Geometric stability of topological lattice phases

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Overview

- Background: Topology and interactions in tight-binding models

- The role of band geometry in the Single Mode Approximation to quantum Hall liquids / fractional Chern insulators

- Role of band geometry for incompressible Hall liquids
  - screening of three target models
Quantum Hall Effect in Periodic Potentials

- quantized Hall response in filled bands:
  \[ \sigma_{xy} = \frac{e^2}{h} \sum_{\text{filled bands}} C_n \]
  Thouless, Kohmoto, Nightingale, de Nijs 1982

- Chern-number for periodic systems
  \[ C = \frac{1}{2\pi} \int_{BZ} d^2 k \mathcal{B}(k) \]

- Hofstadter model (solved 1976):
  \[ \mathcal{H} = -J \sum_{\langle \alpha, \beta \rangle} [\hat{b}_\alpha^\dagger \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c.] \]
  tight-binding model for electrons in bands with finite Chern number

- the Hofstadter spectrum provides bands of all Chern numbers
- filled bands in this spectrum yield a quantized Hall response

\[ \sum A_{\alpha\beta} = 2\pi n_\phi \]
Fractional Quantum Hall Effect in Periodic Potentials

• quantized Hall response in partially filled bands?

\[ \mathcal{H} = -J \sum_{\langle \alpha, \beta \rangle} \left[ b_\alpha^\dagger b_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \sum V_{ij} n_i n_j \]

• THEORY: Kol & Read (1993)

• Confirmations for such states?
Fractional Quantum Hall on lattices: Numerical Evidence

- interest in cold atom community 2000’s:

- realisations of tight-binding models with complex hopping from light-matter coupling:

\[
\mathcal{H} = -J \sum_{\langle \alpha, \beta \rangle} \left[ \hat{b}_{\alpha}^\dagger \hat{b}_{\beta} e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_{\alpha} \hat{n}_{\alpha} (\hat{n}_{\alpha} - 1)
\]

- bosons with onsite U: many-body gap in the half-filled “synthetic Landau-level” persists to large flux density
**Fractional Quantum Hall on lattices with higher Chern-# bands**

- bands of the Hofstadter model go *beyond* the continuum limit and support *new classes* of quantum Hall states

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**theory:**

bosonic Hall states on the lattice

**numerical verification**

for what we would now call FCI states with \( \nu = 1 \)

- \( C=2 \) band
- hardcore bosons

\[
\Delta \quad \begin{array}{c} \begin{array}{c} 1.8 \\ 1.6 \\ 1.4 \\ 1.2 \\ 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0 \\
\end{array} \end{array} \\
\]

\[
\begin{array}{c} n = 1/9: \quad \mathcal{O} = |\langle \Psi_{CF} | GS \rangle|^2 \simeq 0.46 \\
\end{array} \\
\begin{array}{c} n = 1/7: \quad \mathcal{O} = |\langle \Psi_{CF} | GS \rangle|^2 \simeq 0.56 \\
\end{array} \\
\]

GM & NR Cooper, PRL 2009
Chern bands in more general tight binding models

• Original proposal for IQHE without magnetic fields: Haldane (1988)

• 2011: FQHE could naturally in models with spin-orbit coupling + interactions

Chern numbers

Numerical confirmation: D. Sheng; C. Chamon; N. Regnault & A. Bernevig, …
Stability of Fractional Chern Insulators

How to decide which lattice models have stable fractional Chern Insulators?

- single-particle dispersion - want flat bands
  - many groups
  - finite size matter a lot - success by iDMRG A. Grushin et al.

- shape of interactions - clear hierarchy of two-body energies desirable “Pseudopotentials”
  - Läuchli, Liu, Bergholtz, Moessner + other proposals

- band geometry - ideally want even Berry curvature
  - Regnault, Bernevig; Dobardzic, Milovanovic, …
  - no systematic in-depth study of geometric measures

- Full story: all three aspects contribute

This Talk!
**Band Geometry: Berry Curvature and Chern Number**

Basic notations:

Fourier transform:

\[ |k, b\rangle = \frac{1}{\sqrt{N_c}} \sum_{R} e^{i k \cdot (R + d_b)} |R, b\rangle \]

sublattice index

\[ b = 1, \ldots, N \]

Single particle eigenstates:

\[ |k, \alpha\rangle = \sum_{b=1}^{N} u^\alpha_b(k) |k, b\rangle = \hat{\gamma}^\alpha_{k} |\text{vac}\rangle \]

Bloch Hamiltonian:

\[ H_{bc}(k) = \sum_{\alpha=1}^{N} E_{\alpha}(k) u^\alpha_{b}(k) u^\alpha_{c}(k) \]

Berry connection:

\[ A_{\alpha}(k) = -i \sum_{b=1}^{N} u^\alpha_{b} \nabla_k u^\alpha_{b} \]

gauge dependent

Berry curvature:

\[ B_{\alpha}(k) = \nabla \times A_{\alpha}(k) \]

gauge invariant

Chern number:

\[ c_1 = \frac{S_{BZ}}{2\pi} \langle B_{\alpha} \rangle \]

average <-> over BZ, quantized to integer values
Which Berry Curvature?

Gauge invariance of the Bloch functions: one arbitrary $U(1)$ phase for each $k$-point

$$|u_k^\alpha\rangle \rightarrow e^{i\phi_\alpha(k)}|u_k^\alpha\rangle$$

The above manifestly leaves $H$ invariant:

$$H_{bc}(k) = \sum_{\alpha=1}^{N} E_\alpha(k) u_b^{\alpha*}(k) u_c^\alpha(k)$$

However, sublattice dependent phases are not gauges:

$$u_a^\alpha(k) \rightarrow \tilde{u}_b^\alpha(k) = e^{ir_b \cdot k} u_b^\alpha(k)$$

as this substitution yields a modified Berry curvature:

$$\tilde{B}_\alpha(k) - B_\alpha(k) = \sum_{b=1}^{N} r_{b,y} \frac{\partial}{\partial k_x} |u_b^\alpha(k)|^2 - r_{b,x} \frac{\partial}{\partial k_y} |u_b^\alpha(k)|^2$$

There is a unique choice such that the polarisation reduces to the correct semi-classical expression

and canonical position operator

$$\hat{R}_\mu \rightarrow -i \frac{\partial}{\partial k_\mu}$$

see, e.g. Zak PRL (1989)
Example single particle properties

an example: Hofstadter spectrum in magnetic unit cell of 7x1, \( n_\phi = 3/7 \)

Curvature for Fourier transform with respect to unit cell pos

\[ \tilde{\mathcal{B}} = \nabla \times \tilde{\mathcal{A}} \]

Curvature for canonical Fourier transform

\[ \mathcal{B} = \nabla \times \mathcal{A} \]

Magnetic unit cell

\[
\begin{array}{cccccc}
\Phi/7 & \Phi/7 & \Phi/7 & \Phi/7 & \Phi/7 & \Phi/7 & -6\Phi/7
\end{array}
\]

net flux defined only mod \( \Phi_0 \)
GMP Algebra: Generating low-lying excitations

- single mode approximation captures low-lying neutral excitations in quantum Hall systems:

\[ |\Psi_{k}^{SMA}\rangle = \hat{\rho}_{k} |\Psi_{0}\rangle \]

for sp density operators \( \hat{\rho}_{k} = \sum_{q} \gamma^{\dagger}_{k+q} \gamma_{q} \)

GMP algebra (w/LLL form factor):

\[
[\rho_{LLL}(q), \rho_{LLL}(q')] = 2i \sin\left(\frac{1}{2} q \wedge q' \ell_{B}^{2}\right) \exp\left(\frac{1}{2} q \cdot q' \ell_{B}^{2}\right) \rho_{LLL}(q + q')
\]

Chern bands: generalised GMP algebra

- consider band-projected density operators for general Chern bands:

\[
\tilde{\rho}_q \equiv P_\alpha e^{i\mathbf{q}\cdot \mathbf{r}} P_\alpha = \sum_{k} \sum_{b=1}^{N} u_b^{\alpha*}(k + \mathbf{q}/2) u_b^{\alpha}(k - \mathbf{q}/2) \gamma_{k+\mathbf{q}/2}^{\alpha\dagger} \gamma_{k-\mathbf{q}/2}^{\alpha}
\]

- in general, the algebra of density operators does not close, i.e.

\[
[\tilde{\rho}_q, \tilde{\rho}_k] \neq F(k, \mathbf{q}) \tilde{\rho}_{k+\mathbf{q}}
\]

- intuitive consequences for FQH states:
  - no finite, closed set of low-energy excitations corresponding to the GMP single mode states
  - \( \tilde{\rho}_q \) can generate many distinct eigenstates
  - strong violation of the algebra should signal an unstable, gapless phase
Conditions for closure of the generalised GMP algebra I

• conditions for closure can be derived in long-wavelength expansion

i) $\mathcal{O}(k^2)$: 

$$ \sigma_c \equiv \sqrt{\frac{A_{BZ}^2}{4\pi^2} \langle B^2 \rangle - c_1^2} $$

flatness of Berry curvature

ii) $\mathcal{O}(k^3)$: 

Pullback of Hilbert space metric constant over BZ

$$ ds^2 = \langle \delta\psi|\delta\psi \rangle - \langle \delta\psi|\psi \rangle \langle \psi|\delta\psi \rangle $$

$$ g_{\mu\nu} + \frac{i}{2} F_{\mu\nu} $$

$$ = \sum_{\alpha \in \text{occ}} \text{tr} \left( \frac{\partial}{\partial k_{\mu}} P_{\alpha} \right) (1 - P_{\alpha}) \left( \frac{\partial}{\partial k_{\nu}} P_{\alpha} \right) $$

deviations

$$ \sigma_g \equiv \sqrt{\frac{1}{2} \sum_{\mu,\nu} \langle g_{\mu\nu} g_{\nu\mu} \rangle - \langle g_{\mu\nu} \rangle \langle g_{\nu\mu} \rangle} $$
Conditions for closure of the generalised GMP algebra II

iii) closure at all orders if

\[ D(k) \equiv \det g^\alpha(k) - \frac{B_\alpha(k)^2}{4} = 0 \]

• if i), ii) and iii) are met, one obtains a generalised GMP algebra:

\[
[\tilde{\rho}_q, \tilde{\rho}_k] = 2ie\sum_{\mu,\nu} g^\alpha_{\mu\nu} q_\mu k_\nu \sin\left(\frac{B_\alpha}{2} q \wedge k\right) \tilde{\rho}_{q+k}
\]

• under stronger variant of condition iii) the algebra reduces exactly to the GMP algebra, namely if

\[ T(k) \equiv \text{tr} g^\alpha(k) - |B_\alpha(k)| = 0 \]

• Current study: test how violations of the closure constraints correlate with gap

Target models to examine

• Hamiltonian: bosonic states with on-site interactions — defined independent of specific lattice

2-body contact
Laughlin $\nu = \frac{1}{2}$

3-body contact
Moore-Read $\nu = 1$

• lattice geometries to consider:

Haldane model
Kagomé model
Ruby lattice model

$N = 2$
$N = 3$
$N = 6$
Haldane model
**Haldane Model with $t_3=0$**

- Location of max gap for bosonic and fermionic Laughlin agrees with min RMS B
- Band geometry “interpolates” between bosonic, fermionic statistics
- For this model, quantum metric does not provide info beyond that supplied by curvature.

**Haldane Model: Effects of quantum metric for M=0, \( t_3 > 0 \)**

- position of maximum gap appears to be compromise between minimising curvature fluctuations and metric trace inequality (also seen in fermionic Laughlin)
Kagome lattice model
Approximately linear trend which holds from min RMS B point all the way to the phase boundary (gap closure)

Significant scatter, though, which may be explained by quantum metric
Kagome Model: “Shells” of constant RMS Curvature

- Considering models at surfaces of fixed $\sigma_c$, the violation of the metric trace equality $\langle T \rangle$ is highly correlated with the many-body gap

**Bosonic Laughlin on Kagome, trace inequality**

**Bosonic Moore–Read on kagome, trace inequality**

TS Jackson, G. Möller, R. Roy “Geometric stability of topological lattice phases”, arxiv:1408.0843
Ruby lattice model
Ruby Lattice Model: Gap vs RMS Curvature

- Similar linear dependence of gap on RMS B as seen in Kagome model
Ruby Lattice Model: “Shells” of constant RMS Curvature

- Even clearer results for influence of metric trace inequality $\langle T \rangle$

Models with many sub lattices can approximate Landau level physics more closely
**Model Comparison: Gaps vs. RMS B and trace inequality**

- Parameters yielding max gap are always in lower-left corner
- Demonstrates relevance of both band-geometric quantities
Conclusions

• Band geometry provides useful information about stability of fractional Chern insulators

• Berry curvature $O(k^2)$ is the dominant effect (as previously known)
• Trace of the quantum metric $O(k^3)$ provides further information

• Statistically, band geometry is strongly correlated with many-body gap useful for quick exploration of available parameter space
• But: it is only one of three factors, so not the only important measure

TS Jackson, G. Möller, R. Roy “Geometric stability of topological lattice phases”, arxiv:1408.0843

Quantifying degree of correlation on shells of const. RMS B — Spearman $\rho$ monotonicity test

- Nonparametric statistic which is sensitive to any monotonic relationship
- Perfect correlation for $\rho = \pm 1$, no correlation at $\rho = 0$
- Find significant, robust negative correlation between gap and metric inequality on all isosurfaces of constant RMS B, demonstrating importance of trace inequality as a subleading influence on the gap
Plots of quantum metric across the BZ

Standard Haldane

Kagome

Ruby
**NN-only Kagome Model: Gap vs. RMS B and Tr G**

- Looking at (correct) RMS B alone shows two branches
- Pattern holds in both bosonic Laughlin and bosonic Moore-Read states

**Kagome NN only, bosonic Laughlin**

- Branches distinguished by including information about metric trace inequality

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**Kagome NN only, bosonic MR**

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from: Wu, Bernevig & Regnault, PRB (2012)

 vary $\lambda_1$, all other par’s zero

this work

\[ \Delta \]

\[ \sigma_c \]

\[ \langle T \rangle \]

\[ \Delta \]

\[ \sigma_c \]
Geometry & gap for new parametrization of Haldane-t3

- Now have large range of parameters with uniform curvature
- Minimum trace inequality in distinct location from min RMS B