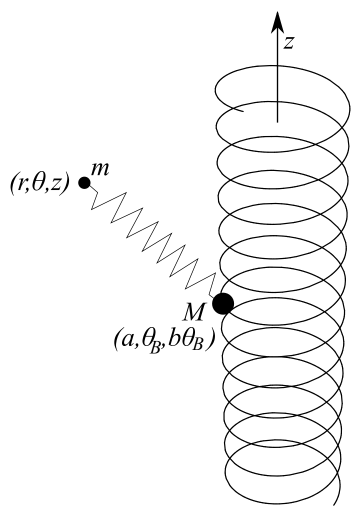
NATURAL SCIENCES TRIPOS Part II

Wednesday 13 January 2016 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **all** *questions to the best of your abilities. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A bead with mass M slides, without friction, along an infinite fixed coil which constrains the bead's cylindrical coordinates, (r_B, θ_B, z_B), to be $(a, \theta_B, b\theta_B)$. A massless spring with zero natural length and spring constant k connects the bead to an unconstrained particle with mass m and cylindrical coordinates (r, θ, z).



(a) Show that, up to irrelevant constants and ignoring gravity, the Lagrangian for the system is: [6]

$$L = \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2 + \frac{M}{m}(a^2+b^2)\dot{\theta}_B^2\right) - \frac{1}{2}k\left(r^2 - 2ar\cos(\theta-\theta_B) + (z-b\theta_B)^2\right).$$

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(b) Find the corresponding equations of motion for the particle and the bead. [5]

(c) The system has a helical symmetry. Find the corresponding conserved quantity. [8]

(d) The particle is released from rest at (r_0, θ_0, z_0). A time T later it is at $(r_0, \theta_0, z_0 + \Delta z)$. If the mass of the bead is negligible, show that

$$\Delta z = -\frac{2A}{b}$$

where A is a geometric property of the particle's trajectory, and give the geometric interpretation of A. [6]

2 (a) Show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \, ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, leads to two of Maxwell equations for a free electromagnetic field. [6]

(b) The electromagnetic stress-energy tensor is

$$T^{\mu\nu} = -F^{\mu}{}_{\lambda} F^{\nu\lambda} - g^{\mu\nu} \mathcal{L} \, .$$

Show that it is conserved. [8]

(c) The Lagrangian density for the interaction of a complex scalar field ϕ with the electromagnetic field A_μ (in natural units) is

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where $D_\mu \phi = (\partial_\mu + iqA_\mu)\phi$ and $(D_\mu \phi)^* = (\partial_\mu - iqA_\mu)\phi^*$. Show that the equation of motion for the electromagnetic field A_μ is $\partial_\mu F^{\mu\nu} = J^\nu$, where [6]

$$J^\mu = iq \left[\phi^* D^\mu \phi - \phi (D^\mu \phi)^* \right] \, .$$

(d) The Lagrangian density $\mathcal{L}$ is invariant under local phase transformation, $\phi'(x) = e^{-iq\alpha(x)} \phi(x)$, provided that a simultaneous Gauge transformation is performed, $A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$, where the function $\alpha(x)$ and the constant q are real. *Use Noether's theorem* to verify that $\partial_\mu J^\mu = 0$. [5]

In some parts of the question you may find it helpful to use Bianchi's identity:

$$\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0 \, .$$

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3 Consider the Klein-Gordon Lagrangian density for a complex scalar field in Minkowski space, coupled to an external vector potential A_μ and to a time-dependent driving force $f(t)$:

$$\mathcal{L} = (\partial_\mu \phi^*)\,(\partial^\mu \phi) - m^2 \phi^* \phi + ieA_\mu \left[\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi\right] + f(t)\,(\phi + \phi^*)$$

where $A_\mu = (V(\boldsymbol{r}), 0, 0, 0)$ and $V(\boldsymbol{r})$ is a real function of the space coordinates $\boldsymbol{r}$ but is independent of time.

(a) Show that the Euler-Lagrange equations can be written as

$$\partial_\mu \partial^\mu \phi + 2ieA^\mu(\boldsymbol{r})\partial_\mu \phi + m^2 \phi = f(t)$$

and equivalently for ϕ^*. [5]

(b) The Green's function $\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t')$ is a solution of the above equation of motion when the right hand side is replaced by $\delta(t - t')\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')$. Using the following sign convention for the Fourier transform,

$$\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} G(\boldsymbol{k}; \omega)\, e^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\,,$$

show that $G(\boldsymbol{k}; \omega)$ satisfies the equation

$$\left[-\omega^2 + k^2 + m^2\right] G(\boldsymbol{k}; \omega) + 2e\omega \int \frac{d^3k'}{(2\pi)^3} V(\boldsymbol{k} - \boldsymbol{k}') G(\boldsymbol{k}'; \omega) = 1$$

where $V(\boldsymbol{k} - \boldsymbol{k}') = \int d^3r\, V(\boldsymbol{r}) e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{r}')}$. [8]

(c) Consider the case where $V(\boldsymbol{k} - \boldsymbol{k}') = -(2\pi)^3\, i\gamma\, \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$, γ real and positive. Show that one can then obtain $G(\boldsymbol{k}; t, t')$ from the integral [3]

$$G(\boldsymbol{k}; t, t') = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-\omega^2 - 2e\gamma i\omega + k^2 + m^2}$$

Discuss the location of the poles as a function of $\boldsymbol{k}$, for fixed m, e, and γ. Draw schematically where they appear in the complex ω plane for $k^2 + m^2 > e^2\gamma^2$ and for $k^2 + m^2 < e^2\gamma^2$. [6]

(d) Assume that $k^2 + m^2 > e^2\gamma^2$. Using contour integration and Cauchy's theorem, compute $G(\boldsymbol{k}; t, t')$ for $t > t'$ as well as $t < t'$. Justify your choice of contour in each case. [3]

4 A ferromagnet consists of a large number, N, of interacting vector spins, $\{\boldsymbol{s}_i\}$, which each have unit length but can point in any direction. Each spin interacts with many other spins via an interaction energy $E = -\boldsymbol{s}_i \cdot \boldsymbol{s}_j$, which

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favors alignment. We propose a Landau theory of the following form to study $\boldsymbol{m} \equiv \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{s}_i$, the average magnetization of the system:

$$f = am + bm^2 + cm^3 + dm^4$$

where $m = |\boldsymbol{m}|$.

(a) Recalling the definition of E, explain which of the above coefficients are permitted, and whether they are positive or negative when the system is aligned and when it is disordered. You may assume no more terms are required in the expansion. [5]

(b) Writing $b = (T - T_c)/T_c$, and ignoring the temperature dependence of the other parameters, find and plot the equilibrium value of m as a function of T. Is the phase transition continuous or discontinuous? What symmetry does the system break at its phase transition? [5]

A nematic liquid crystal is similar to a ferromagnet, in that it consists of a large number, N, of interacting rod shaped molecules each oriented along a vector $\boldsymbol{s}_i$. However, in this case the molecules interact via an energy $E = -\left(\boldsymbol{s}_i \cdot \boldsymbol{s}_j\right)^2$. Nematic liquid crystals also display a transition from disordered to aligned at a given temperature.

(c) By considering the ground state of the nematic energy, explain *qualitatively* why in this case the vector $\boldsymbol{m} \equiv \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{s}_i$ always vanishes, and hence $\boldsymbol{m}$ is not a good order parameter. [3]

(d) We instead use the 3×3 tensor order parameter $S_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N}(3s_{i\alpha}s_{i\beta} - \delta_{\alpha\beta})$, where $s_{i\alpha}$ is the α component of $\boldsymbol{s}_i$, with $\alpha = x, y, z$. Show that $\mathrm{Tr}(S) = 0$. [2]

A Landau theory for liquid crystals requires us to expand out the free-energy in powers of invariants of S, leading to the form

$$f = a\ \mathrm{Tr}\left(S \cdot S\right) + b\ \mathrm{Tr}\left(S \cdot S \cdot S\right) + c\ \mathrm{Tr}\left(S \cdot S \cdot S \cdot S\right).$$

(e) By writing $S_{\alpha\beta} = Q(3n_\alpha n_\beta - \delta_{\alpha\beta})$, where $\boldsymbol{n}$ is a unit vector pointing along the alignment direction, and Q is a scalar measure of the degree of alignment, plot graphs of f as a function of Q for a range of values of $b \leq 0$, assuming $a > 0$ and $c > 0$ are both constant. Is the transition continuous or discontinuous? [5]

(f) Find the critical value of $b \leq 0$ at which the transition occurs, and the value of Q just above and below the transition. *Recall that the equilibrium value of Q minimizes f.* [5]

END OF PAPER