Correlated Phases of Bosons in the Flat Lowest Band of the Dice Lattice

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Outline

• Introduction: Artificial gauge fields and strongly correlated states on lattices

• Lattice geometry as a key to Hofstadter spectra: square vs dice lattice

• Effective low-energy model of bosons on the dice lattice

• Many-body phases in the resulting flat-band model:
  ‣ Gross-Pitaevski mean-field approach
  ‣ Numerical study of phases at low particle density

• Some thoughts about localised states in flat bands
Artificial Gauge Fields on Optical Lattices

experimental realisation:
optical lattice + Raman lasers
⇒ possibility to simulate effect of magnetic field by
imprinting phases for hopping (alternatively: Berry phases)

\[ \sum A_{\alpha \beta} = 2\pi n_V \]

⇒ Bose-Hubbard with a magnetic field (force de Lorentz)

\[ \mathcal{H} = -J \sum_{\langle \alpha, \beta \rangle} \left[ \hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha \beta}} + h.c. \right] + \frac{1}{2} U \sum_{\alpha} \hat{n}_\alpha (\hat{n}_\alpha - 1) - \mu \sum_{\alpha} \hat{n}_\alpha \]

particle density \( n \)    vortex/flux density \( n_V \)    interaction \( U/J \)

Many-body states of bosons on the square lattice

Quantum Hall Regime

- driven by onsite repulsion $U$
- prominent states from the continuum limit: Laughlin state, Jain composite fermion states, Moore-Read state
- novel QH states stabilized by the lattice

Bose Condensates with Vortex Lattices

- always exist for weak repulsion $U$
- can survive to large $U$
- can be thought of as vortex lattices
- states break discrete lattice symmetries
- experimental realization: I. Bloch

Sørensen et al. PRL 2006
Palmer & Jaksch, PRL 2006,
GM & N.R. Cooper, PRL 2009

GM & N.R. Cooper, PRA 2010
Đurić & Lee PRB (2010), ...
Single particle spectrum as the hidden key

square lattice at flux density $n_V$

narrow bands with a gap $\rightarrow$ novel Fractional Quantum Hall states beyond those known in the continuum!

GM & N.R. Cooper, PRL 2009; PRA 2010

narrow band (lowest Landau level) $\rightarrow$ Fractional Quantum Hall states

$n_V=1/2$: wide bands $\rightarrow$ condensates
Lattice geometry as the key to Hofstadter spectrum

’n’-lattice at flux density \( n_{V} \)

Q1: How to generate this model?

Q2: What are the many-body phases of interacting bosons in this flat band?

Vidal, Mosseri, Douçot, PRL (1998)
Realizing a dice lattice with cold atoms

- three standing waves at 120° at antimagic wavelength, e.g. for Yb (mutually incoherent)

- use Raman lasers to drive hoppings between neighboring sites of different species → dice-lattice geometry
Realizing a dice lattice with cold atoms

- three standing waves at 120° at antimagic wavelength, e.g. for Yb (mutually incoherent)
Realizing a dice lattice with cold atoms

- one more laser at antimagic wavelength to break symmetry within magnetic unit cell
Realizing a dice lattice with cold atoms

- a total of 8 Raman lasers is required to drive the transitions between different sublattices

Raman lasers to drive transitions at different detunings
Raman transitions for the Dice lattice set-up

Choice of gauge $A_{ij}$: implementation of Raman driven transitions

S1: standing wave to break mirror symmetry

P1...P6: propagating perpendicular to plane (P3 phase shifted)

L1, L2: propagating with in-plane momentum
Hamiltonian & Single particle wavefunctions

\[ \mathcal{H} = -t \sum_{\langle j, \mu \rangle} (\hat{a}_{\mu}^\dagger \hat{a}_j e^{iA_{\mu j}} + h.c.) \]

1. Choose a gauge \( A_{\mu j} \)
   
   Note 1: can choose \( \mathcal{H} \) real!
   
   Time-Reversal Symmetric
   
   Note 2: magnetic unit cell 
   has 6 sites at flux density 1/2

2. Understand SP states
   
   “Aharonov-Bohm cages”
   Vidal, Mosseri, Douçot, PRL (1998)

Next: add Interactions!
Assume onsite interactions 
can have \( U_6 \neq U_3 \)
**Effective dynamics and projection to lowest band**

Weak interactions: \( nU/t \ll 1 \)

Project to lowest band!

Matrix elements:

\[
V_{ijkl} = U_3 \sum_{\mu} \phi^*_i(\mu)\phi^*_j(\mu)\phi_k(\mu)\phi_l(\mu) + U_6 \sum_{q} \phi^*_i(q)\phi^*_j(q)\phi_k(q)\phi_l(q)
\]

\(i, j, k, l\) label orbitals in lowest band, localized around the sites of a triangular lattice.
Resulting model: “Activated Hopping”

- projected model lives on triangular lattice!

\[ H_{\text{proj}} = \gamma_1 \sum_i \hat{n}_i(\hat{n}_i - 1) + \gamma_2 \sum_{\langle i,j \rangle} \left[ \hat{n}_i \hat{n}_j + \hat{c}_i^\dagger \hat{c}_j^2 + \hat{c}_j^\dagger \hat{c}_i^2 \right] + \gamma_3 \sum_{\Delta(i,j,k)} \left[ \sigma_{kk}^{ij} \hat{c}_i^\dagger \hat{c}_j \hat{c}_k^2 + \sigma_{jk}^{ik} \hat{c}_i^\dagger \hat{c}_j \hat{n}_k + \text{h.c.} \right] \]

where: \[ \gamma_1 = \frac{1}{4} U_6 + \frac{1}{24} U_3, \quad \gamma_2 = \frac{1}{72} U_3, \quad \gamma_3 = \frac{1}{144} U_3 \]
Resulting model: “Activated Hopping”

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- Projected model has finite range interactions
- Time reversal symmetric
- All interactions are of density-density type
- Can tune ratio of onsite terms \( \gamma_1 \) to activated hopping terms \( \gamma_{2,3} \) via the free parameter \( u = U_6/U_3 \)
Many-Body Physics of the Dice Lattice Model

1. Gross-Pitaevski Mean Field Theory

\[ |\Psi\rangle = \exp \left\{ \sum_j \alpha_j \hat{c}_j^\dagger \right\} |\text{vac.}\rangle \]

2. Exact Numerical Diagonalization

\[ \mathcal{H} |\Psi\rangle = E |\Psi\rangle \]

\[ |\Psi\rangle = \sum_\alpha \kappa_\alpha \prod_{i=1}^{N\text{orb}} (\hat{c}_i^\dagger)^{n_i(\alpha)} |\text{vac.}\rangle \]

› Note: \( i,j \) label orbitals in lowest band
Gross-Pitaevski Mean-Field Theory I

• Regime of validity for Ansatz

\[ |\Psi\rangle = \exp \left\{ \sum_j \alpha_j \hat{c}^\dagger_j \right\} |\text{vac.}\rangle \]

  ▸ Large density \( n \gg 1 \)

  ▸ Remain in projected model \( nU \ll t \)

  ▸ Grand canonical ensemble: introduce chemical potential \( \mu \)

• Minimize expectation value of energy given \( \mu \)

\[ \langle H \rangle = \sum_{ijkl} V_{ijkl} \alpha_i^* \alpha_j^* \alpha_k \alpha_l - \mu \sum_j |\alpha_j|^2 \]

  ▸ Note: \( i,j,k,l \) label orbitals in lowest band
Gross-Pitaevski Mean-Field Theory II

- Groundstates are vortex lattices when expanded on full dice lattice (all sites)
- Density: distinct values on 6-fold hubs and 3-fold rims
  
  \[
  \text{for } u = 1: \ n_6 = 2n_3
  \]
- Phases correspond to groundstate solutions of fully-frustrated \(xy\)-model on the dice lattice
  
  

\(\pi\)-Flux vortex lattices with no more than three neighboring plaquettes with same sign of vorticity
Gross-Pitaevski Mean-Field Theory II

- More examples of groundstate vortex patterns

**Low density limit**

\[ \mathcal{H}_{proj} \text{ does not induce any dynamics for particles which are not nearest neighbors} \]

- Densest packing of particles in lowest band yields 3x deg. crystalline groundstate at band filling \( \nu = 1/3 \)

\[ |\Psi_c\rangle = \prod_{n,m} \hat{c}^\dagger [\vec{r}_t + n\vec{u}_1 + m\vec{u}_2]|\text{vac.}\rangle \]
Correlated regimes at intermediate particle density?

- How do fluctuations affect the highly degenerate groundstate?
- How do fluctuations disorder the crystalline phases at small $n$?

ED, using limit of large $u \gg 1$

- analyse different finite size lattices with periodic boundary conditions
- vary band filling $\nu = 3n$

unidentified phase at $\nu = 1/2$

unstable to phase separation

Mott Insulator

Crystal
Characteristics of the phase at filling 1/2

- Reminder: projection yields triangular lattice

- Projection reduces system size to 1/3 of original number of sites.
- Can achieve system size of 16 particles on 32 sites for (hard-core) bosons at half filling, corresponding to 96 sites of the dice lattice!
Characteristics of the phase at filling 1/2

- Particle-Particle Correlations (exact groundstate, 4x2 unit cells)

- symbol size \( \sim \)
  
  \[ \langle \Psi | \hat{n}(\vec{r}) \hat{n}(\vec{0}) | \Psi \rangle \]

- correlations indicate crystalline long-range order with basis
  
  \( (2\eta_1, 2\eta_2) \)
Characteristics of the phase at filling 1/2

- Break translational symmetry explicitly, taking superposition of four lowest-lying eigenstates $|S\rangle$

- Symbol size $\sim$

$$\langle S|\hat{n}(\vec{r})|S'\rangle$$

$$|S\rangle = \sum_{i=1}^{4} c_i |\Psi_i\rangle$$

- Coefficients: see next slide

- Confirms crystalline order anticipated from correlations $(2\vec{n}_1, 2\vec{n}_2)$
Spectrum under twist of boundary conditions

- finite spin stiffness is indicative of a superfluid component
Condensate fraction at filling $1/2$?

- Construction of state $\left| S \right\rangle$ : maximize largest eigenvalue of single-particle density matrix $\rho_{ij} = \langle \hat{c}_i^\dagger \hat{c}_j \rangle$

- blue dot size ~ particle fluctuations

\[ \langle S | \hat{n}^2(\vec{r}) | S \rangle = \bar{n}^2 \]

- red arrows: magnitude and phase of largest eigenvector of $\rho = \text{condensate wavefunction}$

- Suggests simultaneous presence of both crystalline and superfluid fraction!
Finite Condensate fraction in thermodynamic limit?

- Examine finite size scaling of condensate fraction

- confirms the presence of a supersolid ordered phase in the half filled band! Experimental detection: in-situ / expansion
Conclusions I

- Dice lattice is an attractive model to study many-body physics in flat, time-reversal symmetric bands
- The model has a rich phase diagram, including:
  - Crystals, Mott Insulators
  - Supersolids, intricate degenerate vortex lattice phases
- Activated hopping processes emerge as a generic feature of topologically trivial flat band models, and drive the formation of this complex phase diagram

Differences between TRI and Chern bands

- Maximally localized wavefunctions in bands with Chern numbers $C > 0$ are more extended

$C = 0$

$C = 2$

graphics: Wang & Ran, PRB (2011)
Hybrid Wannier functions in Chern bands

- To visualise, think about localising wavefunctions in one direction, first

\[ |W(x, k_y)\rangle = \sum_{k_x} f^{(x, k_y)}_{k_x} |k_x, k_y\rangle \]

- Increase in position for \( k_y \rightarrow k_y + 2\pi \)

= Chern-number \( C \)
Hybrid Wannier functions in Chern bands

- Fully localized state mixes different momenta $k_y$ and thus positions $\langle x \rangle$

\[ |\tilde{W}(x, y)\rangle = \sum_{k_y} g_{k_y} |W(x, k_y)\rangle \]

- Increase in position for $k_y \rightarrow k_y + 2\pi$
  \[ = \text{Chern-number} \ C \]

- Fully localized Wannier states have an extent of ca. $(C+1)$ unit cells
Advert: Analytic Continuation between FQHE and FCI

- Can use single particle Wannier states to construct an analytic continuation between incompressible quantum liquids in C=1 Chern bands (fractional Chern insulators) and the fractional QHE

- for a numerical study of fractional Chern insulators in the hybrid Wannier basis, see Th. Scaffidi & GM, PRL 109, 246805 (2012).