UNIVERSITY OF CAMBRIDGE

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NATURAL SCIENCES TRIPOS Part II

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24 April 2013

THEORETICAL PHYSICS 2

*Answer **three** questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

1 Consider a spin-1/2 in a time varying field, described by the Hamiltonian

$$H(t) = \Omega_0 S_z + \frac{\Omega_\mathrm{R}}{2}\left(S_+ e^{-i\omega t} + S_- e^{i\omega t}\right),$$

where $S_\pm = S_x \pm i S_y$, and $S_i = \frac{\hbar}{2}\sigma_i$, $i = x, y, z$, with σ_i the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Write the state of the system $|\Psi(t)\rangle$ in the form

$$|\Psi(t)\rangle = \exp\left(-i\omega t S_z/\hbar\right)|\Psi_\mathrm{R}(t)\rangle$$

and show that $|\Psi_\mathrm{R}(t)\rangle$ obeys the equation

$$i\hbar \frac{d\,|\Psi_\mathrm{R}(t)\rangle}{dt} = H_\mathrm{Rabi}\,|\Psi_\mathrm{R}\rangle$$

where

$$H_\mathrm{Rabi} = (\Omega_0 - \omega)\,S_z + \Omega_\mathrm{R} S_x.$$

[*You might find it useful to write the Hamiltonian as a* 2×2 *matrix.*] [7]

(b) Find the eigenvalues of H_Rabi and describe the *complete* time evolution of the corresponding eigenstates (you don't need to find the eigenstates explicitly). [8]

(c) Find the evolution of the phase of the eigenstates of H_Rabi after time $2\pi/\omega$. Considering the *adiabatic* limit $\omega \ll \sqrt{\Omega_0^2 + \Omega_\mathrm{R}^2}$, interpret your result in terms of Berry's phase. [9]

(d) Explain how your answers to (a), (b), and (c) would be modified if we have spin-s, rather than spin-1/2. [9]

2 In one dimension, the propagator $K(x,t|x',t')$ is the solution of the equation

$$\left[i\hbar\frac{\partial}{\partial t} - H\right] K(x,t|x',t') = i\hbar\delta(x-x')\delta(t-t') \text{ and}$$
$$K(x,t|x',t') = 0 \text{ for } t < t',$$

where H is the Hamiltonian. The momentum space propagator is defined by

$$K(x,t|x',t') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp dp' \, K(p,t|p',t') \exp\left(ipx/\hbar - ip'x'/\hbar\right).$$

(a) Show that the propagator for a free particle is

$$K_{\text{free}}(x,t|x',t') = \theta(t-t')\left(\frac{m}{2i\pi\hbar(t-t')}\right)^{1/2} \exp\left[-\frac{m(x-x')^2}{2i\hbar(t-t')}\right]$$

[7]

(b) Now consider a particle moving in a linear potential, described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \alpha x.$$

Find the equation satisfied by $K(p,t|p',t')$ and verify that the solution is

$$K(p,t|p',t') = \theta(t-t')\delta(p-p'+\alpha[t-t']) \exp\left(\frac{i\left(p^3 - p'^3\right)}{6\alpha m\hbar}\right).$$

[8]

(c) Use the result of part (b) to obtain $K(x,t|x',t')$ for a particle moving in a linear potential. [9]

(d) By computing the classical action for a trajectory $(x',t') \to (x,t)$, show that the same result follows from the path integral. [9]

3 A particle is initially in a plane wave state $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ and interacts with a scattering potential $V(\boldsymbol{r})$. Its wavefunction is a solution of the Lippmann–Schwinger equation

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}'\,\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi_{\boldsymbol{k}}(\boldsymbol{r}').$$

(a) Show that the differential cross-section in the (first) Born approximation is

$$\frac{d\sigma(\theta,\phi)}{d\Omega} = \left|\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}\; e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}V(\boldsymbol{r})\right|^2$$

where $\boldsymbol{q} = \boldsymbol{k}_f - \boldsymbol{k}$ is the momentum transfer, and $\boldsymbol{k}_f$ is the final momentum, pointing in a direction specified by spherical coordinates (θ,ϕ). [7]

(b) Use this result to show that the total cross-section in the Born approximation is

$$\sigma_{\text{total}} = \frac{m^2}{\pi\hbar^4}\int d^3\boldsymbol{r}\; d^3\boldsymbol{r}'\; V(\boldsymbol{r})V(\boldsymbol{r}')\frac{\sin k\,|\boldsymbol{r}-\boldsymbol{r}'|}{k\,|\boldsymbol{r}-\boldsymbol{r}'|}e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

[8]

(c) By iterating the integral equation a second time find the *second* Born approximation to the scattering amplitude. [9]

(d) Show that the results of parts (b) and (c) are consistent with the optical theorem

$$\operatorname{Im} f(\theta = 0) = \frac{k\sigma_{\text{total}}}{4\pi},$$

where $f(\theta,\phi)$ is the scattering amplitude. [9]

4 A product state of a system of identical bosons or fermions is described in terms of single particle states $\varphi_\alpha(\boldsymbol{r})$, occupation numbers N_α, and creation and annihilation operators $a_\alpha^\dagger$, a_α. The field operator has the form

$$\psi(\boldsymbol{r}) \equiv \sum_\beta \varphi_\beta(\boldsymbol{r}) a_\beta,$$

while the density operator is

$$\hat{\rho}(\boldsymbol{x}) \equiv \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}).$$

(a) Show that in a product state the density-density correlation function has the form (with the plus sign for bosons and minus sign for fermions)

$$C_\rho(\boldsymbol{r},\boldsymbol{r}') \equiv \langle : \rho(\boldsymbol{r})\, \rho(\boldsymbol{r}') : \rangle = \langle \rho(\boldsymbol{r}) \rangle \langle \rho(\boldsymbol{r}') \rangle \pm g(\boldsymbol{r},\boldsymbol{r}') g(\boldsymbol{r}',\boldsymbol{r}).$$

Here $: \cdots :$ denotes normal ordering, $\langle \rho(\boldsymbol{r}) \rangle$ is the expectation value of the density, and

$$g(\boldsymbol{r},\boldsymbol{r}') = \sum_\alpha N_\alpha \varphi_\alpha^*(\boldsymbol{r}) \varphi_\alpha(\boldsymbol{r}')$$

is the single particle density matrix. The term with both creation and both annihilation operators corresponding to the same state may be neglected in the limit of a large system. [10]

(b) Assuming that $g(\boldsymbol{r},\boldsymbol{r}') \to 0$ as $|\boldsymbol{r} - \boldsymbol{r}'| \to \infty$, find the ratio

$$\frac{C_\rho(\boldsymbol{r},\boldsymbol{r})}{\lim\limits_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} C_\rho(\boldsymbol{r},\boldsymbol{r}')}$$

for both bosons and fermions. [4]

(c) Now find the form of the three point correlation function

$$C_\rho^{(3)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3) \equiv \langle : \rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\rho(\boldsymbol{r}_3) : \rangle,$$

expressing your answer in terms of $\langle \rho(\boldsymbol{r}) \rangle$ and $g(\boldsymbol{r},\boldsymbol{r}')$.

[15]

(d) Find the ratio

$$\frac{C_\rho^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})}{\lim\limits_{\substack{|\boldsymbol{r}_1-\boldsymbol{r}_2|\to\infty \\ |\boldsymbol{r}_1-\boldsymbol{r}_3|\to\infty \\ |\boldsymbol{r}_2-\boldsymbol{r}_3|\to\infty}} C_\rho^{(3)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)}$$

for both bosons and fermions. [4]

5 Consider a system of N bosons, each of which can occupy only two states: $|\uparrow\rangle$ and $|\downarrow\rangle$. We can introduce creation and annihilation operators $a_s^\dagger$, a_s, with $s = \uparrow, \downarrow$, to add and remove particles from the two states.

(a) Show that the operators

$$
\begin{aligned}
S_x &= \frac{\hbar}{2}\left(a_\uparrow^\dagger a_\downarrow + a_\downarrow^\dagger a_\uparrow\right) \\
S_y &= -i\frac{\hbar}{2}\left(a_\uparrow^\dagger a_\downarrow - a_\downarrow^\dagger a_\uparrow\right) \\
S_z &= \frac{\hbar}{2}\left(a_\uparrow^\dagger a_\uparrow - a_\downarrow^\dagger a_\downarrow\right)
\end{aligned}
$$

obey the angular momentum ($SU(2)$) commutation relations. [7]

(b) Show that $\boldsymbol{S}^2 \equiv S_x^2 + S_y^2 + S_z^2$ can be expressed in terms of N, the total number of bosons, and find the relationship between the spin quantum number s and N. [8]

(c) A totally symmetric wavefunction of N bosons can be written as $\Psi_{(s_1 s_2 \cdots s_N)}$, where the round brackets denote the operation of symmetrisation:

$$
\Psi_{(s_1 s_2 \cdots s_N)} = \frac{1}{N!}\sum_P \Psi_{s_{P1} s_{P2} \cdots s_{PN}},
$$

and the sum is over all permutations of N objects. How many independent components are needed to describe $\Psi_{(s_1 s_2 \cdots s_N)}$? Interpret this result in terms of angular momentum states. [5]

(d) What are the defining properties of the Lie group $SU(2)$? [3]

(e) Under an element U of $SU(2)$, the components ψ_s of a one boson state transform as $\psi \to \mathsf{U}\psi$. If ϕ_s and χ_s are the components of two one boson states, show that the quantity $\phi_\uparrow \chi_\downarrow - \phi_\downarrow \chi_\uparrow$ is invariant under $SU(2)$ transformations.

[10]

6 The one dimensional Klein–Gordon equation has the form

$$\left[\left(i\hbar\frac{\partial}{\partial t} - V(x)\right)^2 + c^2\hbar^2\frac{\partial^2}{\partial x^2} - m^2c^4\right]\psi(x,t) = 0,$$

where $V(x)$ is an external potential.

(a) When $V(x) = 0$, a general solution of the Klein–Gordon equation can be written

$$\psi(x,t) = \sum_k \sqrt{\frac{1}{2\omega_k}}\left[a_k \exp\left(i\left[kx - \omega_k t\right]\right) + b_k^\dagger \exp\left(-i\left[kx - \omega_k t\right]\right)\right],$$

where $\omega_k = \sqrt{k^2c^2 + m^2c^4/\hbar^2}$. How is this solution modified when $V(x) = V_0$? [6]

(b) Consider the potential step

$$V(x) = \begin{cases} V_0 & x > 0 \\ 0 & x < 0 \end{cases}.$$

Find the transmission and reflection amplitudes for an incoming plane wave of energy $E > mc^2$ incident from $x < 0$. [9]

(c) Paying careful attention to the analytic structure of the reflection and transmission amplitudes, describe their behaviour in the three regimes

$$\begin{aligned} I &: E > V_0 + mc^2 \\ II &: V_0 - mc^2 < E < V_0 + mc^2 \\ III &: E < V_0 - mc^2, \end{aligned}$$

assuming $V_0 > 2mc^2$. [12]

(d) What is the physical interpretation of the behaviour in regime III? [6]

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