UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

23 April 2014

THEORETICAL PHYSICS 2

Answer ***three*** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1

(a) Explain what is meant by the *adiabatic approximation* in quantum mechanics, stating a general condition for its validity. [6]

(b) The state of a system with a time dependent Hamiltonian $H(t)$ may be written in terms of the instantaneous eigenstates $|\varphi_\alpha(t)\rangle$ of $H(t)$

$$|\Psi\rangle = \sum_\alpha a_\alpha(t) \exp\left(-\frac{i}{\hbar}\int^t E_\alpha(t')\,dt'\right)|\varphi_\alpha(t)\rangle\,.$$

Show that the time dependent Schrödinger equation implies that the amplitudes $\{a_\alpha(t)\}$ obey

$$\frac{da_\alpha}{dt} = -\sum_\beta \langle\varphi_\alpha|\left(\frac{d}{dt}|\varphi_\beta\rangle\right) a_\beta \exp\left(\frac{i}{\hbar}\int^t [E_\alpha(t') - E_\beta(t')]\,dt'\right)$$

[8]

(c) Consider a particle in the time-dependent infinite square well potential of width $L(t)$ at time t

$$V(x) = \begin{cases} 0 & 0 < x < L(t) \\ \infty & x < 0 \text{ or } x > L(t). \end{cases}$$

Write the wavefunction in terms of the instantaneous eigenstates

$$\Psi(x,t) = \sqrt{\frac{2}{L(t)}}\sum_{n=1}^{\infty} a_n(t)\exp\left(-\frac{i}{\hbar}\int^t E_n(t')\,dt'\right)\sin\left(\frac{n\pi x}{L(t)}\right),$$

where $E_n(t) = \frac{1}{2m}\left(\frac{\pi\hbar n}{L}\right)^2$. By finding $\langle\varphi_n|\left(\frac{d}{dt}|\varphi_p\rangle\right)$, show that $\{a_n(t)\}$ obey

$$\frac{da_n}{dt} = \frac{\dot{L}}{L}\sum_{p\neq n} a_p(-1)^{n+p}\frac{2np}{p^2-n^2}\exp\left(\frac{i}{\hbar}\int^t [E_n(t') - E_p(t')]\,dt'\right)$$

[10]

(d) By using

$$a_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

on the right hand side of these equations, and integrating, show that for $L(t) = vt$, the probability to make a transition from $n = 1$ at time $t = 0$ to $n = 2$ at time t is approximately

$$\frac{16}{9}\left|\int_0^t \frac{\exp(-i\alpha/t')}{t'}dt'\right|^2$$

where $\alpha = \frac{3\pi^2\hbar}{2mv^2}$. [9]

Solution (a) Bookwork. Some variant of the conditions $\hbar\dot{E} \ll \Delta E$ is required, where $\dot{E}$ is the rate at which energies change and ΔE is their separation. (b) This is a matter of substitution of the expansion into the Schrödinger equation, followed by multiplying by $\langle\varphi_\alpha|$ to take the component. (c) The computation of the matrix element is

$$\begin{aligned}\langle\varphi_n|\,\partial_t\,|\varphi_m\rangle &= -\frac{2}{L}\frac{m\pi\dot{L}}{L^2}\int_0^L x\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} \\ &= -\frac{\dot{L}}{L}(-1)^{n+m}\frac{2nm}{m^2-n^2}\end{aligned}$$

(Time can be saved by realizing that you don't have to differentiate the normalization factor, as this contribution will vanish by orthogonality). Substitution into the previous part yields the answer given. (d) First order perturbation theory applied to the previous part gives

$$\frac{da_2}{dt} = \frac{4}{3t}\exp\left(i\int^t \frac{3\pi^2\hbar}{2mv^2t'^2}\,dt'\right).$$

Integrating and taking the square modulus gives the result.

2

(a) Define the *propagator* and explain why

$$K(x,t|x',t') = \sum_\alpha \varphi_\alpha(x)\varphi^*_\alpha(x')e^{-iE_\alpha(t-t')/\hbar}, \qquad t > t'$$

where E_α and $\varphi_\alpha(x)$ are respectively the eigenvalues and eigenfunctions of a one dimensional Hamiltonian. [5]

(b) Using the above expression, find the propagator for a particle moving on a ring of radius R, with Hamiltonian

$$H = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2},$$

where θ is the angle. Leave your result expressed as a sum. [7]

(c) Find *all* the classical trajectories starting from (θ', t') and finishing at (θ, t). [7]

(d) Find the propagator from the path integral, expressing your result as a sum over classical paths. You may use the result for the propagator in one dimension

$$K(x,t|x',t') = \left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \exp\left[-\frac{m(x-x')^2}{2i\hbar(t-t')}\right], \qquad t > t'.$$

[7]

(e) Use the identity

$$\sum_{p=-\infty}^{\infty} \exp\left[-\frac{\alpha}{2}(x+2\pi p)^2\right] = \sqrt{\frac{1}{2\pi\alpha}} \sum_{q=-\infty}^{\infty} \exp\left[-\frac{q^2}{2\alpha} - iqx\right]$$

to prove that the results of parts (b) and (d) are equal. [7]

Solution (a) Bookwork. (b) The normalized eigenfunctions are

$$\varphi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi R}}$$

with $E_m = \frac{\hbar^2 m^2}{2mR^2}$, $m = 0, \pm1, \pm2, \ldots$. This gives the propagator

$$K(\theta,t|\theta',t') = \frac{1}{2\pi R}\sum_{m=-\infty}^{\infty} \exp\left[im(\theta-\theta') - iE_m(t-t')/\hbar\right]$$

(c) The classical trajectories have a constant velocity v_p with $v_p t/R = \theta - \theta' + 2\pi p$, where $p = 0, \pm1, \pm2, \ldots$.

(d) This gives

$$\begin{aligned} K(\theta, t|\theta', t') &= \left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \sum_{p=-\infty}^{\infty} \exp\left(\frac{imv_p^2[t-t']}{2\hbar}\right) \\ &= \left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \sum_{p=-\infty}^{\infty} \exp\left(\frac{imR^2[\theta-\theta'+2\pi p]^2}{2\hbar(t-t')}\right) \end{aligned} \tag{1}$$

(e) Use the identity with

$$\alpha = -i\frac{imR^2}{\hbar(t-t')}$$

3

(a) At low energies, scattering from a spherically symmetric potential $V(|\mathbf{r}|)$ is dominated by s-wave ($l = 0$) scattering. Explain why this implies that the wavefunction takes the form

$$\psi(\mathbf{r}) \sim 1 - \frac{a}{r}$$

outside the interaction region (i.e. the region where the scattering potential is non-zero), where a is the scattering length. [6]

(b) A pair of scatterers have scattering lengths a_1 and a_2 and are located at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The low energy wavefunction outside the interaction region has the form

$$\psi(\mathbf{r}) = e^{i\mathbf{k}_i\cdot\mathbf{r}} + c_1\frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{k|\mathbf{r}-\mathbf{r}_1|} + c_2\frac{e^{ik|\mathbf{r}-\mathbf{r}_2|}}{k|\mathbf{r}-\mathbf{r}_2|},$$

where $\mathbf{k}_i$ is the wavevector of the incoming particles, and $k = |\mathbf{k}_i|$. Find c_1 and c_2. [12]

(c) Far from the scatterers the wavefunction has the form

$$\psi(\mathbf{r}) \xrightarrow[\mathbf{r}\to\infty]{} e^{i\mathbf{k}_i\cdot\mathbf{r}} + f(\hat{\mathbf{r}})\frac{e^{ikr}}{r},$$

where $\hat{\mathbf{r}}$ is a unit vector parallel to $\mathbf{r}$. Find the scattering amplitude $f(\hat{\mathbf{r}})$ in terms of c_1 and c_2. [6]

(d) Find the total cross section in terms of c_1 and c_2. [9]

Solution (a) Bookwork. (b) Apply the boundary condition from part (a) at each scatterer to give the pair of equations

$$\begin{pmatrix} ika_1+1 & ka_1\varphi \\ ka_2\varphi & ika_2+1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} ka_1e^{i\mathbf{k}_i\cdot\mathbf{r}_1} \\ ka_2e^{i\mathbf{k}_i\cdot\mathbf{r}_2} \end{pmatrix} \qquad (2)$$

where $\varphi = \frac{e^{ikr_{12}}}{kr_{12}}$. The solution is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{k^2a_1a_2\varphi^2 - (ika_1+1)(ika_1+1)}\begin{pmatrix} ika_2+1 & -ka_1\varphi \\ -ka_2\varphi & ika_1+1 \end{pmatrix}\begin{pmatrix} ka_1e^{i\mathbf{k}_i\cdot\mathbf{r}_1} \\ ka_2e^{i\mathbf{k}_i\cdot\mathbf{r}_2} \end{pmatrix}$$

(c) This is independent of the previous part. One needs only to approximate $e^{ik|\mathbf{r}-\mathbf{r}_i|} \sim e^{ikr}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_i}$ to give

$$f(\hat{\mathbf{r}}) = \frac{c_1}{k}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_1} + \frac{c_2}{k}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_2}$$

(d) The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\hat{\mathbf{r}})|^2 = k^{-2}|c_1e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_1}+c_2e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_2}|^2 = k^{-2}\left[|c_1|^2+|c_2|^2+2\mathrm{Re}\,c_1c_2^*e^{-ik\hat{\mathbf{r}}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\right].$$

V7.1

Integrating over the unit sphere gives

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\left[|c_1|^2 + |c_2|^2 + 2\mathrm{Re}\, c_1 c_2^* \frac{\sin kr_{12}}{kr_{12}}\right].$$

The forward scattering amplitude is

$$f(\hat{\mathbf{r}} = \mathbf{k}_i/|\mathbf{k}_i|) = \frac{c_1}{k}e^{-i\mathbf{k}_i\cdot\mathbf{r}_1} + \frac{c_1}{k}e^{-i\mathbf{k}_i\cdot\mathbf{r}_2}$$

Verifying the optical theorem (**not part of the question**) is straightforward for $a_1 = a_2$ by multiplying the original Eq. (**??**) by $(\bar{c}_1, \bar{c}_2)$ to give

$$|c_1|^2 + |c_2|^2 + ka(i|c_1|^2 + i|c_2|^2 + 2\varphi\mathrm{Re}\,\bar{c}_1 c_2) = -ka\bar{c}_1 e^{i\mathbf{k}_i\cdot\mathbf{r}_1} - ka\bar{c}_2 e^{i\mathbf{k}_i\cdot\mathbf{r}_2}.$$

Taking the imaginary part immediately gives the optical theorem

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\mathrm{Im}\, f(\hat{\mathbf{r}} = \mathbf{k}_i/|\mathbf{k}_i|)$$

It seems to be harder to see for $a_1 \neq a_2$

4

(a) Give the form of the density matrix in thermal equilibrium of a system with Hamiltonian H in the canonical ensemble at temperature T. [4]

(b) For a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

show that in thermal equilibrium the average kinetic energy and average potential energy are equal at all temperatures. You may use the standard result

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$
$$p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right)$$

with $[a, a^\dagger] = 1$. [8]

(c) Show that

$$\overline{\langle x^2 \rangle} = \frac{\hbar}{2m\omega}\coth\left(\frac{\hbar\omega}{2k_B T}\right)$$

where $\overline{\langle \cdots \rangle}$ denotes the quantum and thermal average. [10]

(d) Use the identity

$$\exp\left[\beta\left(A + B\right) + \beta^2\left[A, B\right]/2 + O(\beta^3)\right] = \exp(\beta A)\exp(\beta B)$$

to find the high temperature form of the density matrix $\langle x|\rho|x'\rangle$ in the position representation, when the β^2 and higher terms in the exponent on the left hand side are neglected. [11]

Solution (a) $\rho = \exp(-\beta H)/Z$, where the partition function $Z = \operatorname{tr} e^{-\beta H}$.
(b) Many ways to do this, but perhaps the simplest is to represent x and p in terms of oscillator variables and note that since the density matrix is diagonal in the oscillator basis, only terms with $a^\dagger a$ or $aa^\dagger$ contribute. Thus

$$\overline{\langle T \rangle} = \frac{\hbar\omega}{4}\overline{\langle a^\dagger a + aa^\dagger \rangle} = \overline{\langle V \rangle}$$

(c) Now we actually have to evaluate one of these averages

$$\overline{\langle x^2 \rangle} = \frac{\hbar}{2m\omega}\overline{\langle 2N + 1 \rangle}$$

The average number is the Bose function

$$\frac{\sum_n n e^{-\beta\hbar\omega n}}{\sum_n e^{-\beta\hbar\omega n}} = \frac{1}{e^{\beta\hbar\omega} - 1}$$

(d) We are splitting the density matrix into two factors, one of which is diagonal is position and the other was calculated in the lectures

$$\begin{aligned}\langle x|\rho|x'\rangle &= Z^{-1}\exp(-\beta m\omega^2 x^2/2)\,\langle x|\exp(-\beta p^2/2m)|x'\rangle \\ &= Z^{-1}\exp(-\beta m\omega^2 x^2/2)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{1/2}\exp\left[-\frac{m(x-x')^2}{2\hbar^2\beta}\right]\end{aligned} \tag{3}$$

Normalization is a straightforward Gaussian integral

$$\langle x|\rho|x'\rangle = \left(\frac{m\omega}{2\pi\hbar}\right)\exp(-\beta m\omega^2 x^2/2)\exp\left[-\frac{m(x-x')^2}{2\hbar^2\beta}\right]$$

5 A system of bosons moving on a ring of radius R is described by the Hamiltonian $H = H_1 + H_2$, where

$$H_1 = \int_0^{2\pi} \left[\frac{\hbar^2}{2mR^2} \frac{d\psi^\dagger}{d\theta} \frac{d\psi}{d\theta} \right] d\theta$$

is the single particle Hamiltonian and

$$H_2 = \frac{U}{2} \int_0^{2\pi} \psi^\dagger \psi^\dagger \psi \psi \, d\theta$$

describes interactions between the particles. $\psi^\dagger(\theta)$ and $\psi(\theta)$ satisfy

$$\left[\psi(\theta), \psi^\dagger(\theta')\right] = \delta(\theta - \theta')$$
$$\left[\psi(\theta), \psi(\theta')\right] = \left[\psi^\dagger(\theta), \psi^\dagger(\theta')\right] = 0$$

(a) In the basis of angular momentum eigenstates

$$\varphi_l(\theta) = \frac{e^{il\theta}}{\sqrt{2\pi}}, \qquad l = 0, \pm 1, \pm 2, \dots$$

$\psi(\theta)$ may be expressed

$$\psi(\theta) = \sum_{l=-\infty}^{\infty} \varphi_l(\theta) a_l,$$

where a_l annihilates a particle in state l. By considering *only* states $l = 0$ and 1, show that the Hamiltonian takes the form

$$H = \mathcal{E} a_1^\dagger a_1 + \frac{U}{4\pi} \left[a_0^\dagger a_0^\dagger a_0 a_0 + a_1^\dagger a_1^\dagger a_1 a_1 + 4 a_1^\dagger a_0^\dagger a_0 a_1 \right]$$

and identify $\mathcal{E}$. [10]

(b) Find the energy of the product state

$$|N_0, N_1\rangle = \frac{1}{\sqrt{N_0! N_1!}} \left(a_0^\dagger\right)^{N_0} \left(a_1^\dagger\right)^{N_1} |\mathrm{VAC}\rangle \, .$$

[6]

(c) Show that in the state

$$|\chi\rangle = \frac{1}{\sqrt{N!}} \left[\cos \frac{\chi}{2} a_0^\dagger + \sin \frac{\chi}{2} a_1^\dagger \right]^N |\mathrm{VAC}\rangle \, .$$

the occupation of the states $l = 0$ and 1 follows a binomial distribution, with average occupancies $\overline{N}_0 = N \cos^2(\chi/2)$ and $\overline{N}_1 = N \sin^2(\chi/2)$. [10]

(d) Show that for a large number of particles N, the expectation value of the energy *per particle* is approximately

$$E(\chi)/N = \mathcal{E}\sin^2(\chi/2) + \frac{nU}{2}\left[1 + \frac{1}{2}\sin^2\chi\right],$$

where $n = N/(2\pi)$ is the density. [7]

Solution (a) This is a question of substituting the expansion in terms of $a_l^\dagger$, a_l and integrating over θ

$$H = \sum_l E_l a_l^\dagger a_l + \frac{U}{4\pi}\sum_{l_1+l_2=l_3+l_4} a_{l_1}^\dagger a_{l_2}^\dagger a_{l_3} a_{l_4}$$

where $E_l = \frac{\hbar^2 l^2}{2mR^2}$. Thus $\mathcal{E} = E_1$, and careful counting of the number of interaction terms with two 0's and two 1's is required. (b) This is a straightforward expectation value

$$E(N_0, N_1) = \mathcal{E}N_1 + \frac{U}{4\pi}\left[N_0(N_0-1) + N_1(N_1-1) + 4N_0N_1.\right]$$

(c) Expanding out $|\chi\rangle$ gives the contribution to the state $|N_0, N_1\rangle$ as

$$\frac{1}{\sqrt{N!}}\frac{N!}{N_0!N_1!}\cos^{N_0}(\chi/2)\sin^{N_1}(\chi/2)(a_0^\dagger)^{N_0}(a_1^\dagger)^{N_1}|\mathrm{VAC}\rangle =$$
$$\sqrt{\frac{N!}{N_0!N_1!}}\cos^{N_0}(\chi/2)\sin^{N_1}(\chi/2)\,|N_0, N_1\rangle\,.$$

Thus the probability of having (N_0, N_1) is

$$\frac{N!}{N_0!N_1!}\cos^{2N_0}(\chi/2)\sin^{2N_1}(\chi/2),$$

i.e. binomial with $p_0 = \cos^2(\chi/2)$, $p_1 = \sin^2(\chi/2)$

(d) This is a matter of realizing that for large N, we can replace occupancies by their mean values $N_l \to \bar{N}_l = Np_l$

6 Consider 2×2 complex matrices of the form

$$\mathsf{M} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1$$

(a) Explain why these matrices form a Lie group. [6]

(b) By considering M close to the identity $\mathsf{M} = \mathbb{1} + \mathsf{m} + \cdots$, show that an element m of the Lie algebra of this group may be written

$$\mathsf{m} = \lambda \mathsf{m}_1 + \mu \mathsf{m}_2 + \nu \mathsf{m}_3$$

where λ, μ, and ν are real, and

$$\mathsf{m}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{m}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathsf{m}_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the commutation relations of the m_a. [11]

(c) A pair of bosons (or oscillator variables) $a^\dagger$, a and $b^\dagger$, b satisfying the usual commutation relations are used to define

$$K_1 = \frac{1}{2}\left[a^\dagger b^\dagger + ab\right], \qquad K_2 = -\frac{i}{2}\left[a^\dagger b^\dagger - ab\right], \qquad K_3 = \frac{1}{2}\left[a^\dagger a + bb^\dagger\right].$$

Show that iK_a for $a = 1, 2, 3$ have the same commutation relations as the m_a in the previous part. [8]

(d) Relate $C = K_3^2 - K_1^2 - K_2^2$ to the number of a and b quanta, and explain why it commutes with the K_a. [8]

Solution (a) Main thing is to verify that $\mathsf{M}_1\mathsf{M}_2$ satisfies the above property if M_1 and M_2 do. A good answer would also point out that the existence of an inverse follows from unit determinant.

(b) Expanding M near the identity

$$\mathsf{M} = \mathbb{1} + \begin{pmatrix} \epsilon & \delta \\ \delta^* & \epsilon^*. \end{pmatrix}$$

Further, the determinant condition in the infinitesimal gives ϵ imaginary. Matrices of this form may be written in terms of the basis given. The commutation relations are

$$[\mathsf{m}_1, \mathsf{m}_2] = \mathsf{m}_3$$
$$[\mathsf{m}_3, \mathsf{m}_1] = -\mathsf{m}_2$$
$$[\mathsf{m}_2, \mathsf{m}_3] = -\mathsf{m}_1$$

(c) The commutation relations are

$$[K_1, K_2] = -iK_3$$
$$[K_3, K_1] = iK_2$$
$$[K_2, K_3] = iK_1$$

so that the commutation relations of the iK_a coincide with those of the m_a i.e. they form a unitary representation. (d) We have

$$C = K_3^2 - K_1^2 - K_2^2 = \frac{1}{4}\left([N_a - N_b]^2 - 1\right)$$

In this form the fact that C commutes is obvious, as $K_{1,2}$ only add and remove quanta in pairs.

END OF PAPER