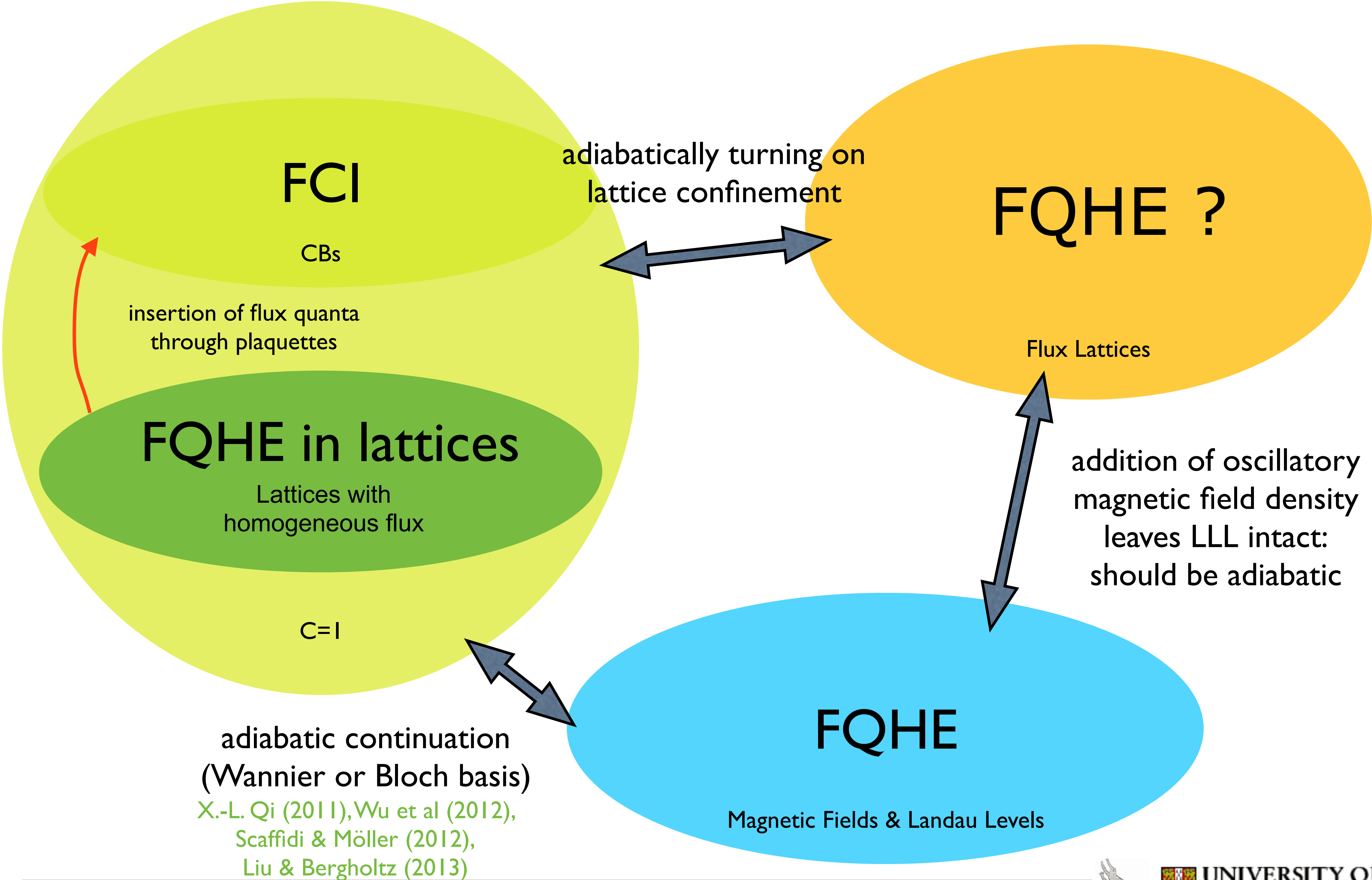
real space



absorption of photons = momentum + state transfers

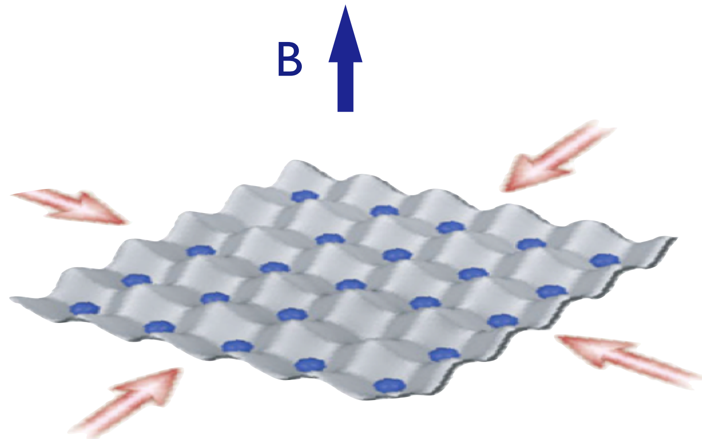
reciprocal space

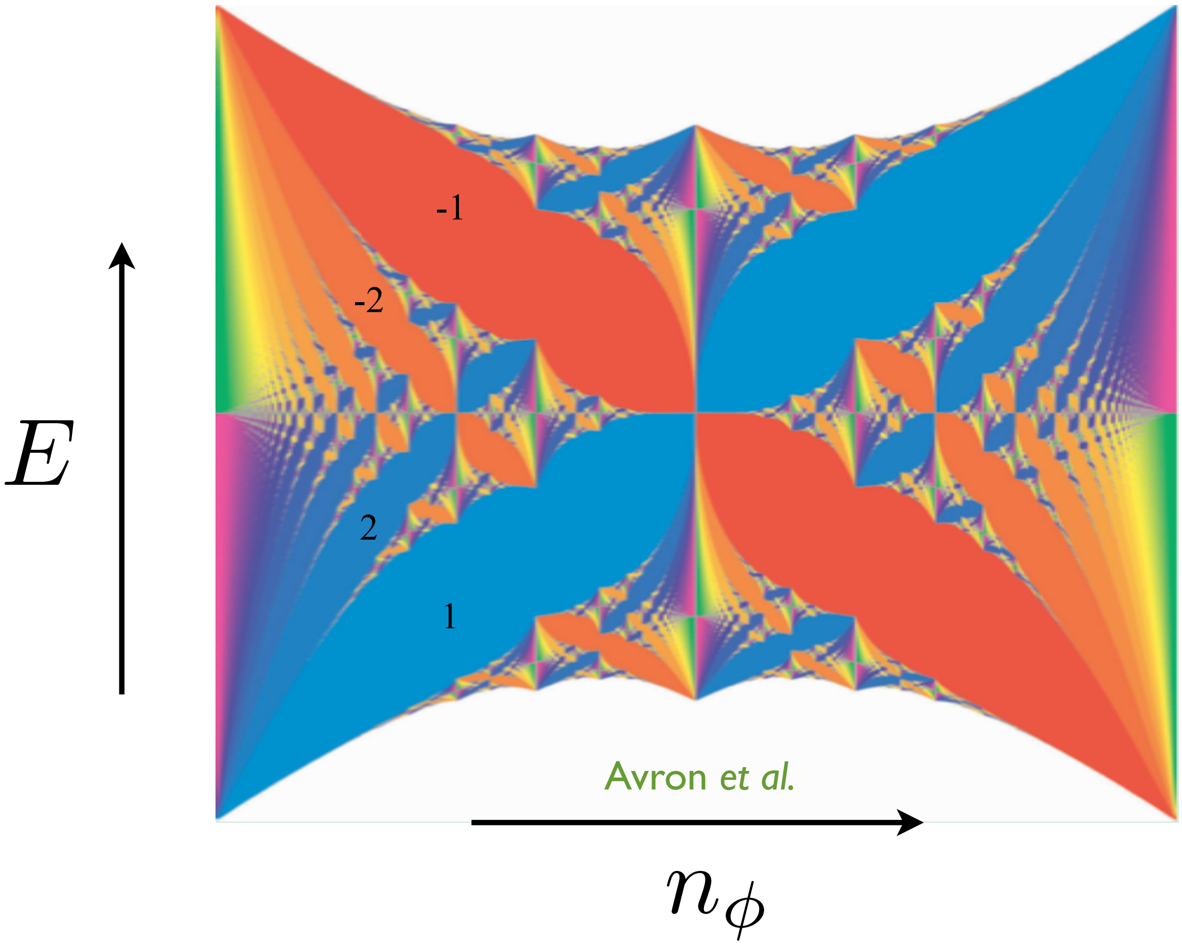
Interrelations between interacting flat-band problems



Is there new physics in Chern Insulators?

- YES: e.g., higher Chern-numbers
- characteristic system featuring bands of any Chern number: the square lattice with constant magnetic flux



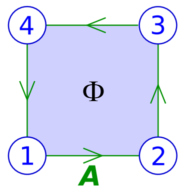


color-coding of gaps by Chern numbers - blue: positive, red: negative integers

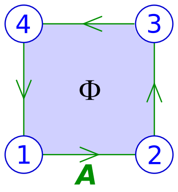
- use Aharonov-Bohm effect to emulate flux

$$\mathcal{H}_c = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right]$$

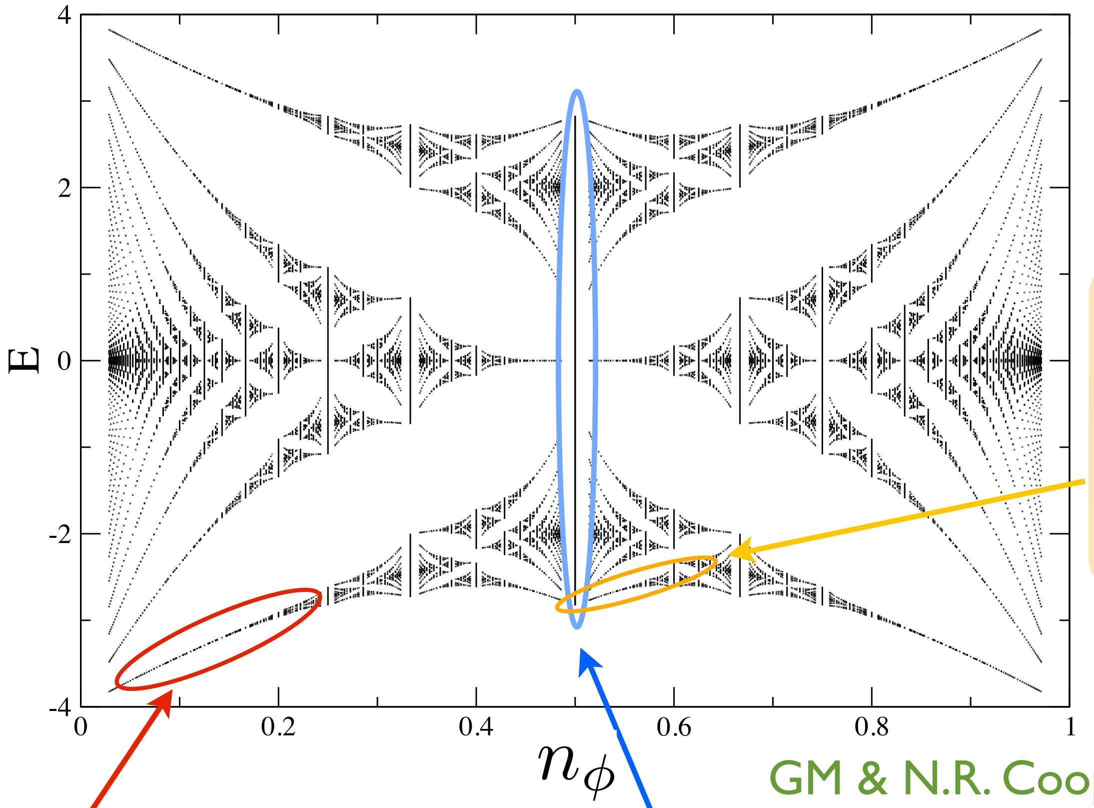
$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$



Many-body phases of the Hofstadter spectrum (for bosons)



square lattice at flux density n_ϕ



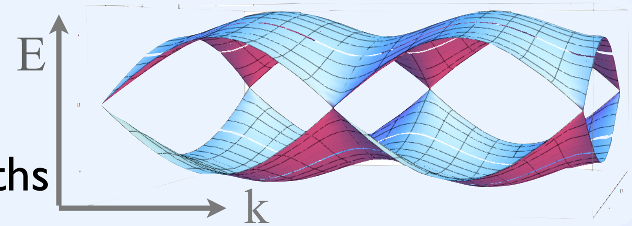
narrow bands with a gap
➜ novel Fractional Quantum Hall states beyond those known in the continuum!

GM & N.R. Cooper, PRL 2009; PRA 2010

narrow band (lowest Landau level)
➜ Fractional Quantum Hall states

A. Sørensen et al., PRL 2005

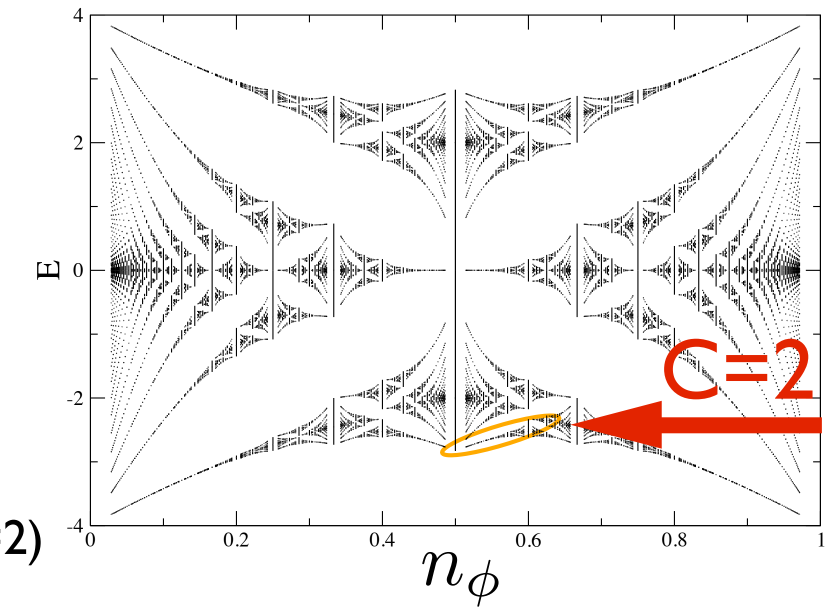
n_ϕ=1/2: wide bands
➜ condensates at all interaction strengths



Strongly correlated phases in the Hofstadter hierarchy

Interacting Bose-Hubbard model with flux n_ϕ

$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right] + \frac{1}{2}U \sum_\alpha \hat{n}_\alpha(\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$



A) $n_\phi = \frac{1}{2} + \epsilon$ and limit of small U

physics ~ QHE in Landau levels with subband index (C=2)

Halperin states: R. Palmer & D. Jaksch, PRL 2006, PRA 2008

Double Pfaffian (paired) state: L. Hormozi, GM & S.H. Simon, PRL 2012

B) hardcore limit $U \to \infty$: physics of composite fermions

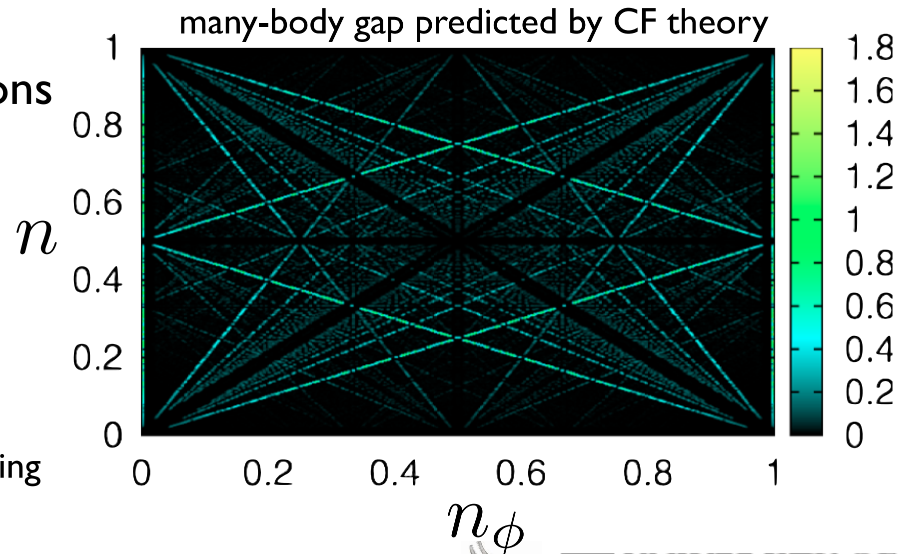
Interactions favour flux attachment, suppressing any double occupation at expense of kinetic energy

$$\Psi_\mathrm{B}(\{\mathbf{r}_i\}) \propto \underbrace{\psi_\mathrm{J}^{(\phi_x,\phi_y)}(\{\mathbf{r}_i\})}_{\text{equivalent of a Jastrow-factor}} \; \psi_\mathrm{CF}^{(-\phi_x,-\phi_y)}(\{\mathbf{r}_i\})$$

equivalent of a Jastrow-factor

GM & N.R. Cooper, PRL 2009

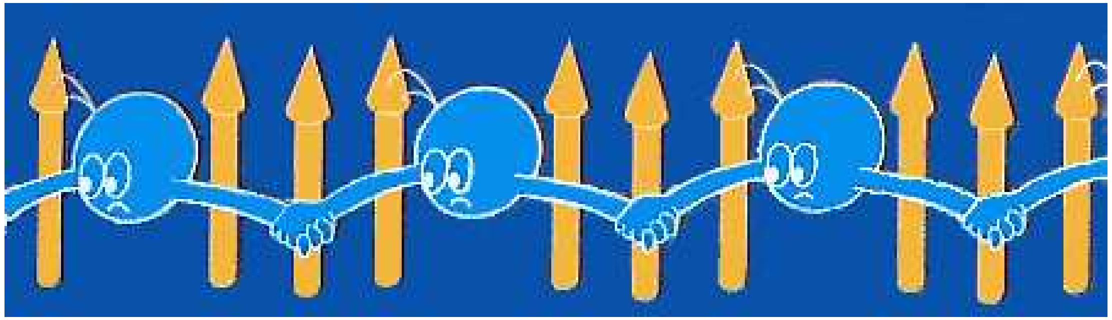
Composite Fermions filling Hofstadter bands



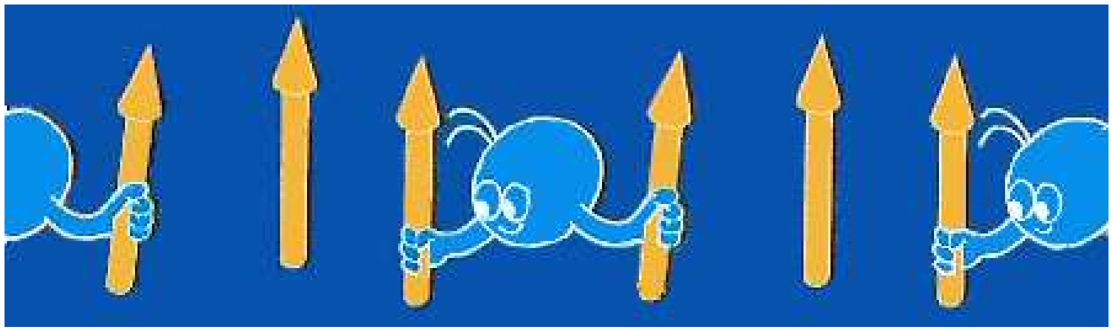
The Composite Fermion Approach

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

Account for repulsive interactions $U > 0$ by "flux-attachment" (Fradkin 1988, Jain 1989)



Continuum Landau-level for fermions at filling 1/3: three flux per particle



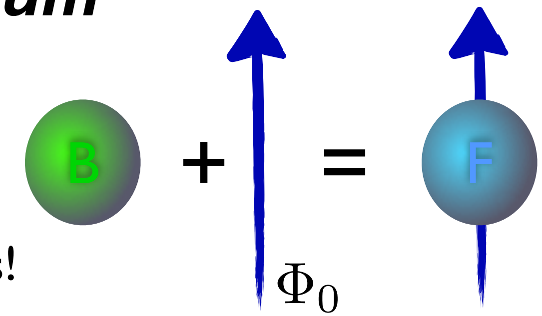
Composite fermions = electron + 2 flux quanta

$$\Psi \propto \prod_{i<j} (z_i - z_j)^2 \Psi_{\mathrm{CF}}$$

1 flux per composite particle

drawings: K. Park

Composite Fermions in the Hofstadter Spectrum



1. Flux attachment for bosonic atoms: $n_\phi^* = n_\phi \mp n$

$$\Psi_B \propto \prod_{i<j}(z_i - z_j)\Psi_{\mathrm{CF}}$$ $\Rightarrow$ transformation of statistics!

2. Effective spectrum at flux n_ϕ^* is again a Hofstadter problem

$\Rightarrow$ weakly interacting CF will fill bands, so obtain density n

by counting bands using fractal structure

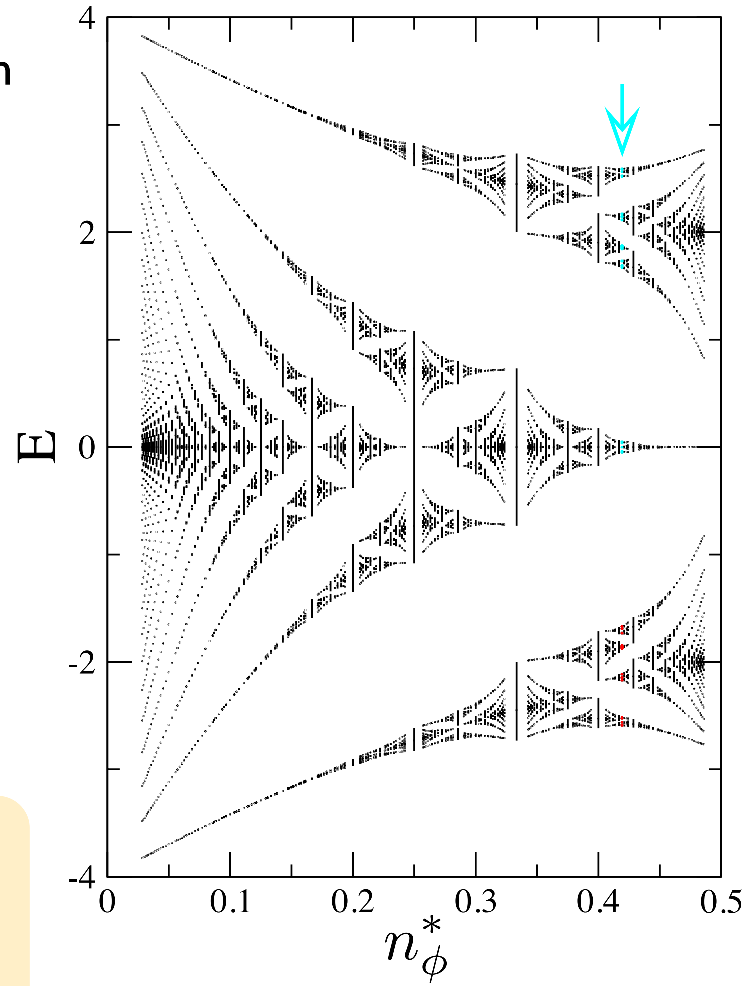
$\Rightarrow$ linear relation of flux and density for bands under a gap

$$n = \alpha n_\phi^* + \delta$$

3. Construct Composite Fermion wavefunction

continuum: $$\Psi_{\mathrm{B}}(\{\mathbf{r}_i\}) \propto \mathcal{P}_{LLL} \underbrace{\prod_{i<j}(z_i - z_j)}_{\text{Vandermonde / Slater determinant of LLL states}} \psi_{\mathrm{CF}}(\{\mathbf{r}_i\})$$

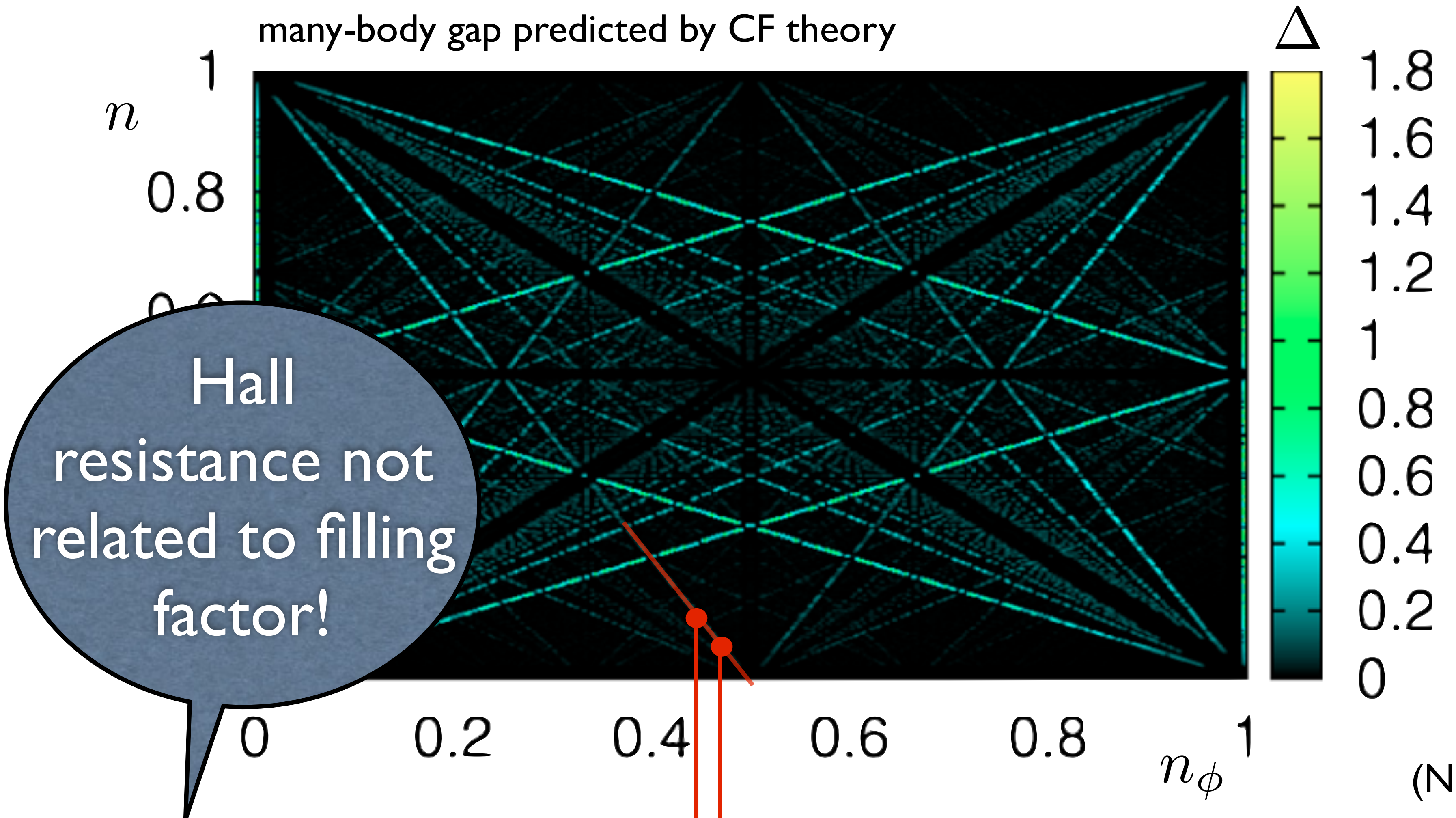
lattice: $$\Psi_{\mathrm{B}}(\{\mathbf{r}_i\}) \propto \underbrace{\psi_{\mathrm{J}}^{(\phi_x,\phi_y)}(\{\mathbf{r}_i\})}_{\text{Slater determinant of Hofstadter orbitals at flux density } n_\phi^0 = n} \psi_{\mathrm{CF}}^{(-\phi_x,-\phi_y)}(\{\mathbf{r}_i\})$$



GM & N. R. Cooper, PRL 2009

Composite Fermion Theory: Predictions & Verification

many-body gap predicted by CF theory



(N=5 particles)

testing CF states numerically at

$$n = 1 - 2n_\phi$$

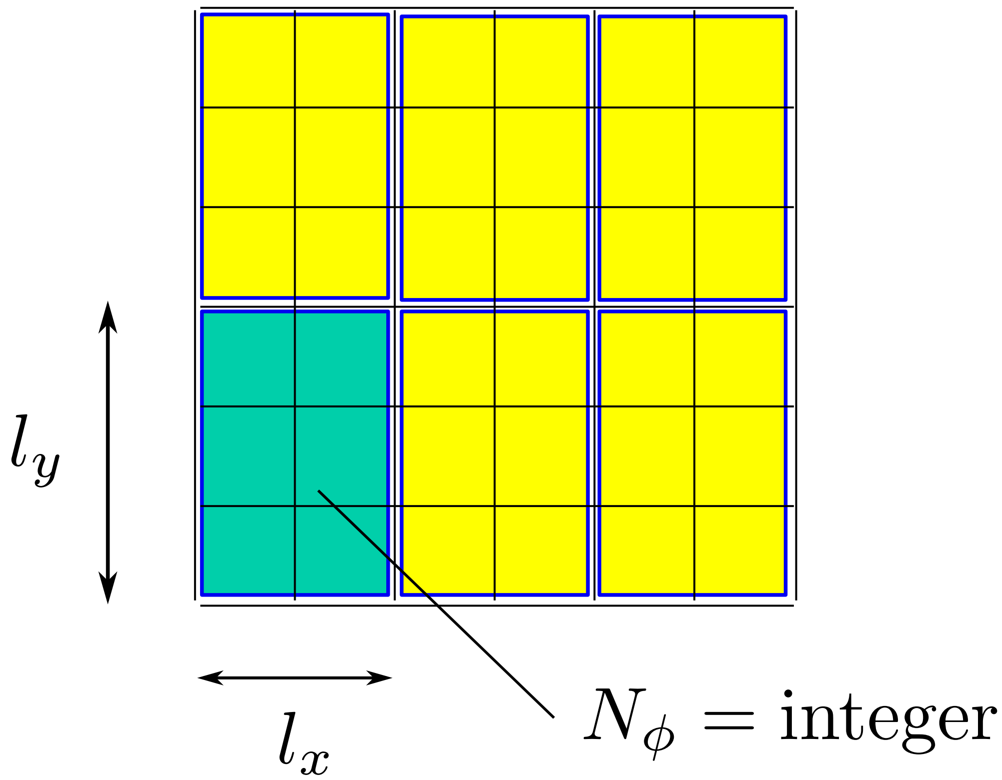
$$n = 1/9:\ \mathcal{O} = |\langle \Psi_{\rm CF} | {\rm GS} \rangle|^2 \simeq 0.46$$

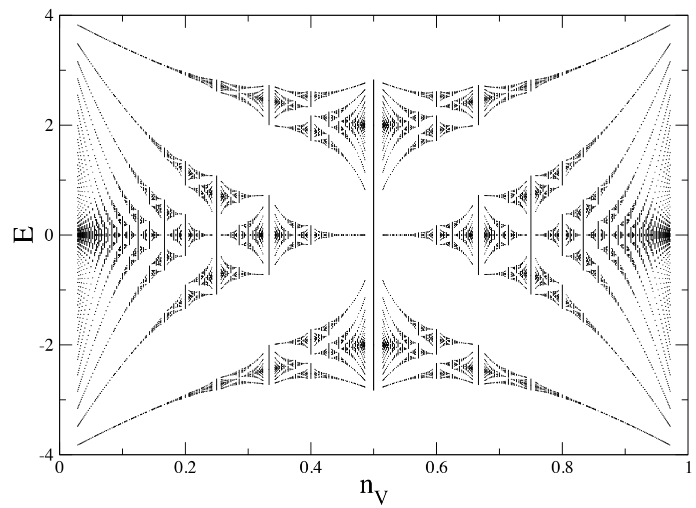
$$n = 1/7:\ \mathcal{O} = |\langle \Psi_{\rm CF} | {\rm GS} \rangle|^2 \simeq 0.56$$

⇒ *unprojected* CF state captures overall structure of correlations well (d=1.2M for result at n=1/9)

Symmetries of lattices in magnetic flux

- finite simulation cell: L_x x L_y copies of the magnetic unit cell l_x x l_y
- Hofstadter bands at $n_\phi = p/q$ is a Chern insulator with q sublattices





- overall translation group reduced to magnetic translations

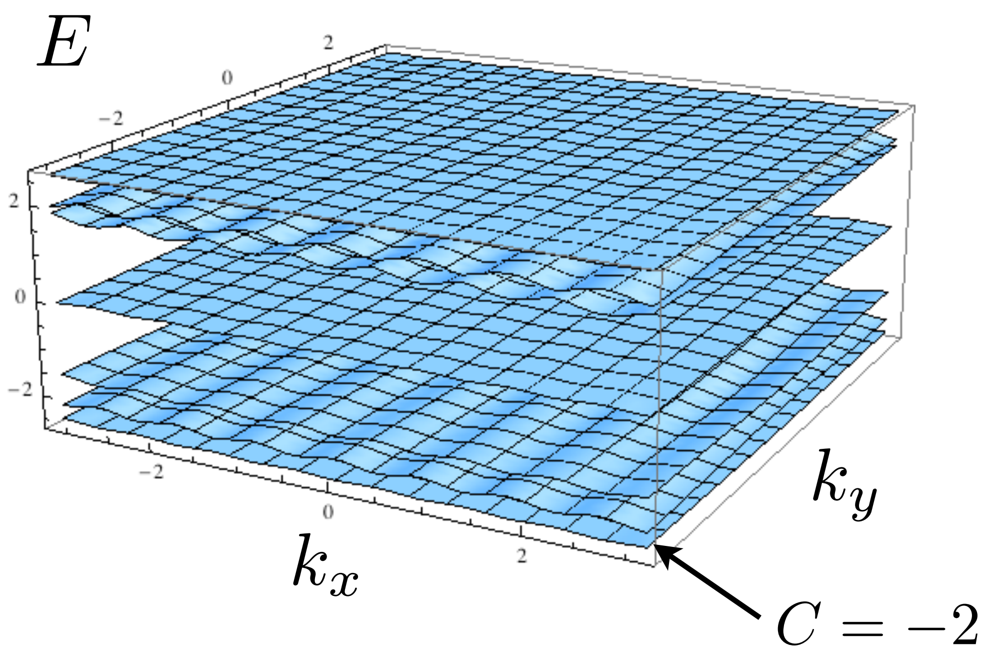
but: can still trivially translate magnetic unit cell enclosing an integer number of flux quanta

→ treat sites in magnetic unit cell as sublattices and project to resulting bands in *k*-space

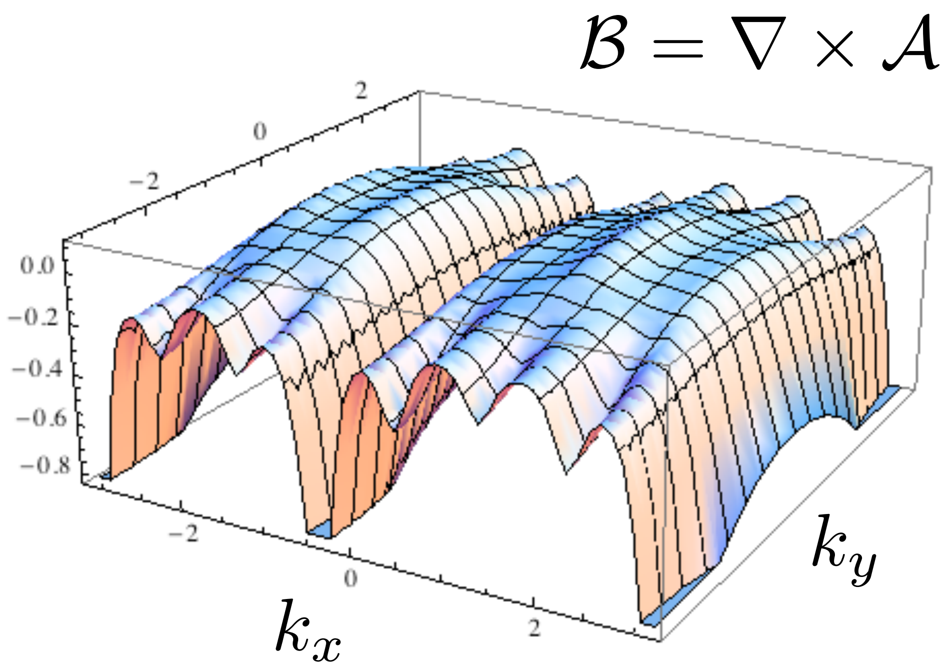
Example single particle properties

an example: Hofstadter spectrum in magnetic unit cell of 7x1, $n_\phi = 3/7$

Energy spectrum



Berry curvature (lowest band)



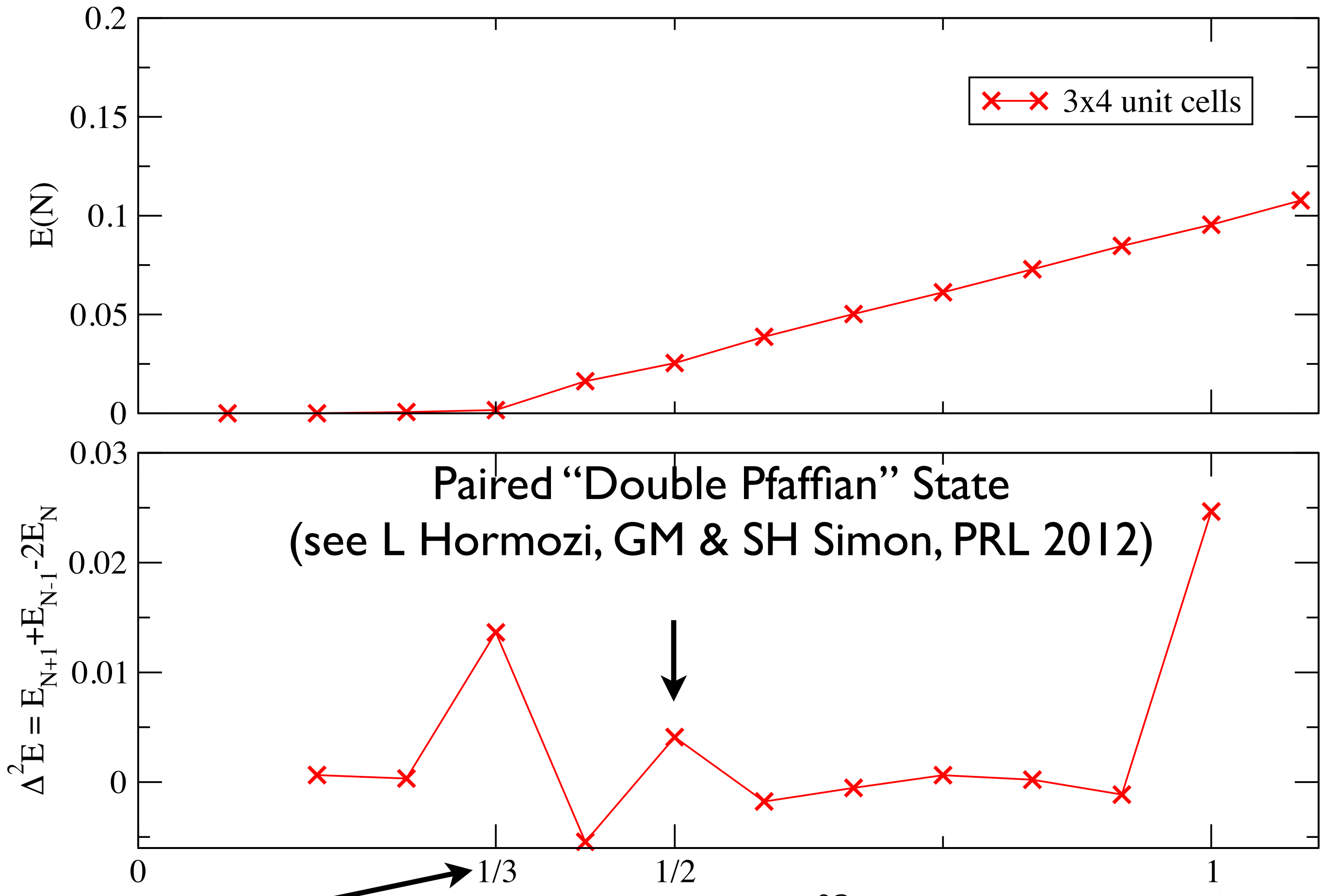
Magnetic unit cell

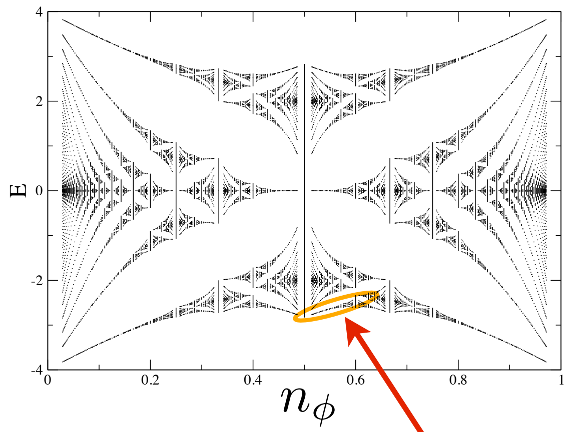
$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$-\frac{6\Phi}{7}$

Composite Fermion States in the band projected model

take projection to the Chern-# C=2 subband dominating the $n = 1 - 2n_\phi$ CF states

$n_\phi = 4/9$ for bosons, using 3x4 *k*-points (N_s=12), magnetic unit cell of 3x3 sites





C=2

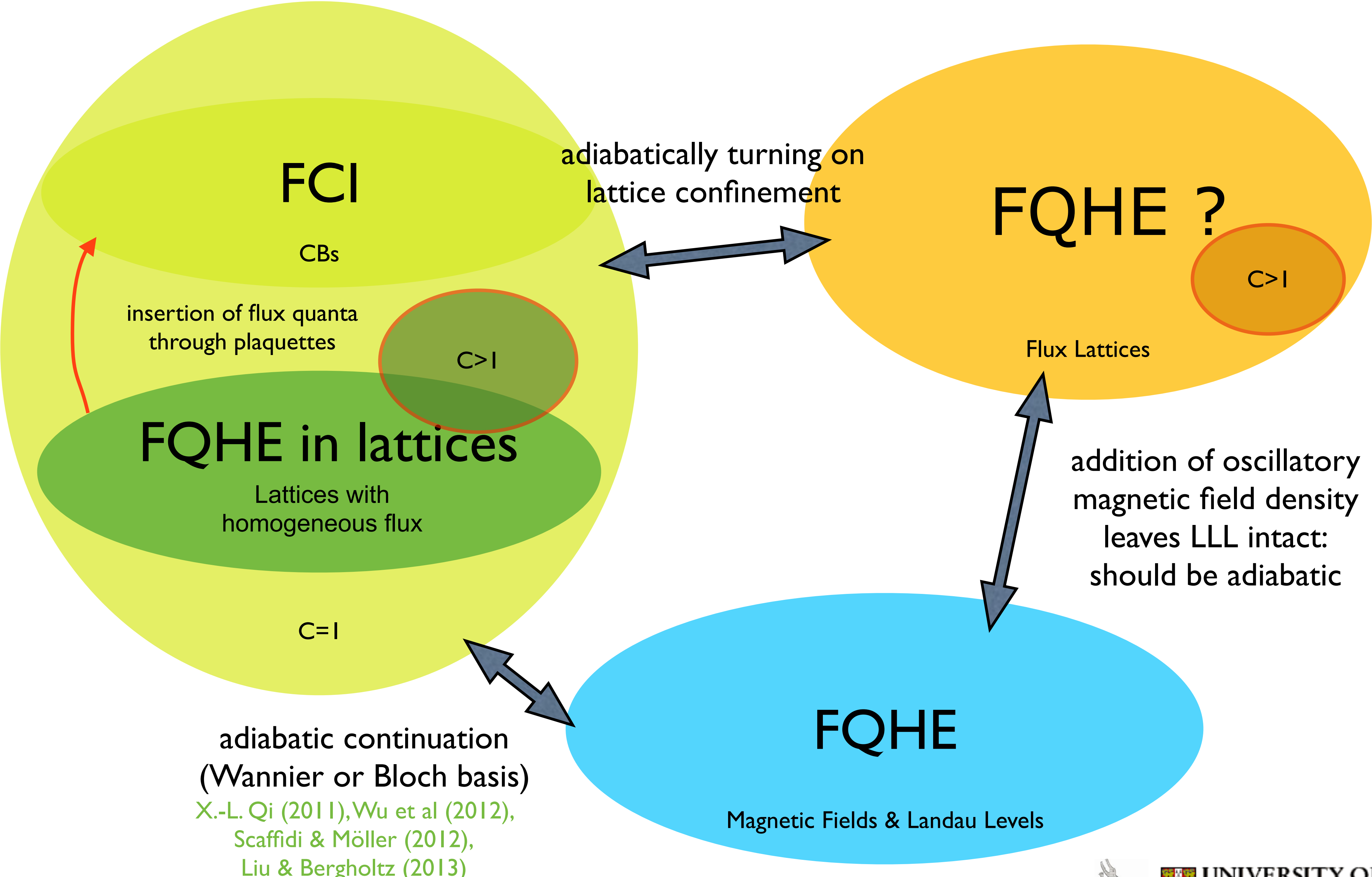
composite fermion states with both positive and negative flux attachment seen - and a paired state, too!

CF state at positive flux attachment

$$\tilde{\nu} = \frac{n}{1 - 2n_\phi}$$

CF state at negative flux attachment

New physics in Chern bands



Conclusions

- Many new systems in which FQHE-like physics of Chern-Insulators can be induced:
  - spin-orbit coupled materials
  - optical flux lattices or artifical gauge fields in cold atomic gases

- For C=1 bands, can identify quantum liquids in FCI models by analytic continuation to usual FQHE in the continuum lowest Landau level - useful tool for identifying phases

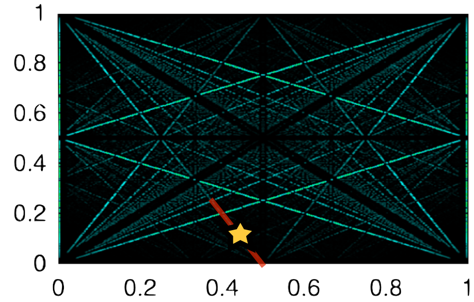
T. Scaffidi & GM, Phys. Rev. Lett. 109, 246805 (2012).

- New physics can be found in Chern bands with higher $C > 1$
  - can use picture of LL + subband / color index to describe $C > 1$ bands, but color index is not conserved!
  - many-body phases in $C > 1$ bands first analyzed for the Hofstadter spectrum:

GM & N. R. Cooper, "Composite Fermion Theory for Bosonic Quantum Hall States on Lattices", Phys. Rev. Lett. 103, 105303 (2009).

Supplementary Material

Composite Fermions or Chern Insulators



Matching the language: an example $n = 1/7,\ n_\phi = 3/7$

1) composite fermion theory

$n_\phi^* = n_\phi - n$ with filled CF bands

e.g. $n_\phi = \frac{1}{2}(1-n)$

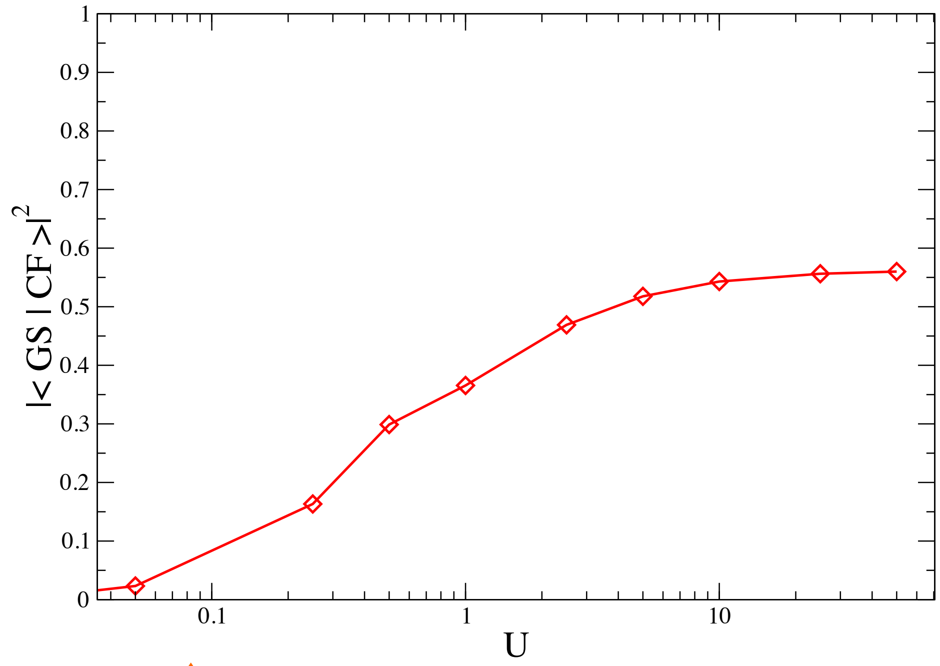
2) language of Chern insulators

e.g. $C = 2,\ \nu = 1$

3) subband projection:

$\nu_{\text{eff}} = n/\epsilon$ e.g. $\nu_{\text{eff}} = 2$

overlaps with CF state (no projection)



Data for N=5 particles on a lattice with 7x5 sites
Magnetic unit cell: 3 flux per 7 sites = 7x1 sites
Repetitions of magnetic unit cell: 1x5 unit cells

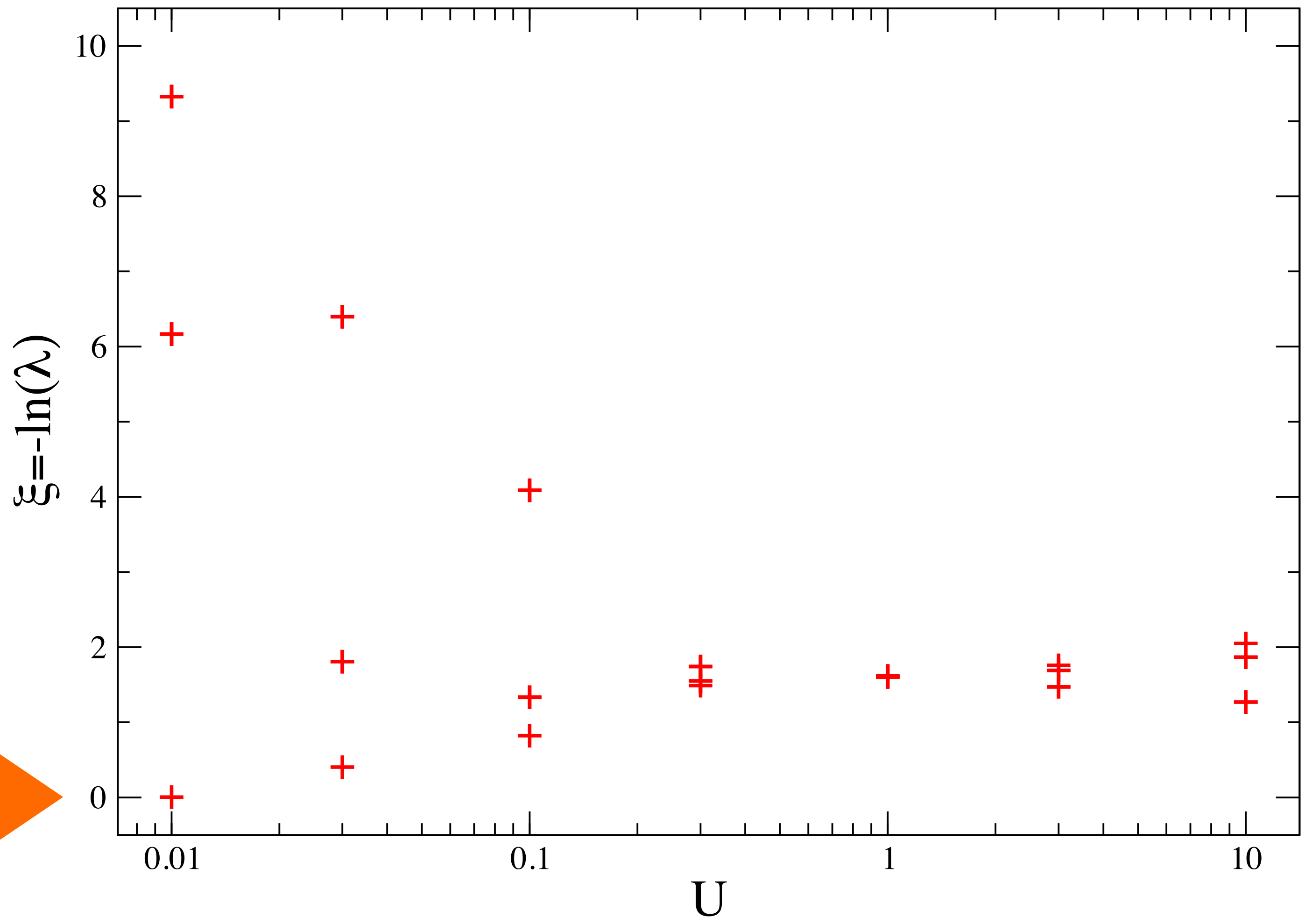
other order at small U

CF state good at large U

Groundstate properties: Single particle density matrix

Eigenvalues of $(\rho_1)_{ij} = \langle \hat{c}_i^\dagger \hat{c}_j \rangle$ (N=5, band projected, not flattened)

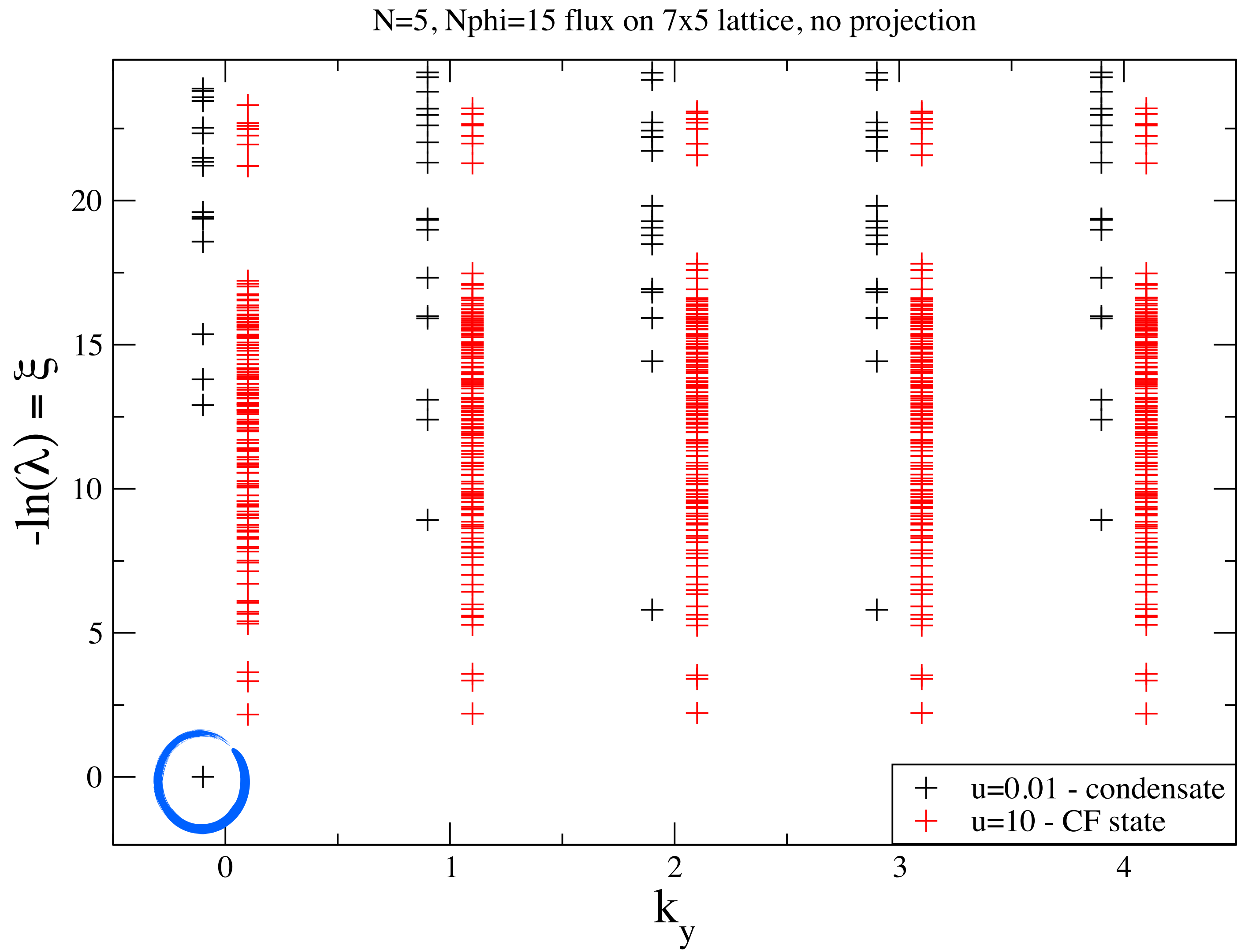
99% weight in single state!



Single particle density matrix (= PES with N_A=1) reveals state at small U as a condensate!

Groundstate: PES at $N_A=3$

$n_\phi = 3/7$ N=5



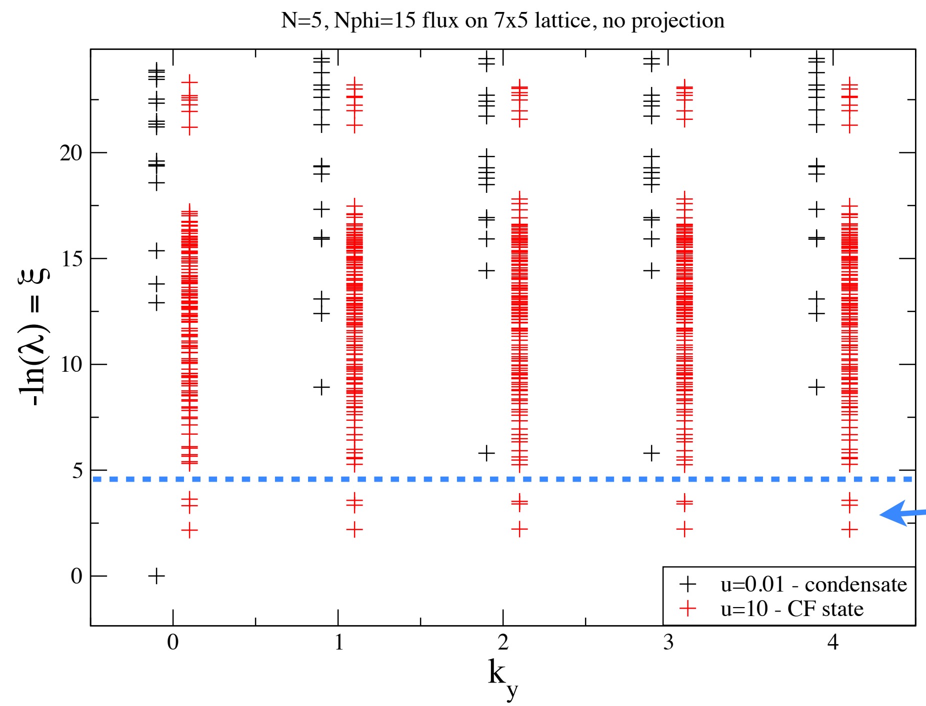
PES with $N_A=3$ confirms state at small U as a condensate: state dominated by single Schmidt EV!

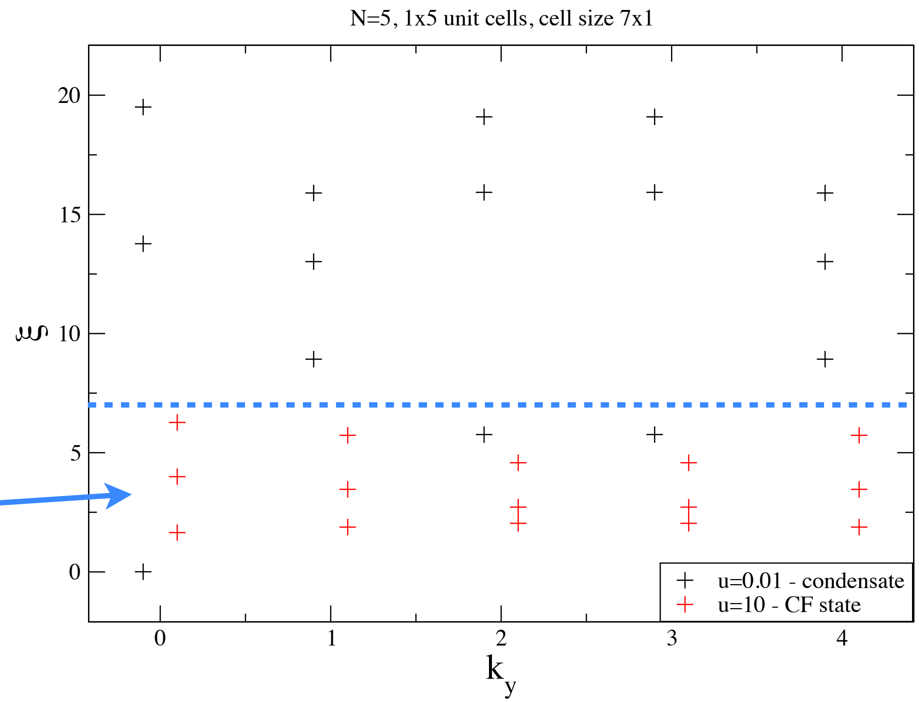
Groundstate: PES at N_A=3

$n_\phi = 3/7$ N=5

No Projection to lowest band

With Projection to lowest band



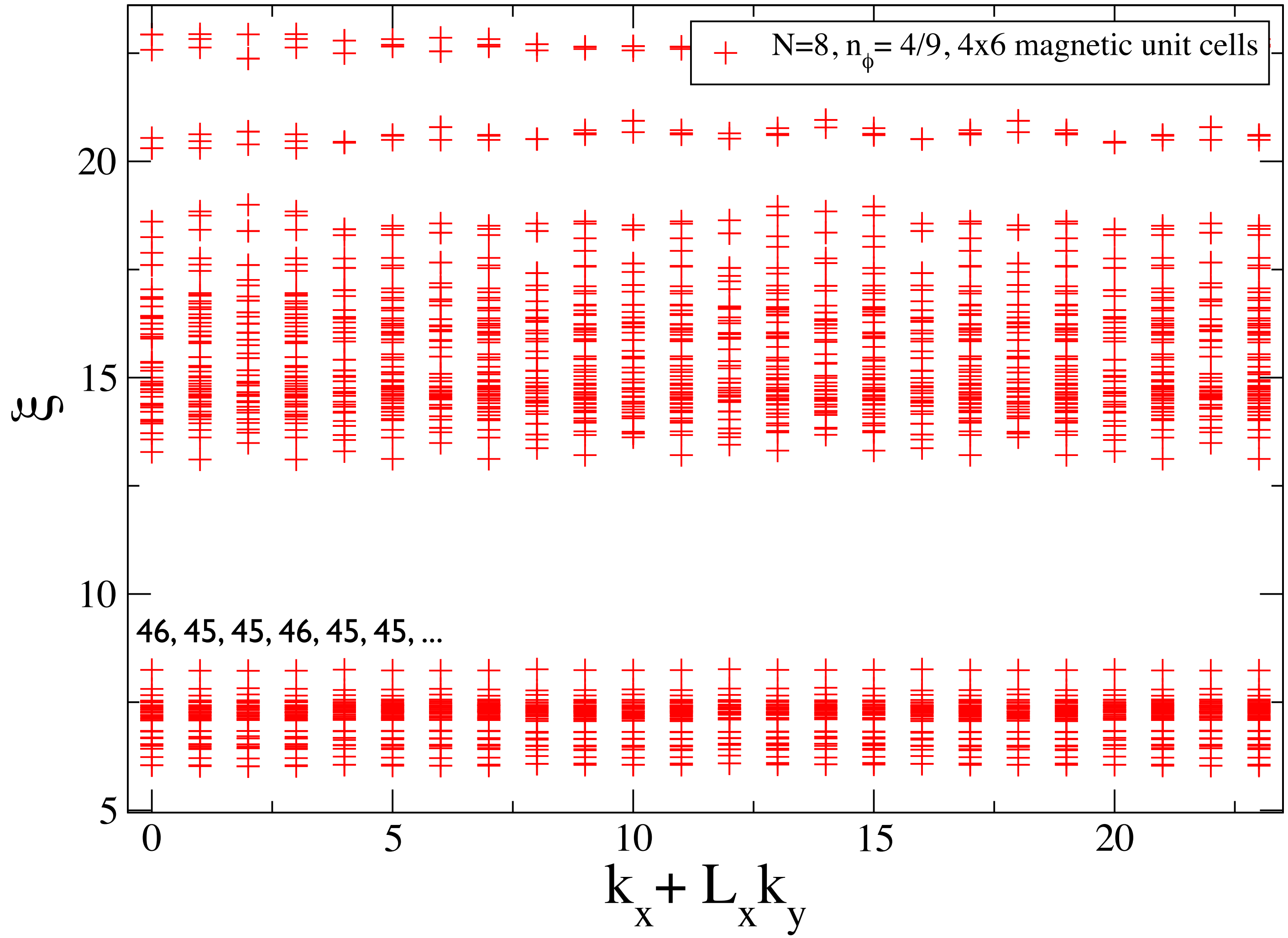


PES counting at low entanglement energies: groundstate in band projected matches unprojected model

Suggests: Hard-Core limit = Flat Band Limit! (n=1/7 small)

PES for larger system sizes: ν=1/3 - C=2 band

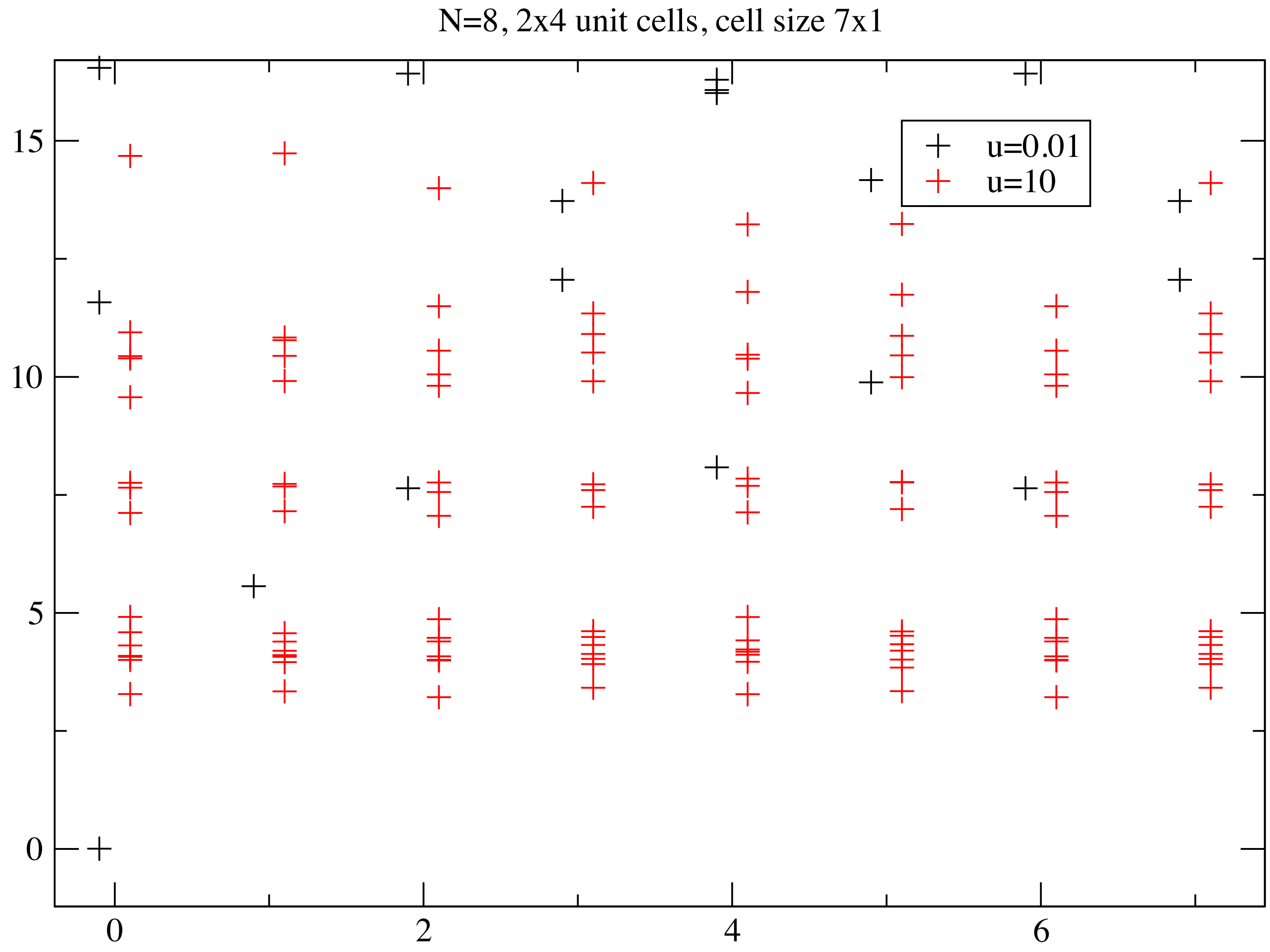
$n_\phi = 4/9$ N=8, band projected, d_{GS}=3, positive flux CFs: $n_\phi = \frac{1}{2}(1-3n)$



3 x degenerate ground state at ν=1/3: its PES shows a clear entanglement gap

PES for larger system sizes: ν=1 - filled C=2 band

$n_\phi = 3/7$ N=8, band projected, d_{GS}=1, CF state at neg flux: $n_\phi = \frac{1}{2}(1-n)$



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