Geometric stability of topological lattice phases

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arxiv:1408.0843

Advanced Numerical Algorithms for Strongly Correlated Quantum Systems, Universität Würzburg, February 2015

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Overview

• Background: Topology and interactions in tight-binding models

• The role of band geometry in the Single Mode Approximation to quantum Hall liquids / fractional Chern insulators

• Role of band geometry for incompressible Hall liquids

→ screening of three target models

Quantum Hall Effect in Periodic Potentials

- quantized Hall response in filled bands:

$$\sigma_{xy} = \frac{e^2}{h} \sum_{\text{filled bands}} C_n$$

Thouless, Kohmoto, Nightingale, de Nijs 1982

- Chern-number for periodic systems

$$C = \frac{1}{2\pi} \int_{BZ} d^2\mathbf{k}\, \mathcal{B}(\mathbf{k})$$

- Hofstadter model (solved 1976): *tight-binding* model for electrons in bands with finite Chern number

$$\mathcal{H} = -J \sum_{\langle\alpha,\beta\rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.\right]$$

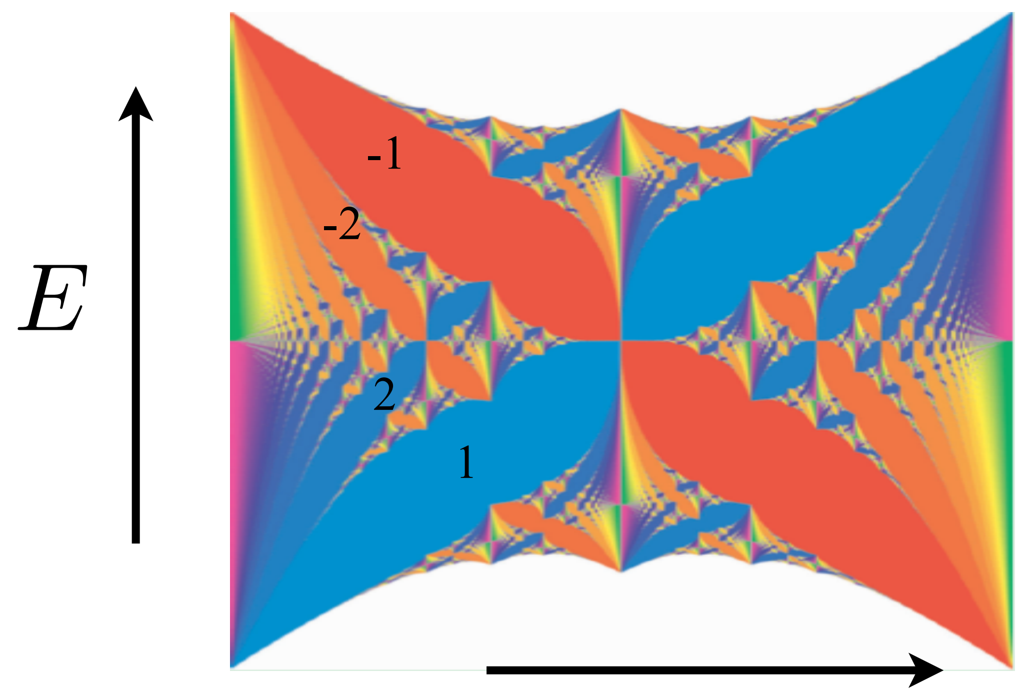
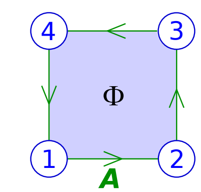


figure: Avron *et al. (2003)* n_ϕ



$$\sum_{\square} A_{\alpha\beta} = 2\pi n_\phi$$

- the Hofstadter spectrum provides bands of all Chern numbers
- filled bands in this spectrum yield a quantized Hall response

Fractional Quantum Hall Effect in Periodic Potentials

- quantized Hall response in *partially* filled bands?
- THEORY: Kol & Read (1993)

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \sum V_{ij} \hat{n}_i \hat{n}_j$$

PHYSICAL REVIEW B VOLUME 48, NUMBER 12 15 SEPTEMBER 1993-II

Fractional quantum Hall effect in a periodic potential

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(Received 28 May 1993)

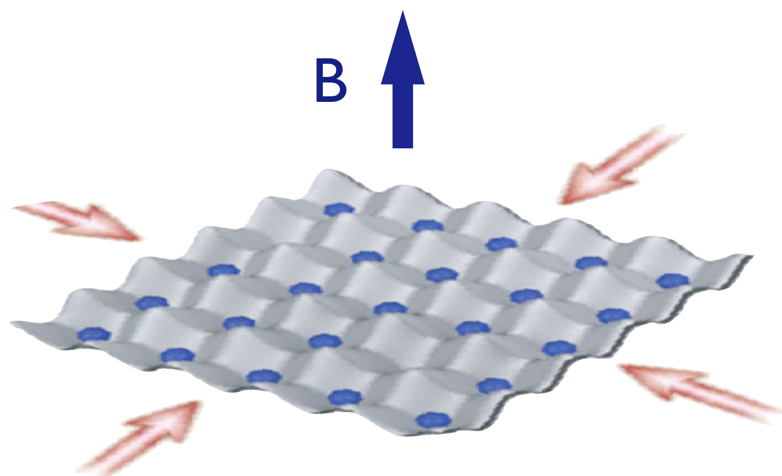
The fractional quantum Hall effect in a periodic potential or modulation of the magnetic field is studied by symmetry, topological, and Chern-Simons field-theoretic methods. With periodic boundary conditions, the Hall conductance in a finite system is known to be a fraction whose denominator is the degeneracy of the ground state. We show that in a finite system, translational symmetry predicts a degeneracy that varies periodically with system size and equals 1 for certain commensurate cases which we argue are physically representative. However, this analysis may overlook gaps due to finite-size effects that vanish in the thermodynamic limit. This possibility is addressed using a fermionic Chern-Simons field theory in the mean-field approximation. In addition to solutions describing the usual Laughlin or Jain states whose properties are only weakly modified by the periodic background, we also find solutions whose existence depends on the presence of the background. In these incompressible states, the Hall conductance is a fraction not equal to the filling factor, and its denominator is the same as that of the fractional charge and statistics of the elementary quasiparticle excitations.

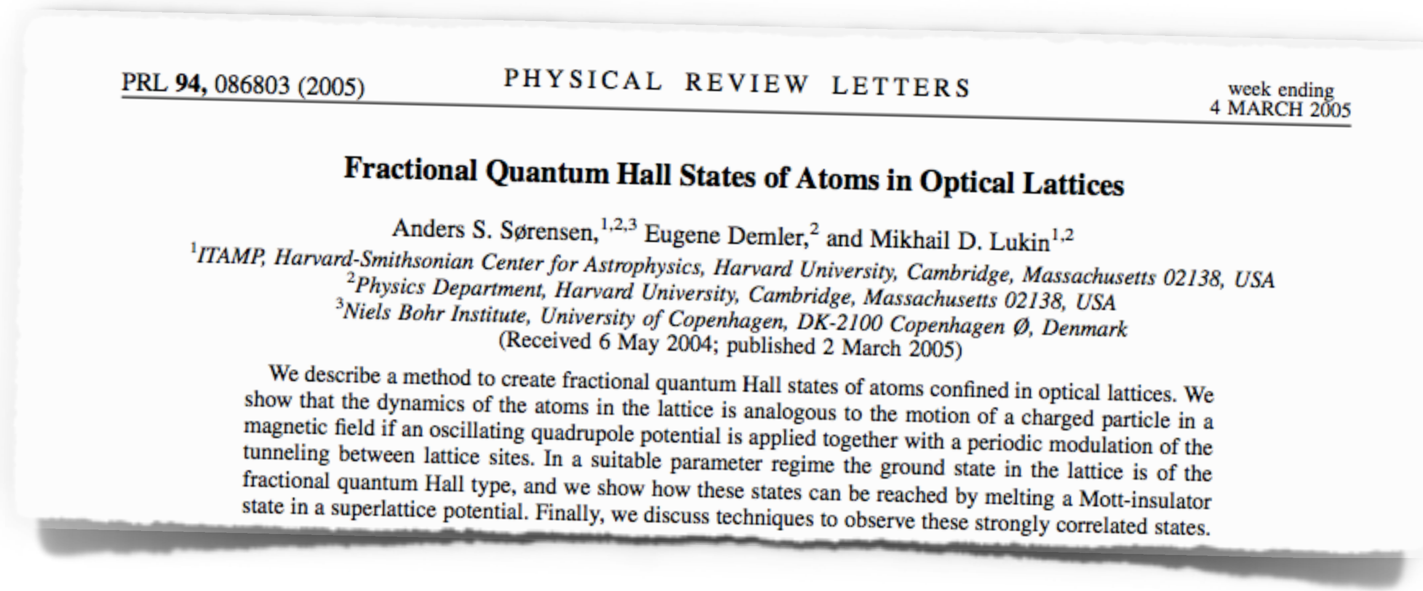
- Confirmations for such states?

Fractional Quantum Hall on lattices: Numerical Evidence

- interest in cold atom community 2000's:

- realisations of tight-binding models with complex hopping from light-matter coupling:





PRL **94**, 086803 (2005) PHYSICAL REVIEW LETTERS week ending 4 MARCH 2005

Fractional Quantum Hall States of Atoms in Optical Lattices

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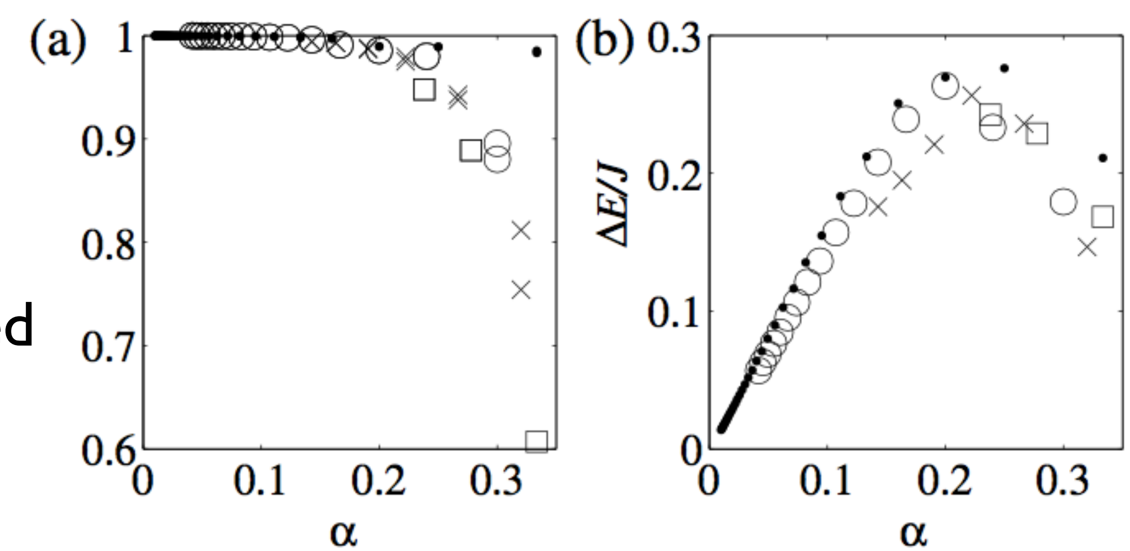
(Received 6 May 2004; published 2 March 2005)

We describe a method to create fractional quantum Hall states of atoms confined in optical lattices. We show that the dynamics of the atoms in the lattice is analogous to the motion of a charged particle in a magnetic field if an oscillating quadrupole potential is applied together with a periodic modulation of the tunneling between lattice sites. In a suitable parameter regime the ground state in the lattice is of the fractional quantum Hall type, and we show how these states can be reached by melting a Mott-insulator state in a superlattice potential. Finally, we discuss techniques to observe these strongly correlated states.

$$\mathcal{H} = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1)$$

- bosons with onsite U: many-body gap in the half-filled "synthetic Landau-level" persists to large flux density



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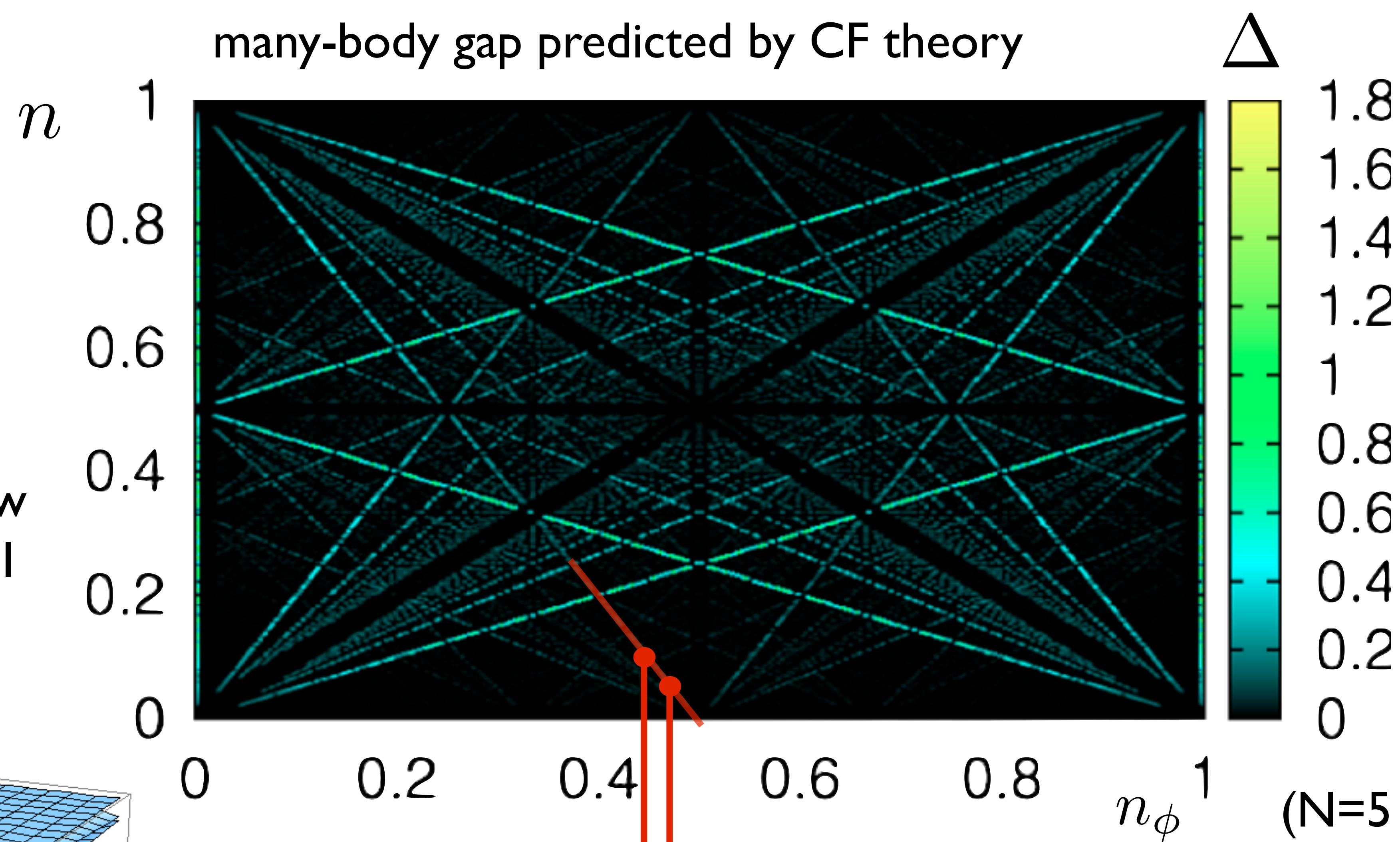
Fractional Quantum Hall on lattices with higher Chern-# bands

- bands of the Hofstadter model go *beyond* the continuum limit and support *new classes* of quantum Hall states

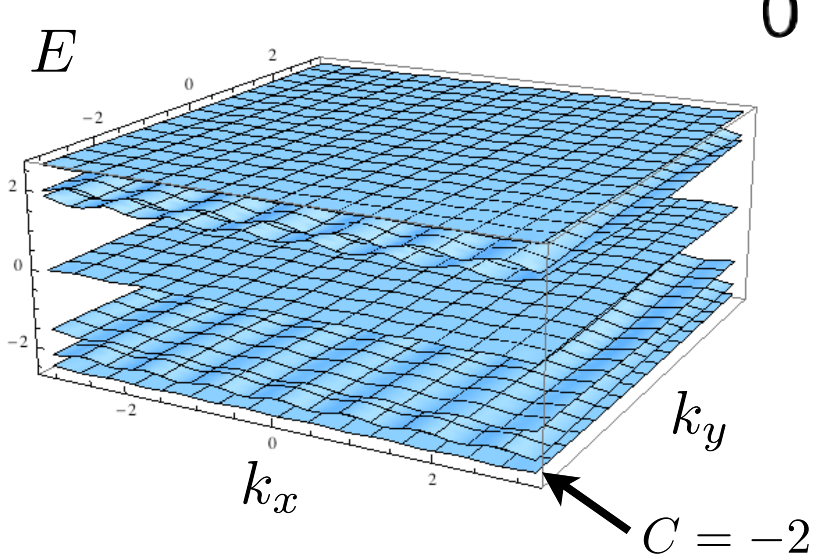
theory:
bosonic Hall states on the lattice

numerical verification
for what we would now call FCI states with ν=1
- C=2 band
- hardcore bosons



$n = 1/9$: $\mathcal{O} = |\langle \Psi_{\rm CF} | {\rm GS} \rangle|^2 \simeq 0.46$

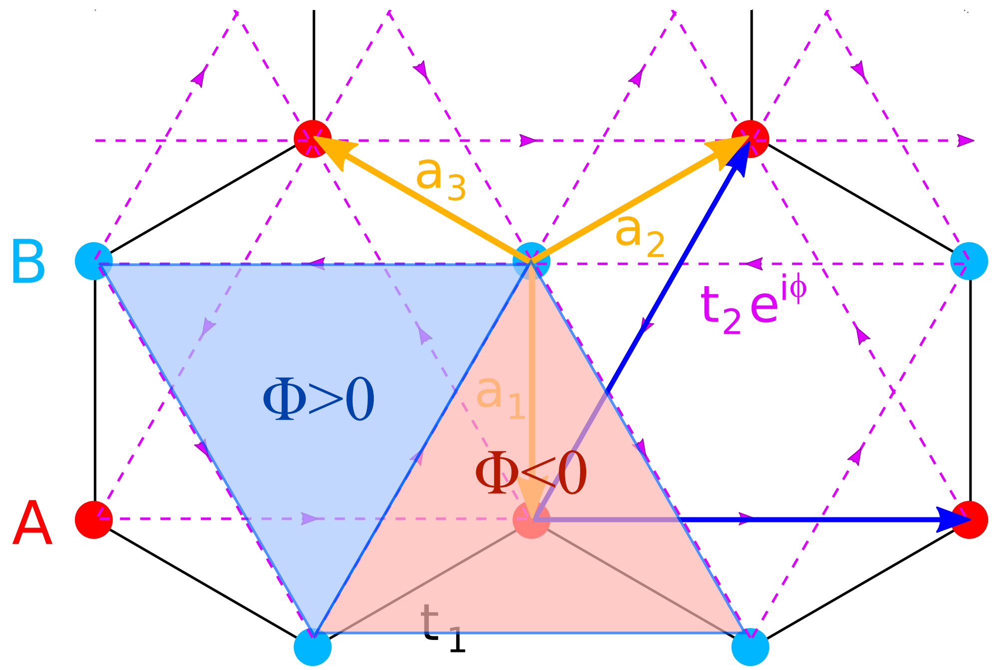
$n = 1/7$: $\mathcal{O} = |\langle \Psi_{\rm CF} | {\rm GS} \rangle|^2 \simeq 0.56$

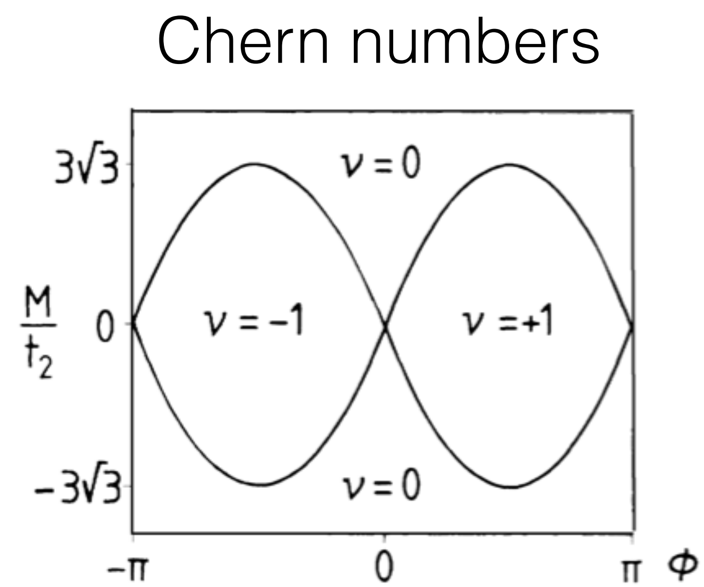


GM & NR Cooper, PRL 2009

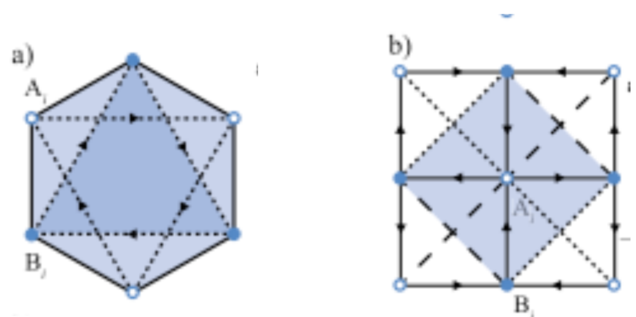
Chern bands in more general tight binding models

- Original proposal for IQHE without magnetic fields: Haldane (1988)

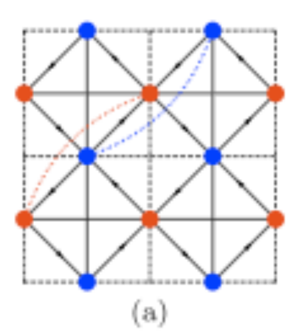




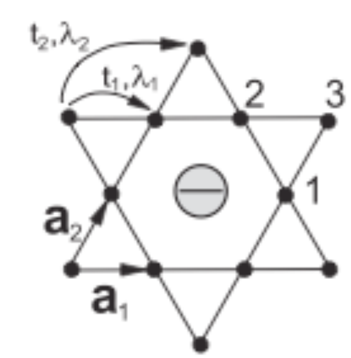
- 2011: FQHE could naturally in models with spin-orbit coupling + interactions



T. Neupert et al.



K. Sun et al.



E. Tang et al.

Numerical confirmation: D. Sheng; C. Chamon; N. Regnault & A. Bernevig, …

Stability of Fractional Chern Insulators

How to decide which lattice models have stable fractional Chern Insulators?

- single-particle dispersion - want flat bands

  many groups

  finite size matter a lot - success by iDMRG A. Grushin et al.

- shape of interactions - clear hierarchy of two-body energies desirable "Pseudopotentials"

  Läuchli, Liu, Bergholtz, Moessner + other proposals

- band geometry - ideally want even Berry curvature

  Regnault, Bernevig; Dobardzic, Milovanovic, …

  no systematic in-depth study of geometric measures

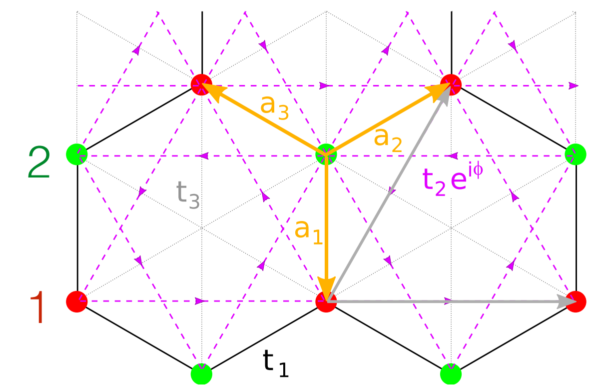
This Talk!

- Full story: all three aspects contribute

Band Geometry: Berry Curvature and Chern Number

Basic notations:

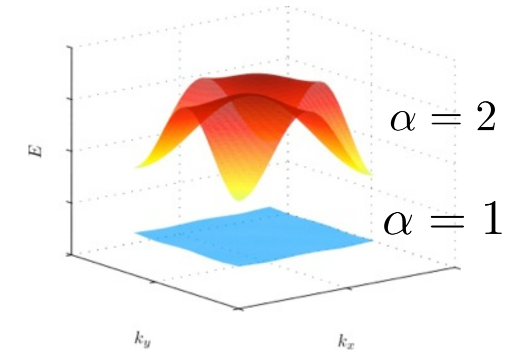
Fourier transform: $$|\mathbf{k}, b\rangle = \frac{1}{\sqrt{N_c}} \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot(\mathbf{R}+\mathbf{d}_b)} |\mathbf{R}, b\rangle$$ sublattice index $b = 1, \ldots, \mathcal{N}$



Single particle eigenstates:

$$|\mathbf{k}, \alpha\rangle = \sum_{b=1}^{\mathcal{N}} u_b^\alpha(\mathbf{k}) |\mathbf{k}, b\rangle = \hat{\gamma}_{\mathbf{k}}^{\alpha\dagger} |\text{vac}\rangle$$

band index α



Bloch Hamiltonian:

$$H_{bc}(\mathbf{k}) = \sum_{\alpha=1}^{\mathcal{N}} E_\alpha(\mathbf{k}) u_b^{\alpha *}(\mathbf{k}) u_c^\alpha(\mathbf{k})$$

Berry connection:

$$\mathcal{A}_\alpha(\mathbf{k}) = -i \sum_{b=1}^{\mathcal{N}} u_b^{\alpha *} \nabla_{\mathbf{k}} u_b^\alpha$$ gauge dependent

Berry curvature: $$B_\alpha(\mathbf{k}) = \nabla \times \mathcal{A}_\alpha(\mathbf{k})$$ gauge invariant

Chern number: $$c_1 = \frac{S_{BZ}}{2\pi} \langle B_\alpha \rangle$$ average <> over BZ, quantized to integer values

Which Berry Curvature?

Gauge invariance of the Bloch functions: one arbitrary U(1) phase for each **k**-point

$$|u_{\mathbf{k}}^{\alpha}\rangle \to e^{i\phi_\alpha(\mathbf{k})}|u_{\mathbf{k}}^{\alpha}\rangle$$

The above manifestly leaves H invariant:

$$H_{bc}(\mathbf{k}) = \sum_{\alpha=1}^{\mathcal{N}} E_\alpha(\mathbf{k}) u_b^{\alpha*}(\mathbf{k}) u_c^{\alpha}(\mathbf{k})$$

However, sublattice dependent phases are *not gauges*:

$$u_a^{\alpha}(\mathbf{k}) \to \widetilde{u}_b^{\alpha}(\mathbf{k}) = e^{i\mathbf{r}_b\cdot\mathbf{k}} u_b^{\alpha}(\mathbf{k})$$

as this substitution yields a modified Berry curvature:

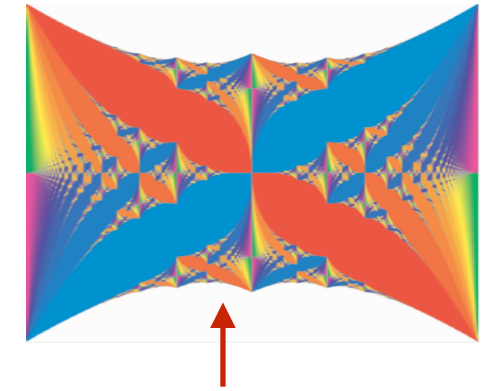
$$\widetilde{B}_\alpha(\mathbf{k}) - B_\alpha(\mathbf{k}) = \sum_{b=1}^{\mathcal{N}} r_{b,y}\frac{\partial}{\partial k_x}|u_b^{\alpha}(\mathbf{k})|^2 - r_{b,x}\frac{\partial}{\partial k_y}|u_b^{\alpha}(\mathbf{k})|^2$$

There is a unique choice such that the polarisation reduces to the correct semi-classical expression

and canonical position operator $\hat{R}_\mu \to -i\dfrac{\partial}{\partial k_\mu}$

see, e.g. Zak PRL (1989)

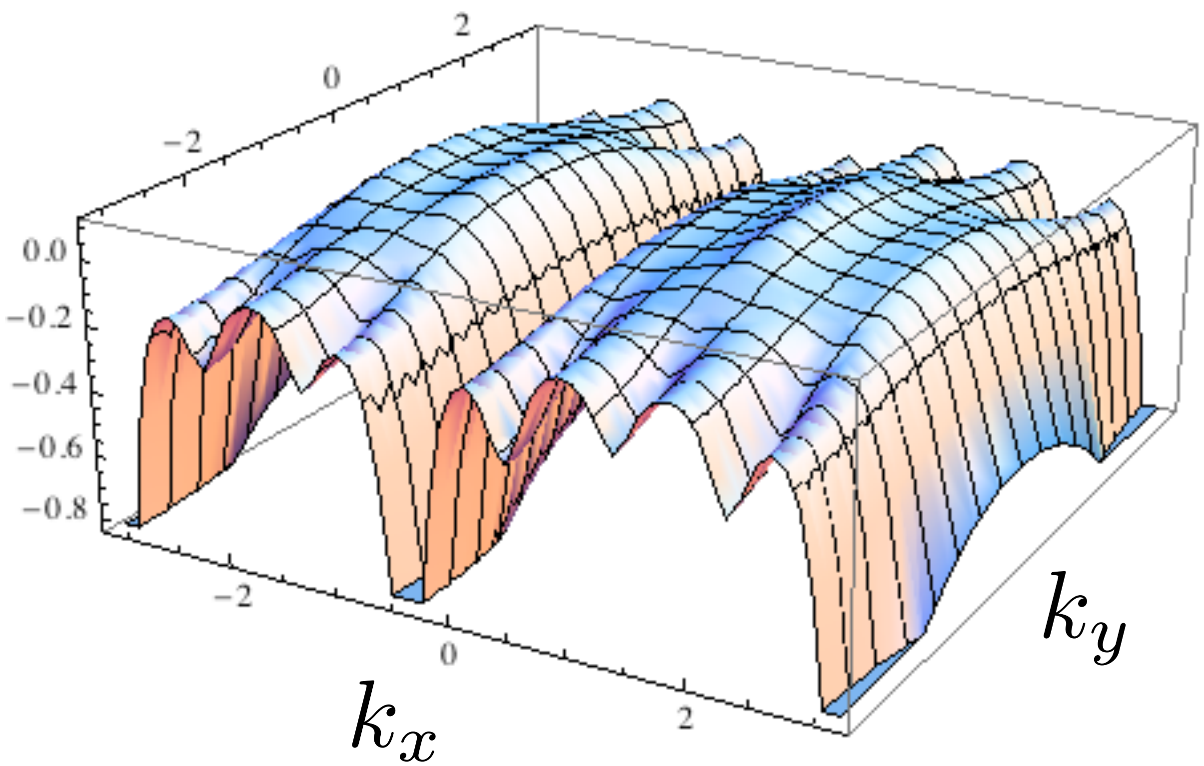
Example single particle properties



an example: Hofstadter spectrum in magnetic unit cell of 7x1, $n_\phi = 3/7$

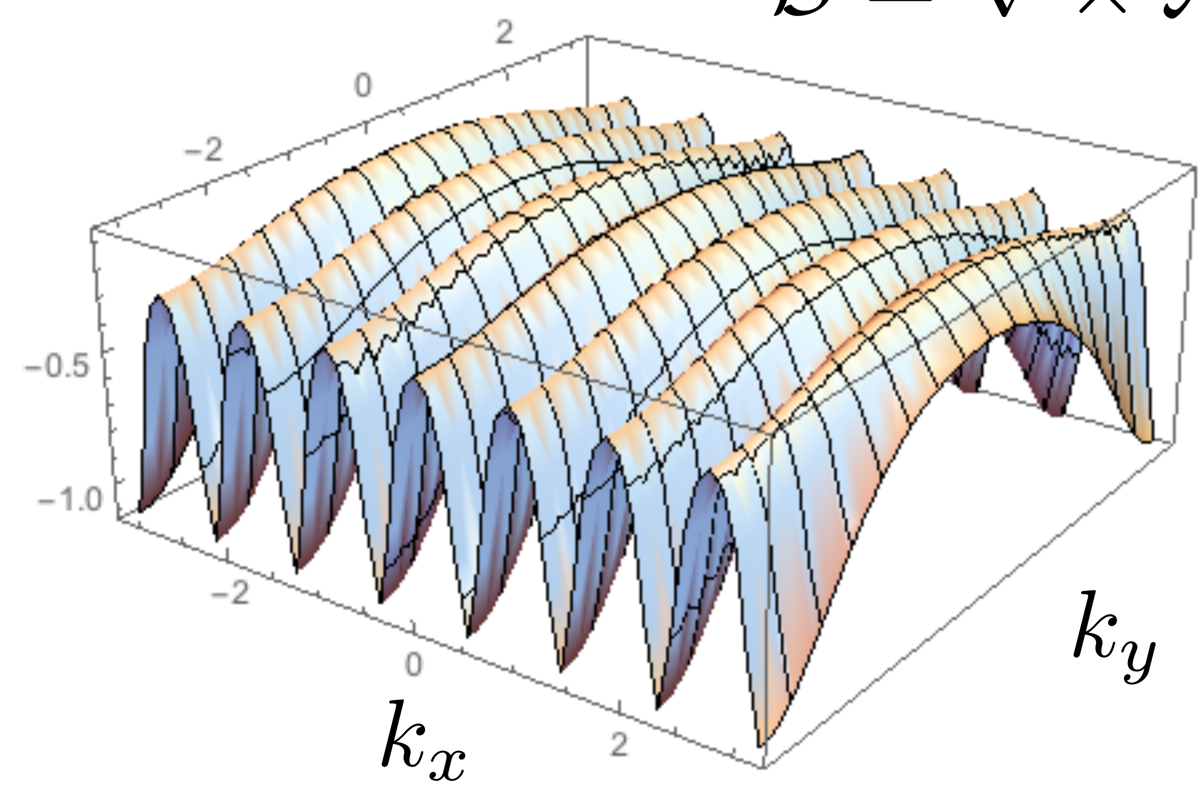
Curvature for Fourier transform with respect to unit cell pos

$$\tilde{\mathcal{B}} = \nabla \times \tilde{\mathcal{A}}$$



Curvature for canonical Fourier transform

$$\mathcal{B} = \nabla \times \mathcal{A}$$



Magnetic unit cell

$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$\frac{\Phi}{7}$	$-\frac{6\Phi}{7}$

net flux defined only mod Φ_0

GMP Algebra: Generating low-lying excitations

Girvin, MacDonald and Platzman, PRB **33**, 2481 (1986).

- single mode approximation captures low-lying neutral excitations in quantum Hall systems:

$$|\Psi^{\mathrm{SMA}}_{\mathbf{k}}\rangle = \hat{\rho}_{\mathbf{k}}|\Psi_0\rangle$$

for sp density operators $\hat{\rho}_{\mathbf{k}} = \sum_q \hat{\gamma}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{\gamma}_{\mathbf{q}}$

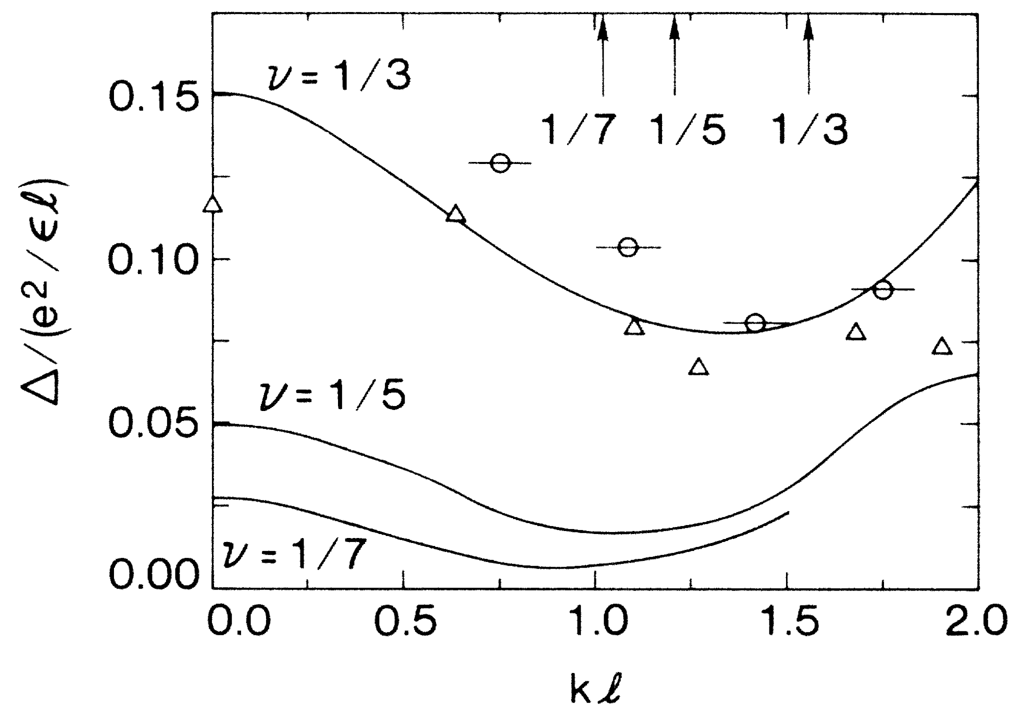


FIG. 4. Comparison of SMA prediction of collective mode energy for $\nu=\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ with numerical results of Haldane and Rezayi (Ref. 20) for $\nu=\frac{1}{3}$. Circles are from a seven-particle spherical system. Horizontal error bars indicate the uncertainty

GMP algebra (w/LLL form factor):

$$[\rho_{\mathrm{LLL}}(\mathbf{q}), \rho_{\mathrm{LLL}}(\mathbf{q}')] = 2i\sin\left(\tfrac{1}{2}\mathbf{q}\wedge\mathbf{q}'\ell_B^2\right)\exp\left(\tfrac{1}{2}\mathbf{q}\cdot\mathbf{q}'\ell_B^2\right)\rho_{\mathrm{LLL}}(\mathbf{q}+\mathbf{q}')$$

SMA carries over to Chern bands: Repellin, Neupert, Papić, Regnault, Phys. Rev. B 90 (2014)

Chern bands: generalised GMP algebra

- consider band-projected density operators for general Chern bands:

$$\widetilde{\rho}_{\mathbf{q}} \equiv P_\alpha e^{i\mathbf{q}\cdot\widehat{\mathbf{r}}} P_\alpha = \sum_{\mathbf{k}} \sum_{b=1}^{\mathcal{N}} u_b^{\alpha *}(\mathbf{k}+\mathbf{q}/2) u_b^{\alpha}(\mathbf{k}-\mathbf{q}/2) \gamma^{\alpha\dagger}_{\mathbf{k}+\mathbf{q}/2} \gamma^{\alpha}_{\mathbf{k}-\mathbf{q}/2}$$

- in general, the algebra of density operators does not close, i.e.

$$[\widetilde{\rho}_{\mathbf{q}}, \widetilde{\rho}_{\mathbf{k}}] \neq F(\mathbf{k},\mathbf{q}) \widetilde{\rho}_{\mathbf{k}+\mathbf{q}}$$

- intuitive consequences for FQH states:
  - no finite, closed set of low-energy excitations corresponding to the GMP single mode states
  - $\widetilde{\rho}_{\mathbf{q}}$ can generate many distinct eigenstates
  - strong violation of the algebra should signal an unstable, gapless phase

Conditions for closure of the generalised GMP algebra I

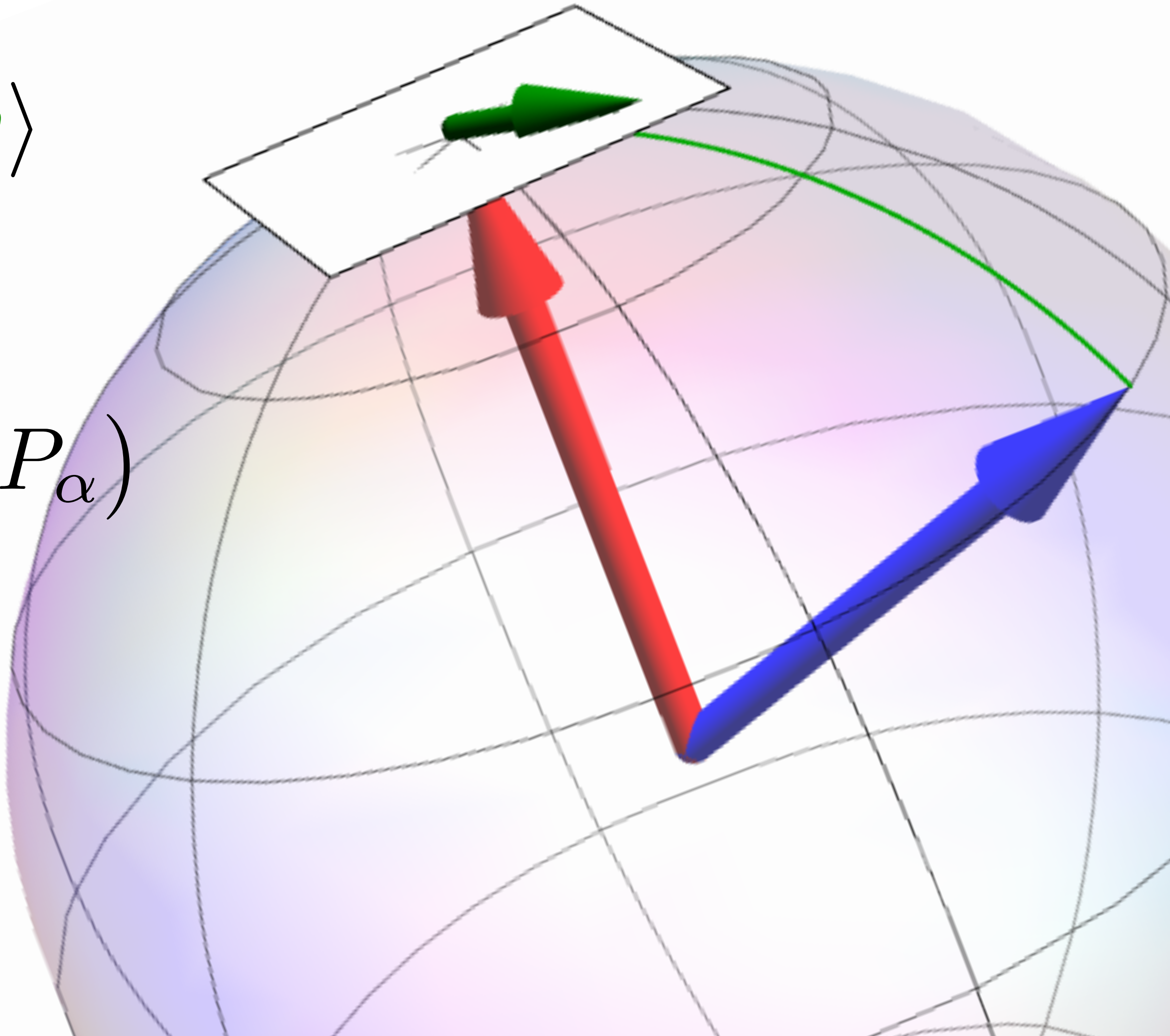
- conditions for closure can be derived in long-wavelength expansion

i) $\mathcal{O}(k^2)$: $$\sigma_c \equiv \sqrt{\frac{A_{BZ}^2}{4\pi^2}\langle B^2\rangle - c_1^2}$$ *flatness of Berry curvature*

ii) $\mathcal{O}(k^3)$: Pullback of Hilbert space metric constant over BZ

$$ds^2 = \langle\delta\psi|\delta\psi\rangle - \langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle$$

$$g_{\mu\nu} + \frac{i}{2}F_{\mu\nu} = \sum_{\alpha\in\text{occ}} \text{tr}\left(\frac{\partial}{\partial k_\mu}P_\alpha\right)(1-P_\alpha)\left(\frac{\partial}{\partial k_\nu}P_\alpha\right)$$



deviations $$\sigma_g \equiv \sqrt{\frac{1}{2}\sum_{\mu,\nu}\langle g_{\mu\nu}g_{\nu\mu}\rangle - \langle g_{\mu\nu}\rangle\langle g_{\nu\mu}\rangle}$$

Conditions for closure of the generalised GMP algebra II

iii) closure at all orders if

$$D(\mathbf{k}) \equiv \det g^{\alpha}(\mathbf{k}) - \frac{B_{\alpha}(\mathbf{k})^2}{4} = 0$$

- if i), ii) and iii) are met, one obtains a generalised GMP algebra:

$$[\widetilde{\rho}_{\mathbf{q}}, \widetilde{\rho}_{\mathbf{k}}] = 2ie^{\sum_{\mu,\nu} g^{\alpha}_{\mu\nu}\mathbf{q}_{\mu}\mathbf{k}_{\nu}} \sin\left(\frac{B_{\alpha}}{2}\mathbf{q} \wedge \mathbf{k}\right)\widetilde{\rho}_{\mathbf{q}+\mathbf{k}}$$

- under stronger variant of condition iii) the algebra reduces exactly to the GMP algebra, namely if

$$T(\mathbf{k}) \equiv \mathrm{tr}\ g^{\alpha}(\mathbf{k}) - |B_{\alpha}(\mathbf{k})| = 0$$

- Current study: test how violations of the closure constraints correlate with gap

R. Roy, arxiv:1208.2055 (PRB 2014); Parameswaran, Roy, Sondhi C. R. Physique (2013)

Target models to examine

- Hamiltonian: bosonic states with on-site interactions — defined independent of specific lattice

2-body contact

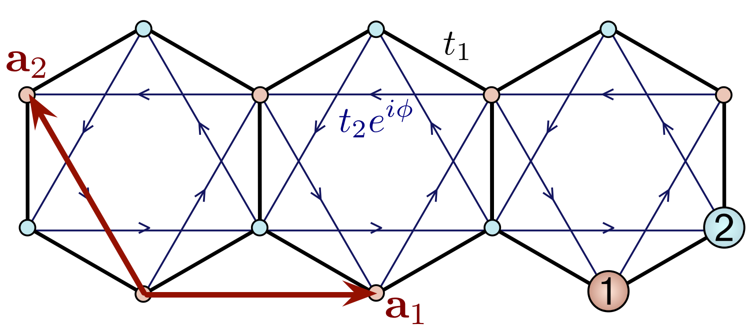
Laughlin $\nu = \frac{1}{2}$

3-body contact

Moore-Read $\nu = 1$

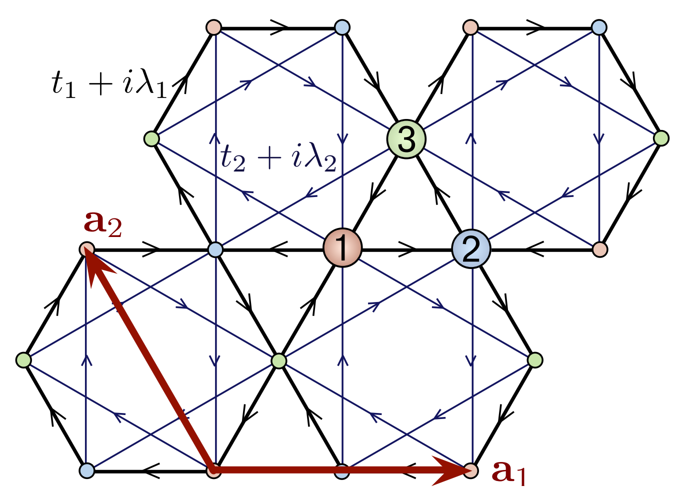
- lattice geometries to consider:

Haldane model



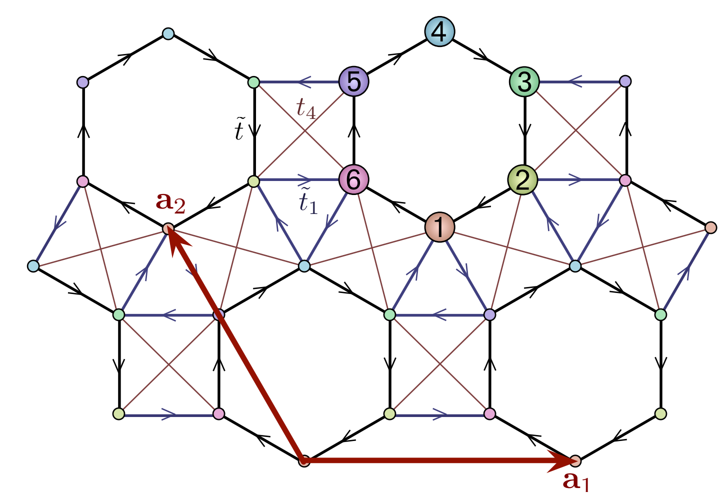
$\mathcal{N} = 2$

Kagomé model



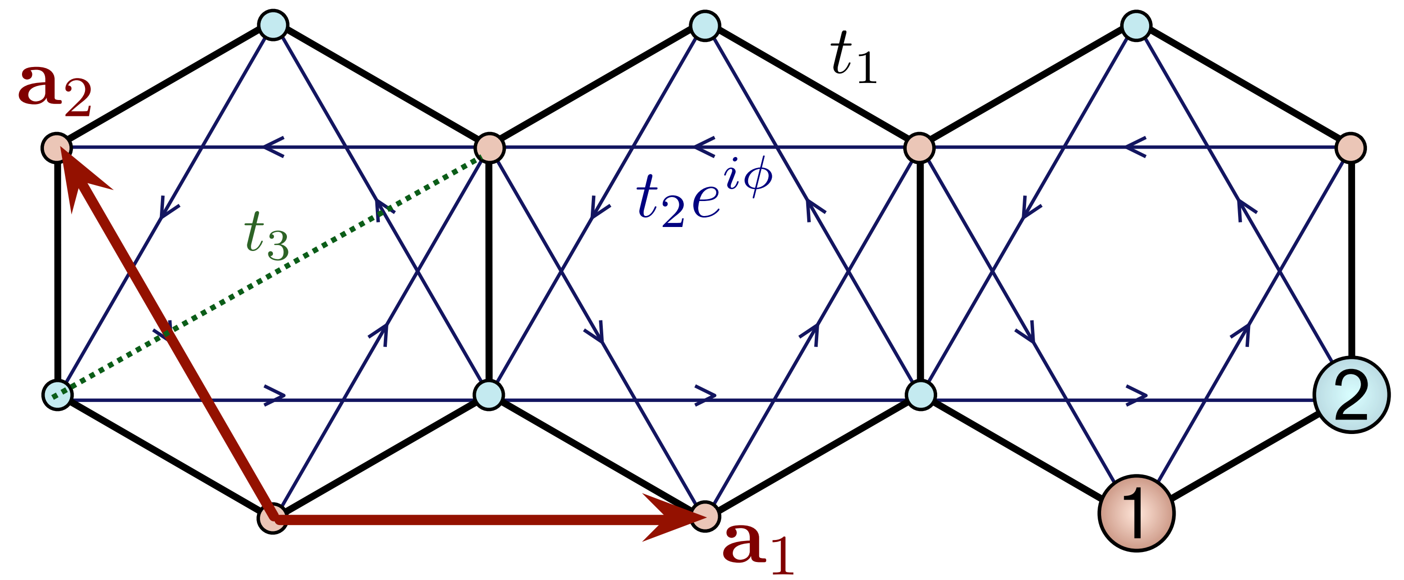
$\mathcal{N} = 3$

Ruby lattice model



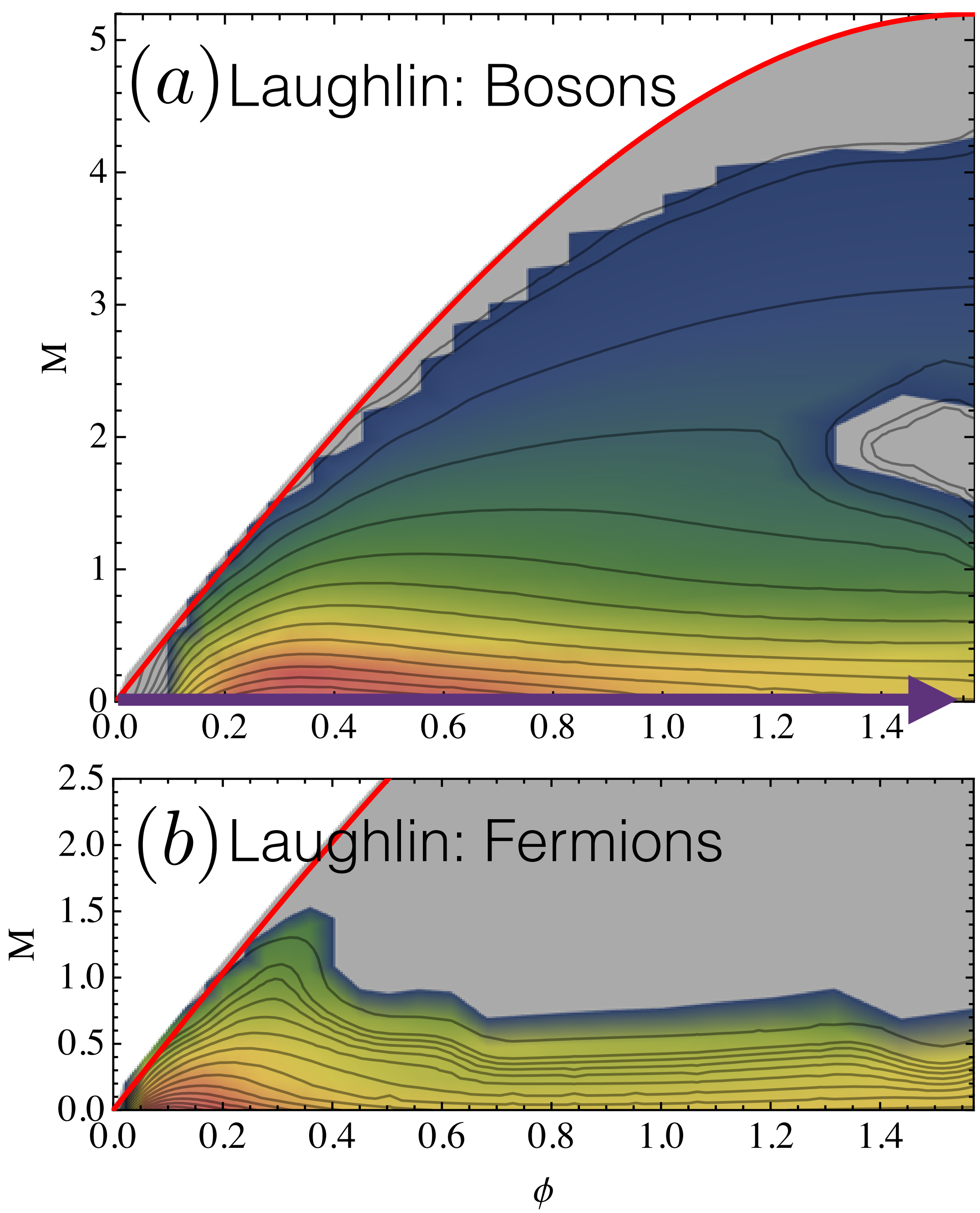
$\mathcal{N} = 6$

Haldane model



Haldane Model with t_3=0

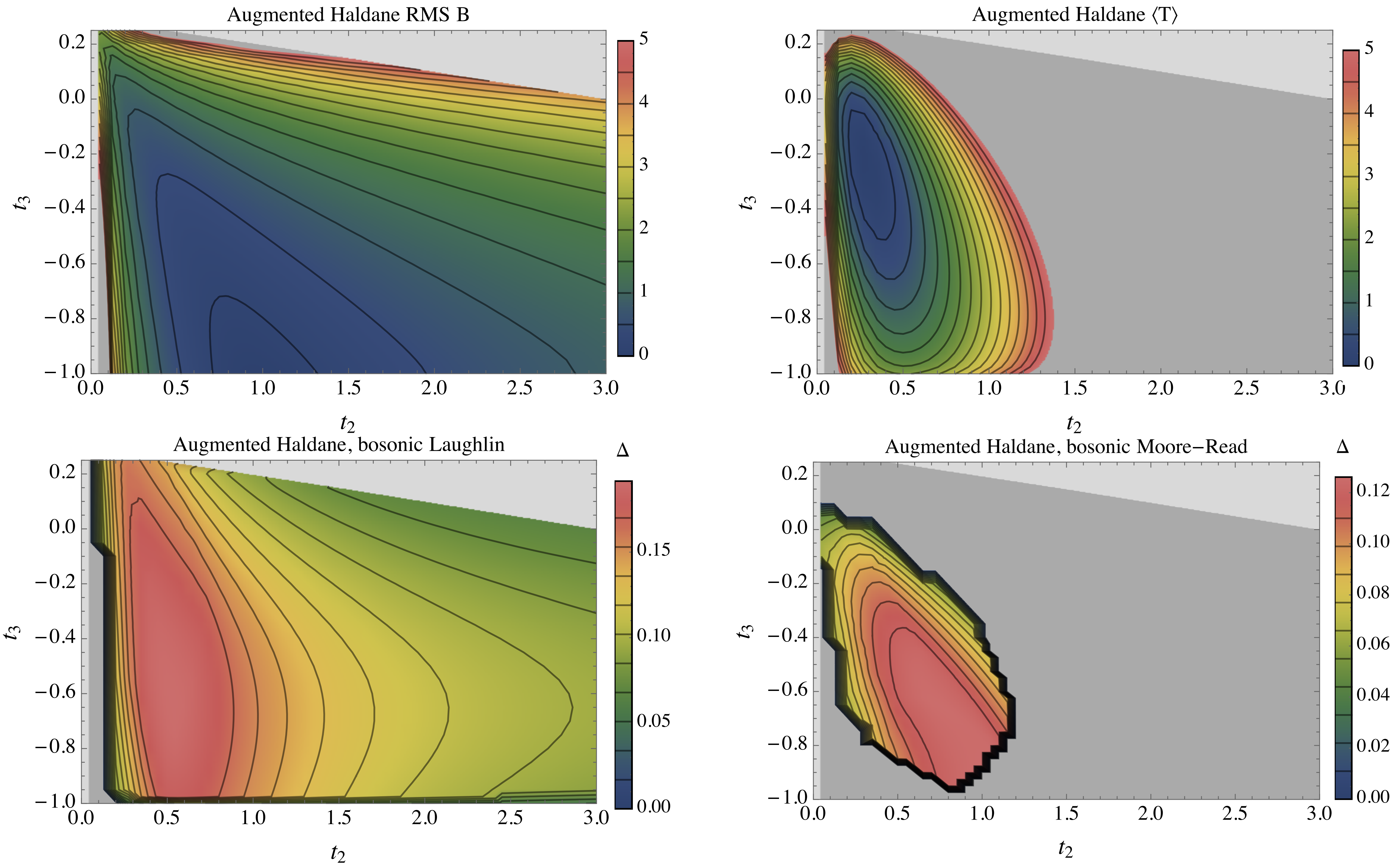
- Location of max gap for bosonic and fermionic Laughlin agrees with min RMS B
- Band geometry “interpolates” between bosonic, fermionic statistics
- For this model, quantum metric does not provide info beyond that supplied by curvature.



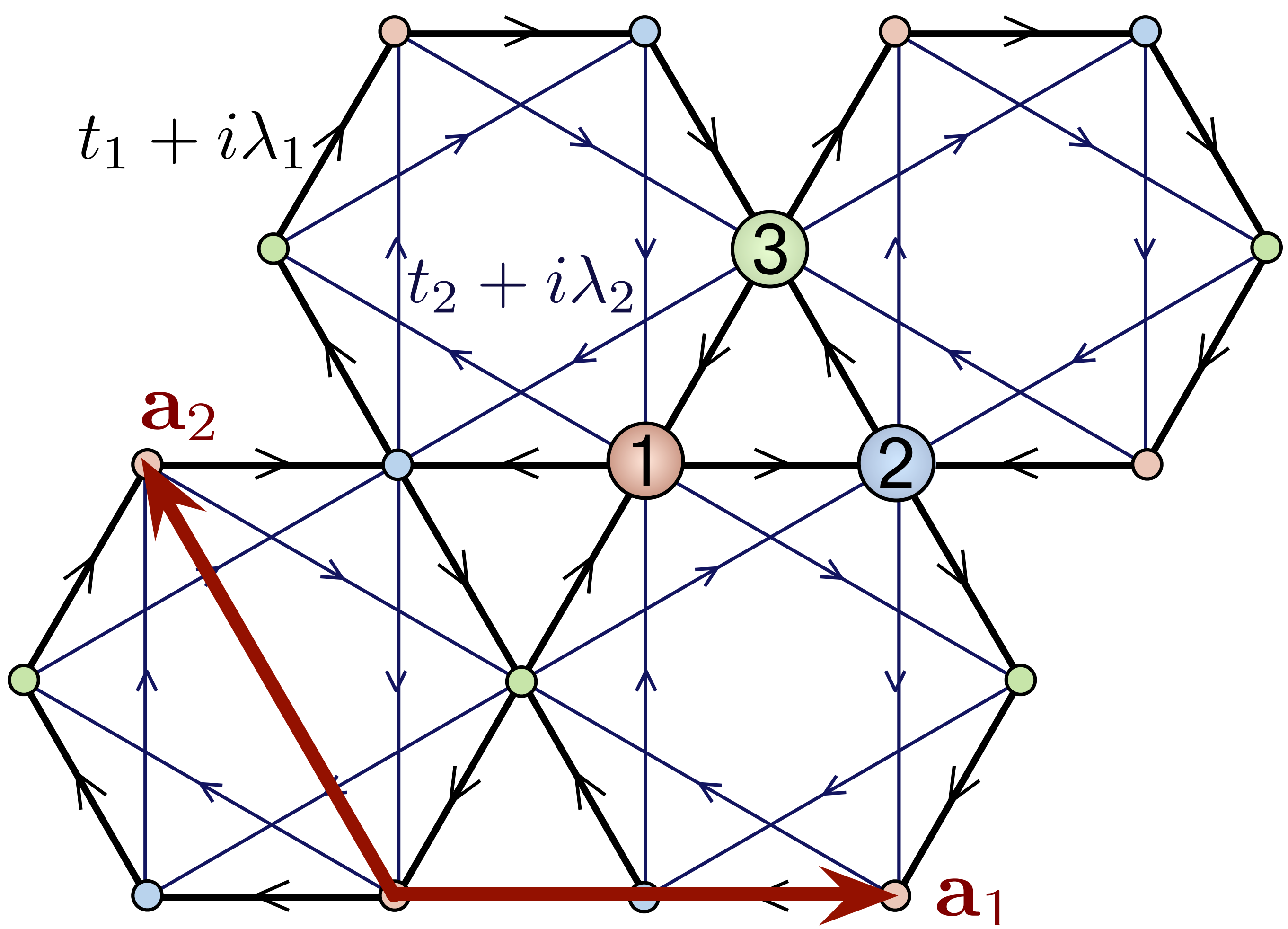
Jackson, GM, Roy, arxiv:1408.0843; cf. Dobardzic, Milovanovic and Regnault, PRB 2013)

Haldane Model: Effects of quantum metric for M=0, t_3>0

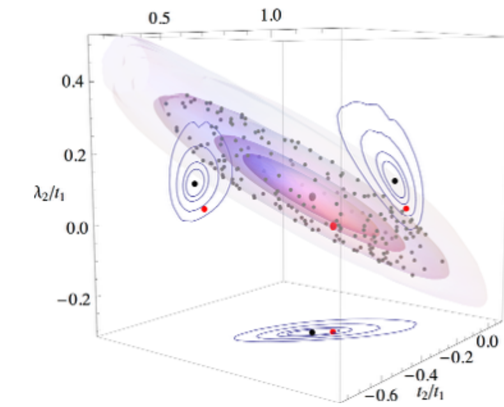
- position of maximum gap appears to be compromise between minimising curvature fluctuations and metric trace inequality (also seen in fermionic Laughlin)



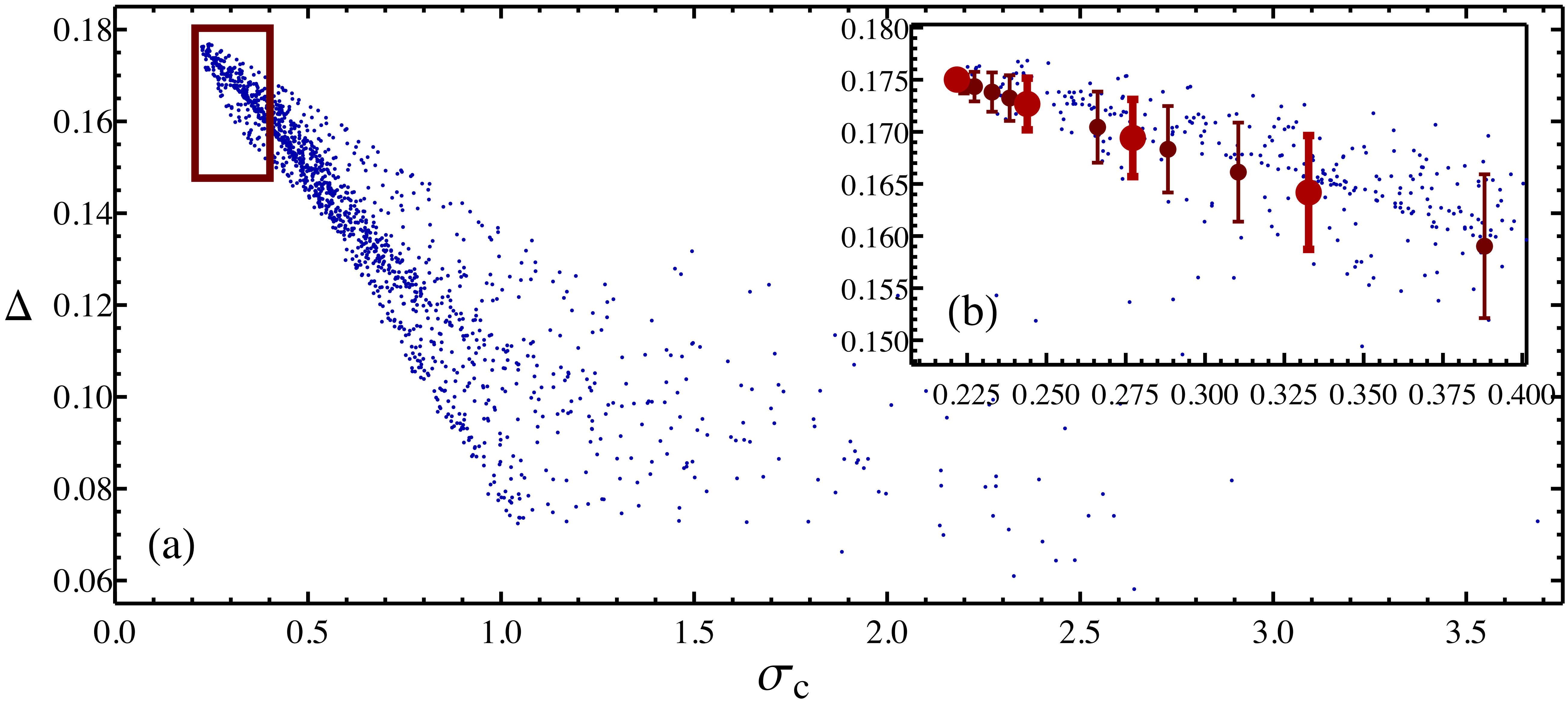
Kagome lattice model



Full parameter space for Kagome Model: Gap vs. RMS B



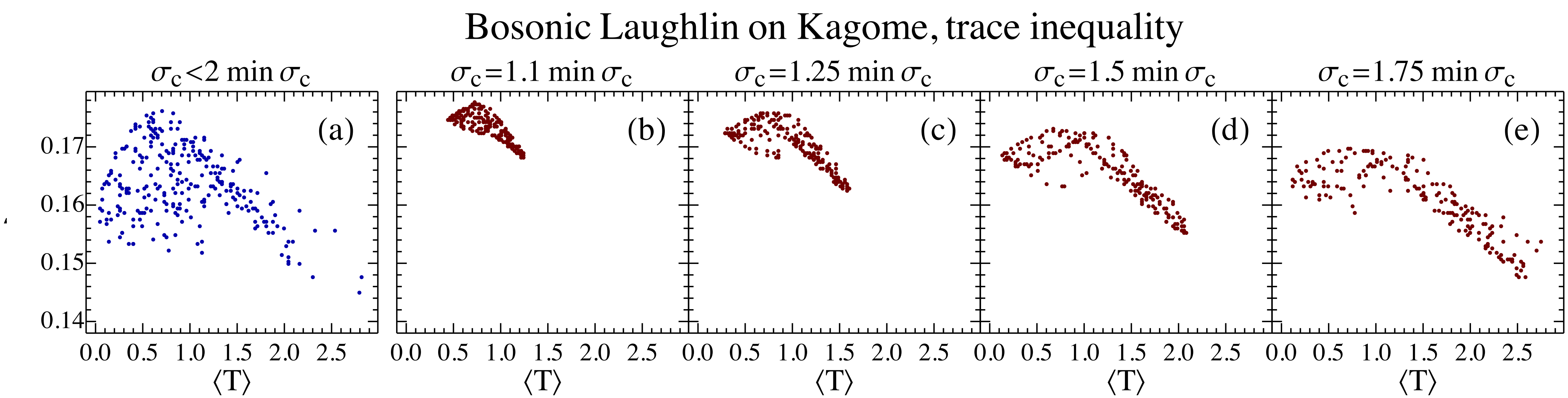
- Randomly sample points in parameters space of $t_1(=1)$, t_2, λ_1, λ_2
- Gap data for the bosonic Laughlin state

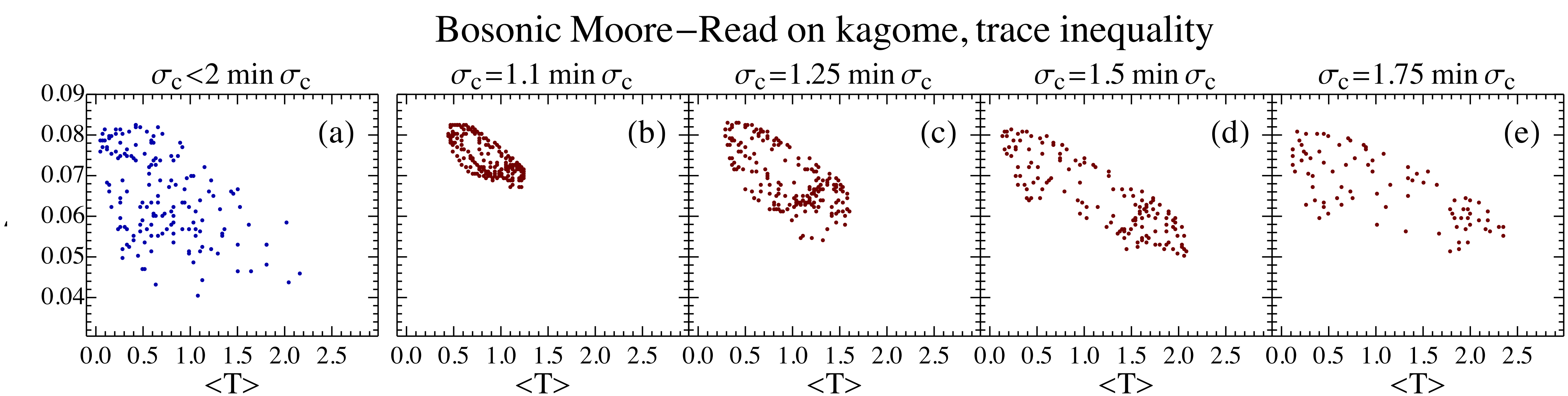


- Approximately linear trend which holds from min RMS B point all the way to the phase boundary (gap closure)
- Significant scatter, though, which may be explained by quantum metric

Kagome Model: “Shells” of constant RMS Curvature

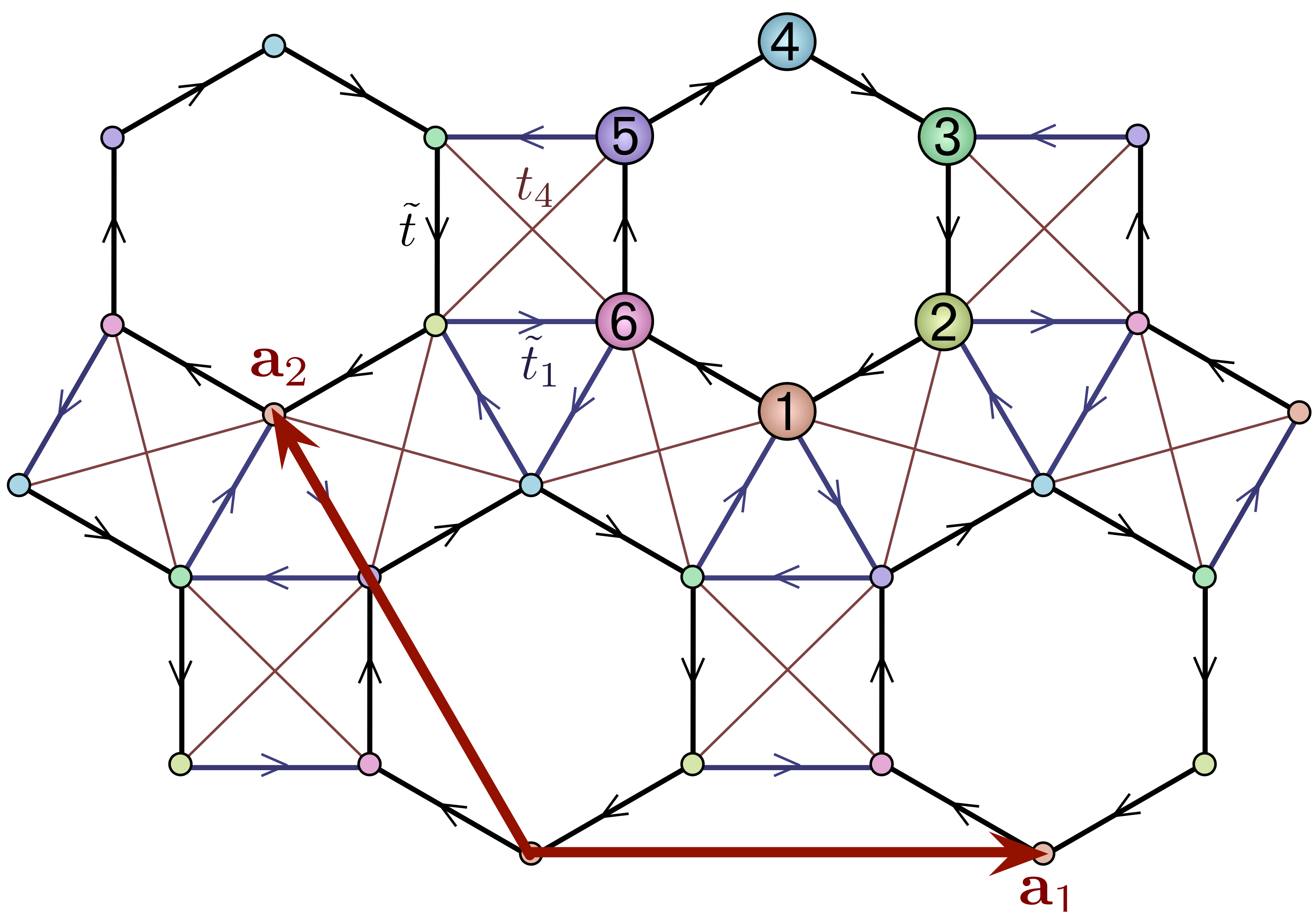
- Considering models at surfaces of fixed σ_c, the violation of the metric trace equality ⟨T⟩ is highly correlated with the many-body gap





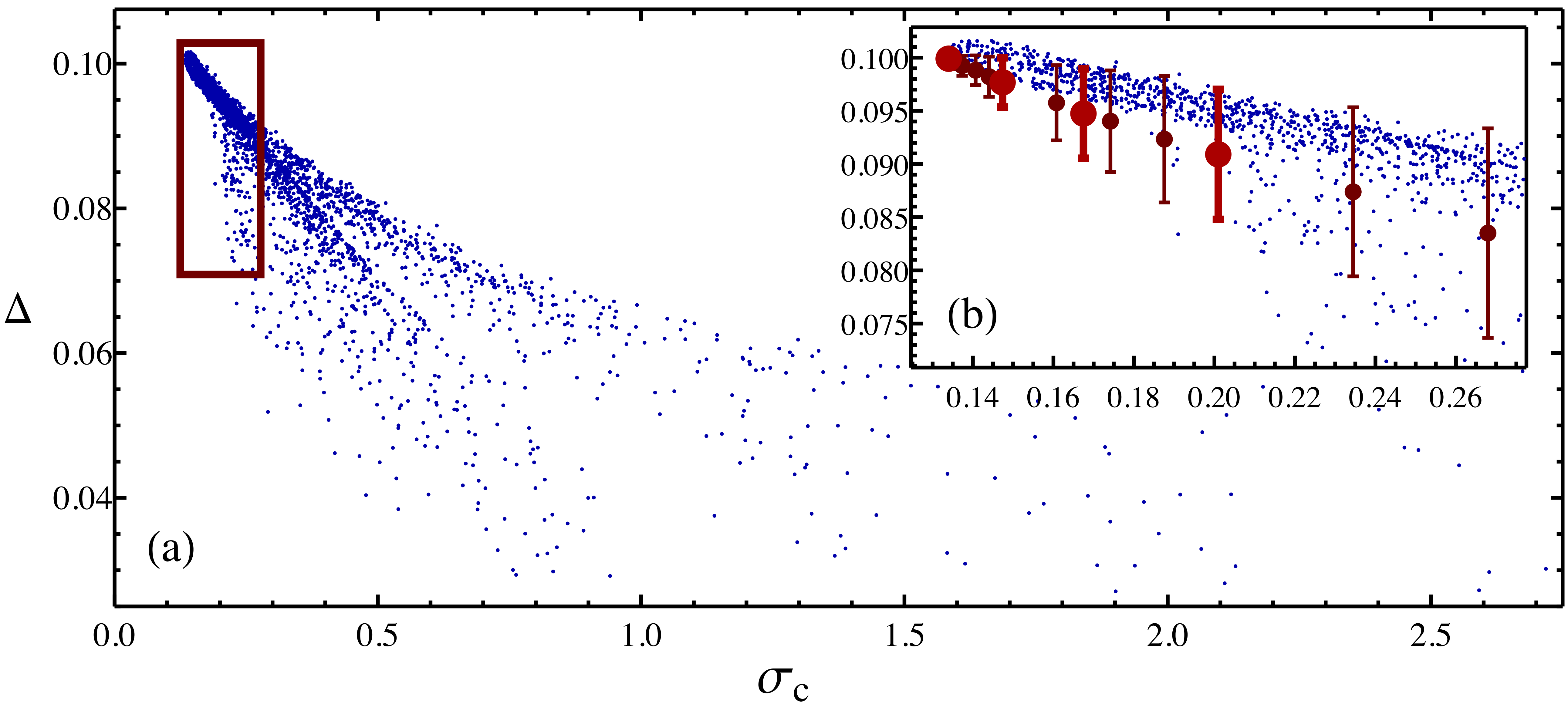
TS Jackson, G. Möller, R. Roy “Geometric stability of topological lattice phases”, arxiv:1408.0843

Ruby lattice model



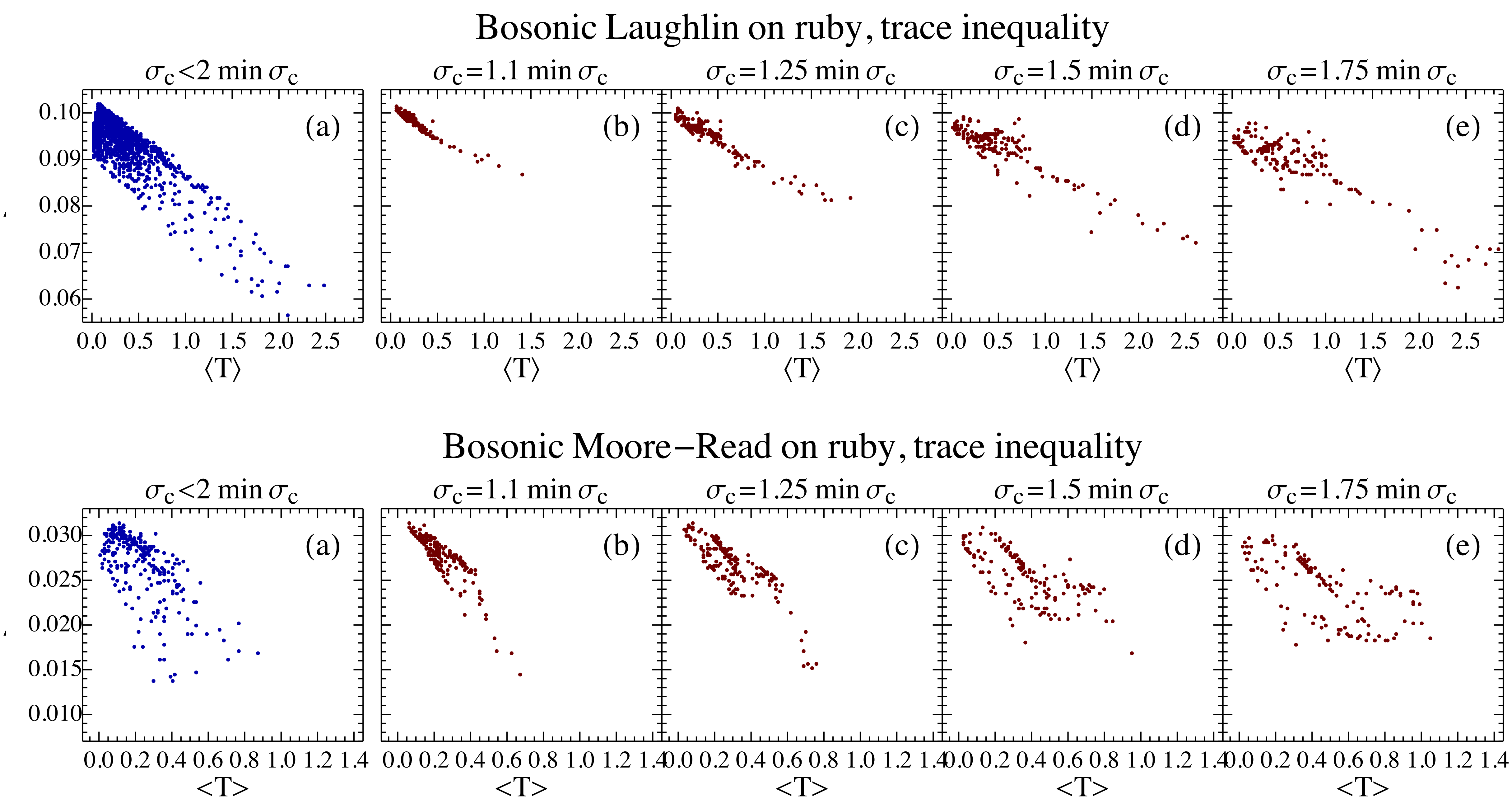
Ruby Lattice Model: Gap vs RMS Curvature

- Similar linear dependence of gap on RMS B as seen in Kagome model



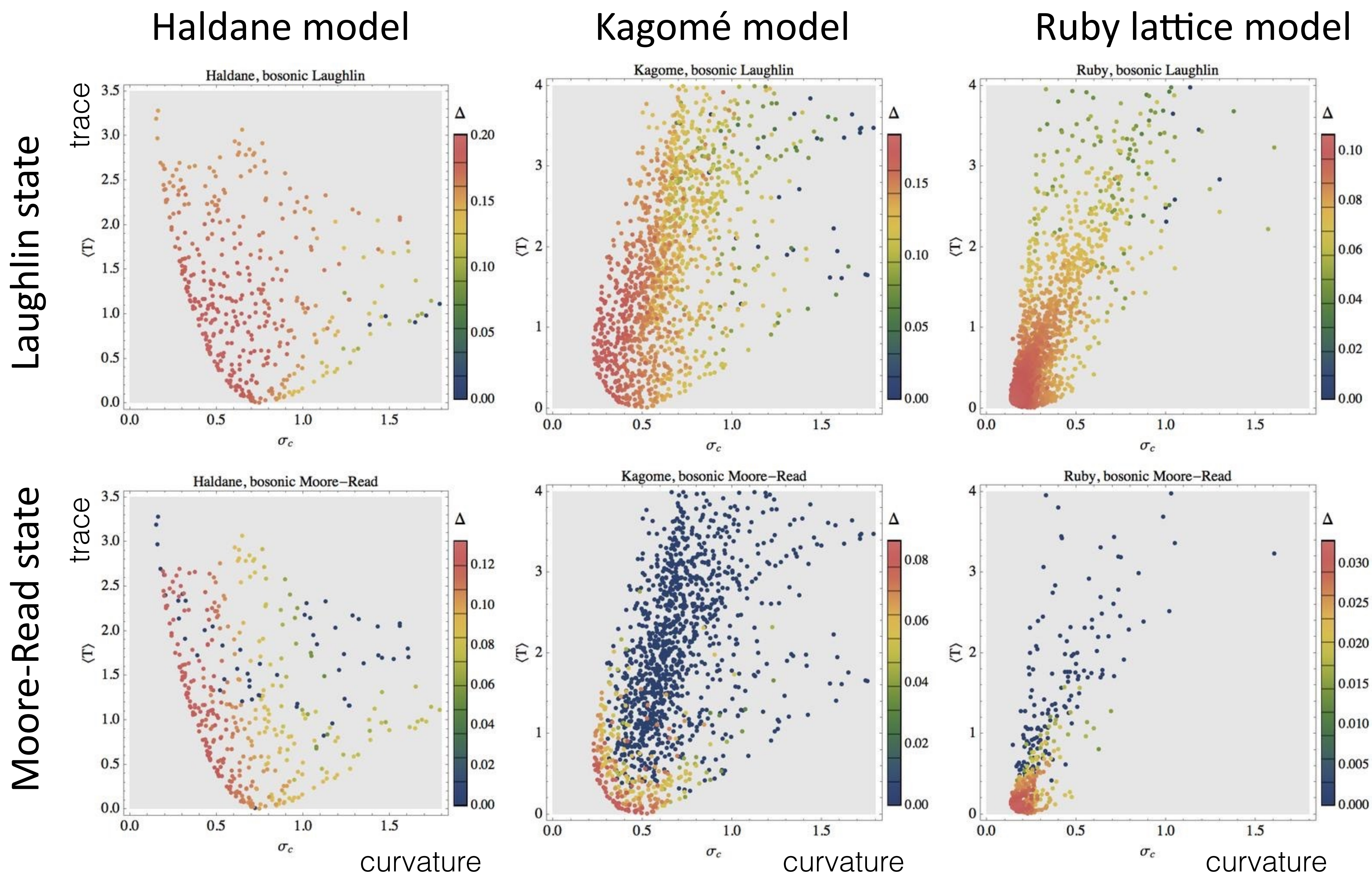
Ruby Lattice Model: “Shells” of constant RMS Curvature

- Even clearer results for influence of metric trace inequality ⟨T⟩



Models with many sub lattices can approximate Landau level physics more closely

Model Comparison: Gaps vs. RMS B and trace inequality



- Parameters yielding max gap are always in lower-left corner
- Demonstrates relevance of both band-geometric quantities

Conclusions

- Band geometry provides useful information about stability of fractional Chern insulators

- Berry curvature $O(k^2)$ is the dominant effect (as previously known)
- Trace of the quantum metric $O(k^3)$ provides further information

- Statistically, band geometry is strongly correlated with many-body gap

→ useful for quick exploration of available parameter space

- But: it is only one of three factors, so not the only important measure

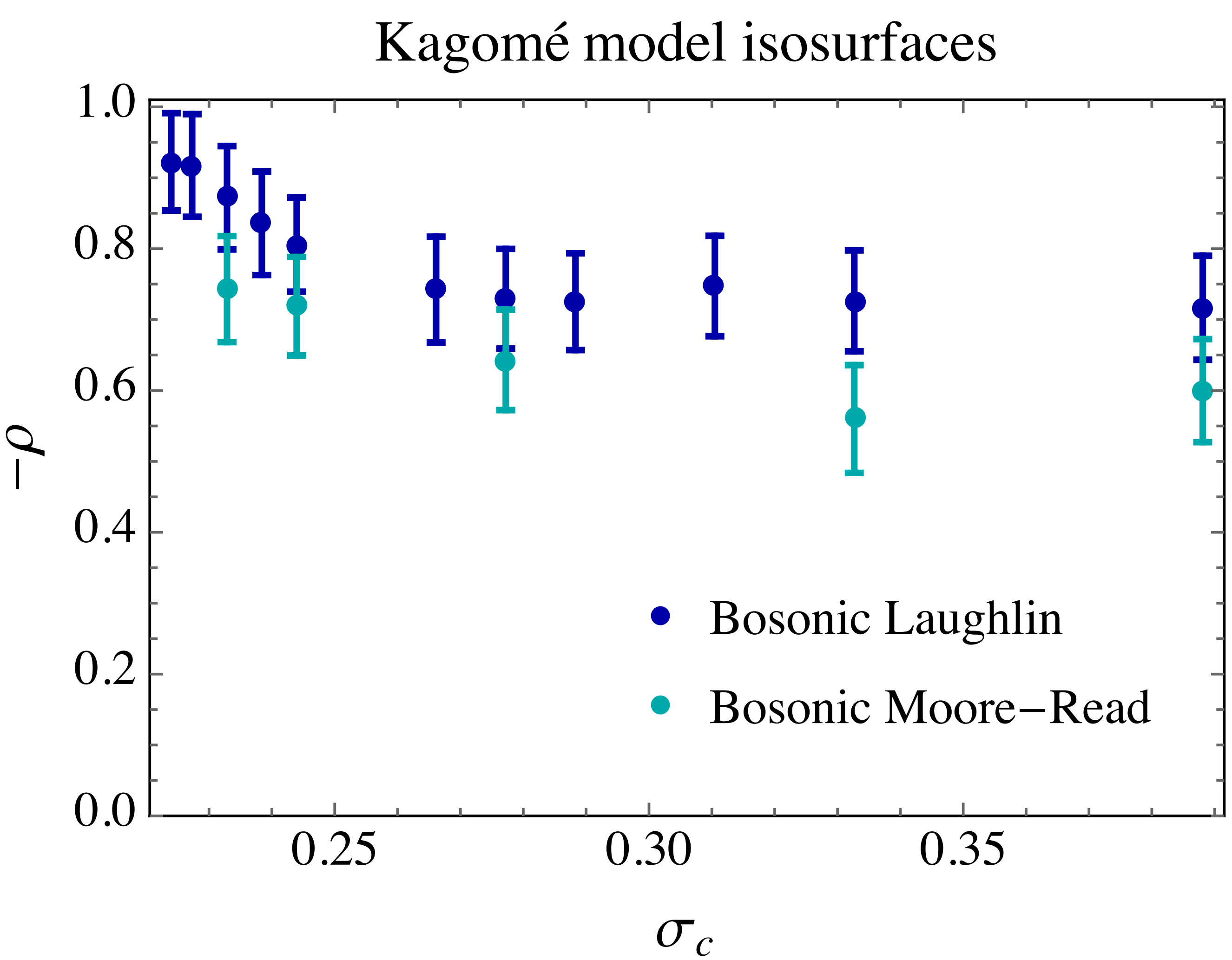
TS Jackson, G. Möller, R. Roy "Geometric stability of topological lattice phases", arxiv:1408.0843

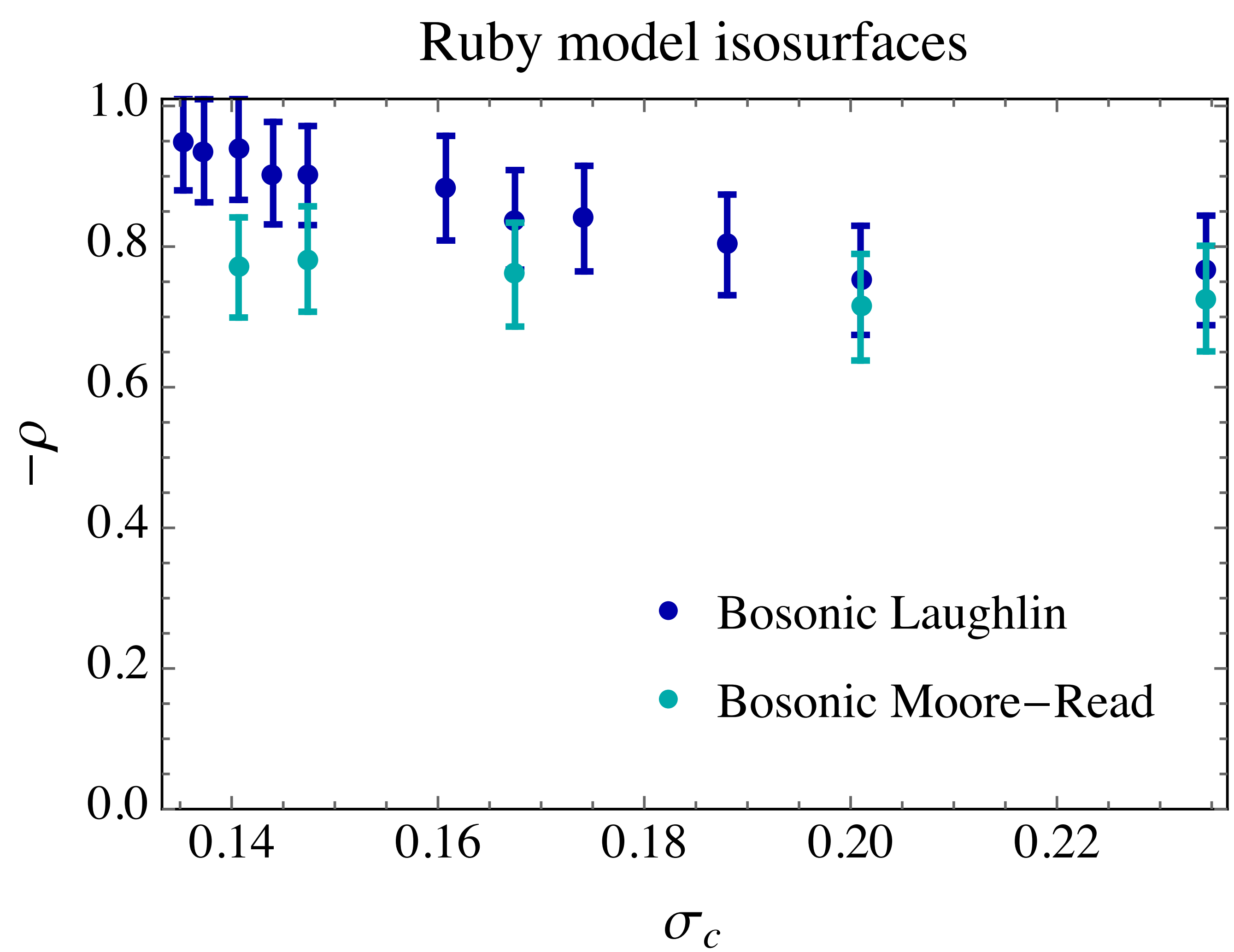
Related works: Adiabatic continuity T. Scaffidi & GM, Phys. Rev. Lett. **109**, 246805 (2012)

FCI in the Hofstadter model GM & N. R. Cooper, Phys. Rev. Lett. **103**, 105303 (2009)

Quantifying degree of correlation on shells of const. RMS B — Spearman ρ monotonicity test

- Nonparametric statistic which is sensitive to any monotonic relationship
- Perfect correlation for ρ = ±1, no correlation at ρ = 0
- Find siginifcant, robust negative correlation between gap and metric inequality on all isosurfaces of constant RMS B, demonstrating importance of trace inequality as a subleading influence on the gap



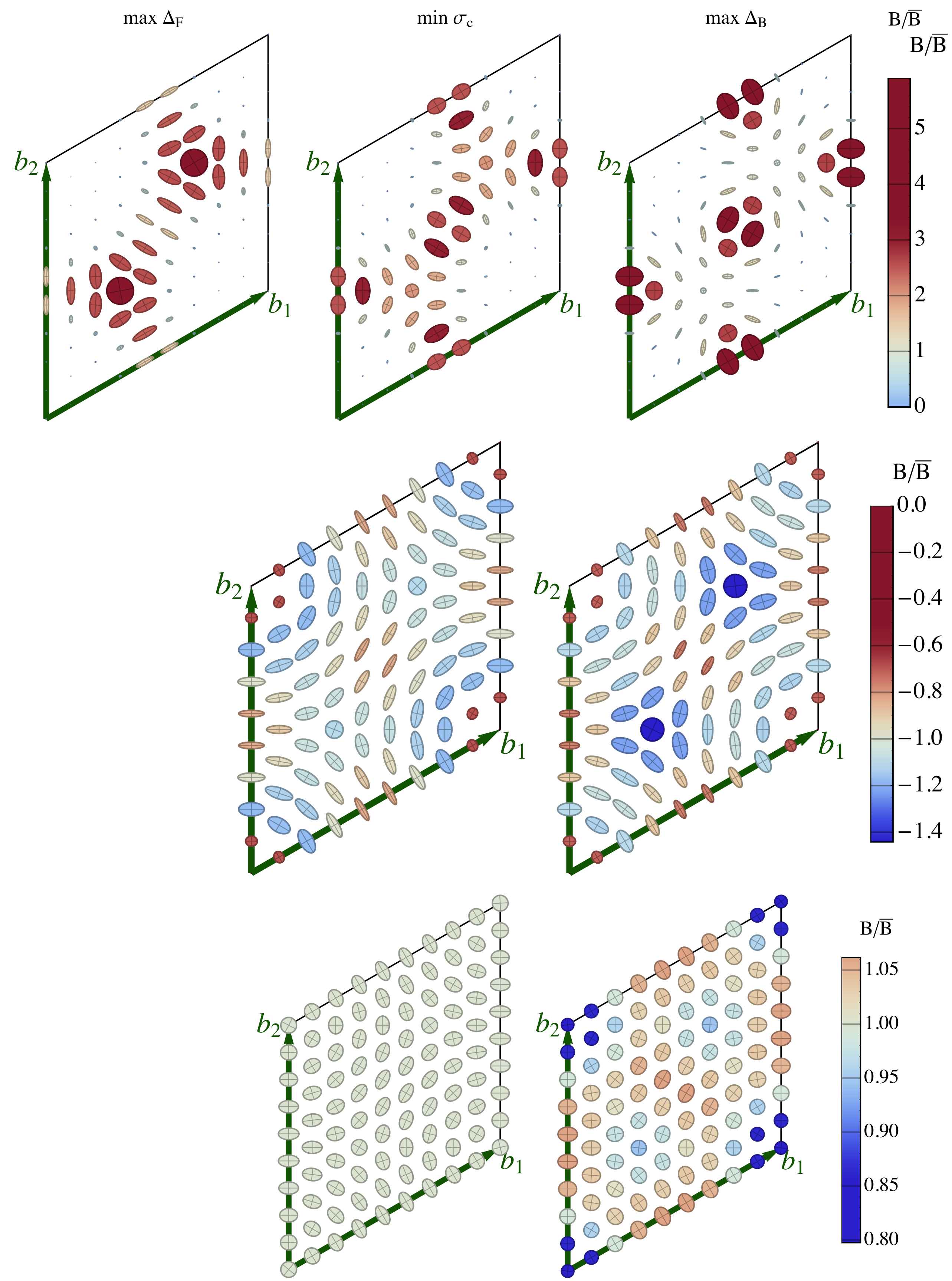


Plots of quantum metric across the BZ

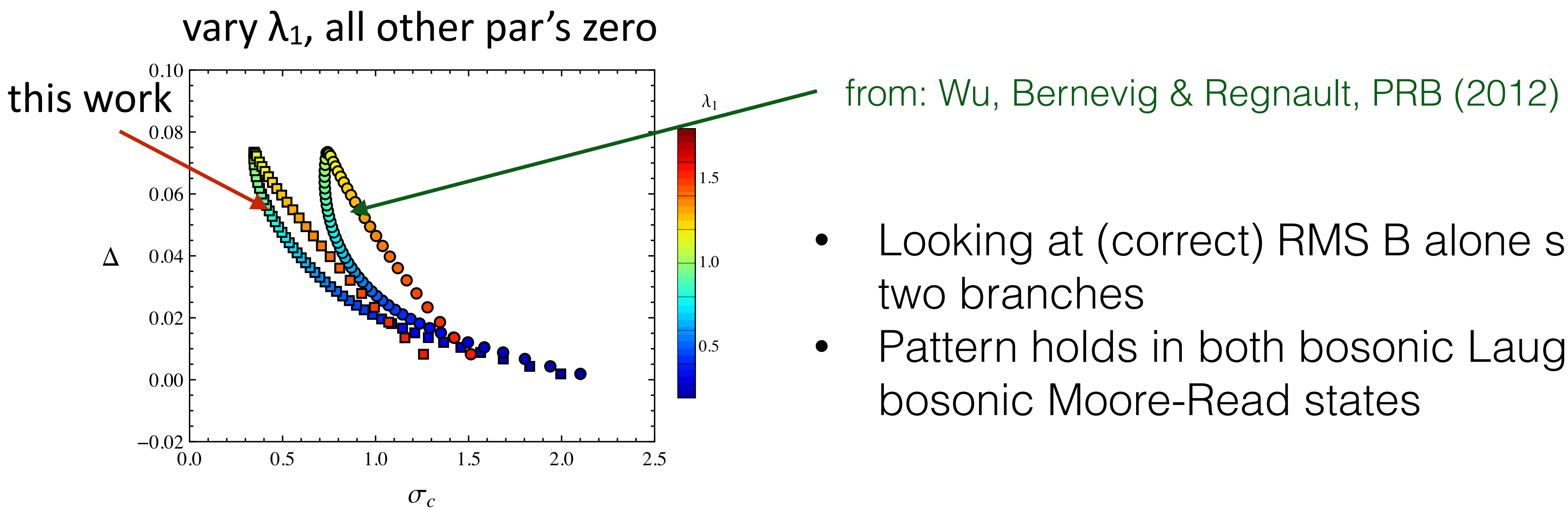
Standard Haldane

Kagome

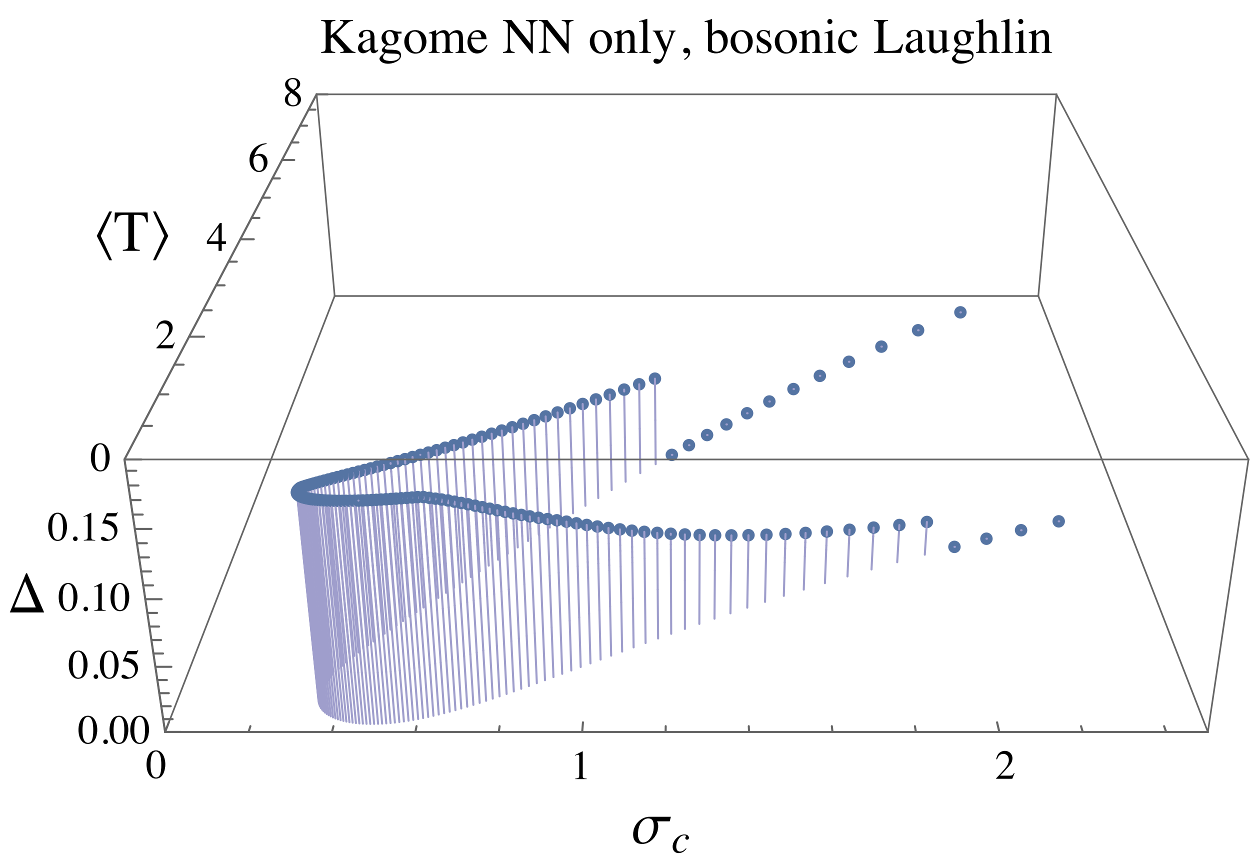
Ruby

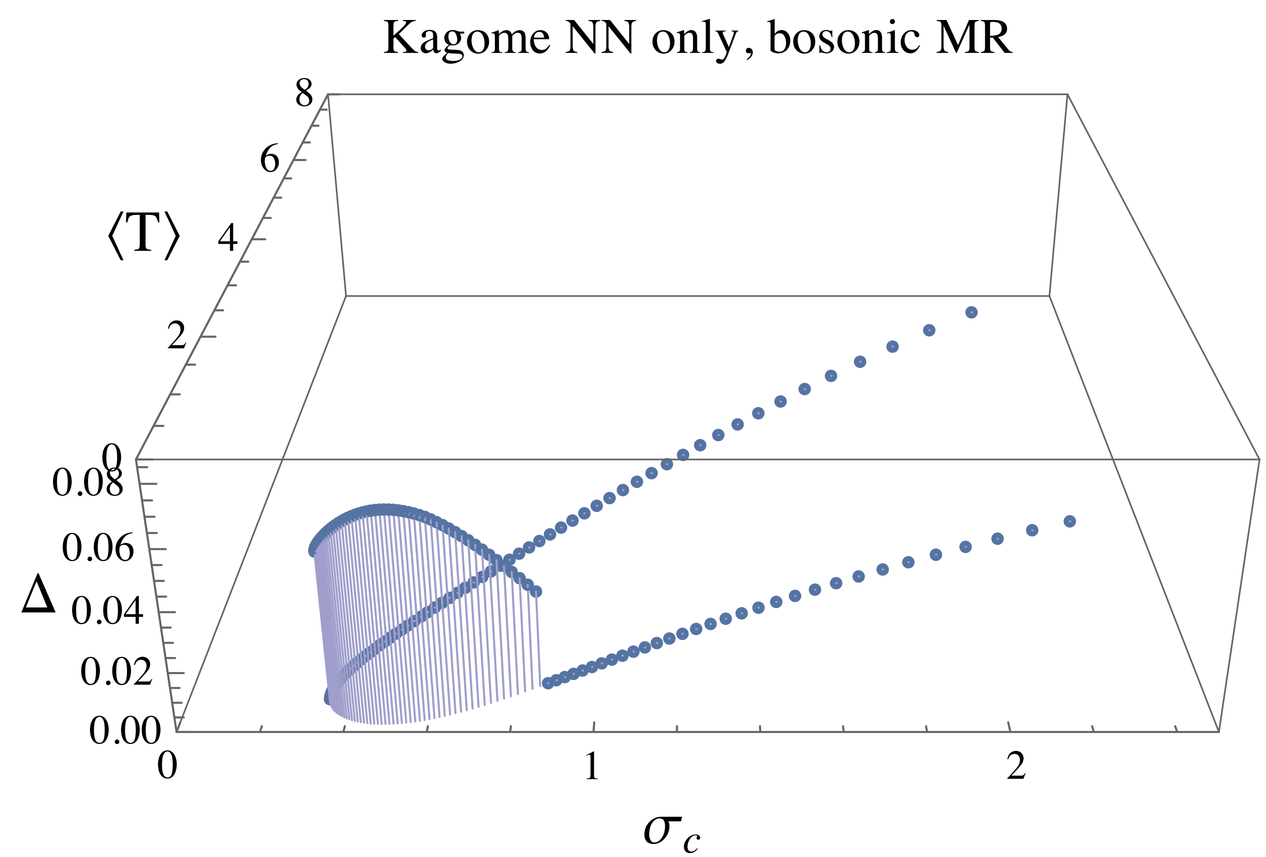


NN-only Kagome Model: Gap vs. RMS B and Tr G



- Looking at (correct) RMS B alone shows two branches
- Pattern holds in both bosonic Laughlin and bosonic Moore-Read states





- Branches distinguished by including information about metric trace inequality

Geometry & gap for new parametrization of Haldane-t3

- Now have large range of parameters with uniform curvature
- Minimum trace inequality in distinct location from min RMS B