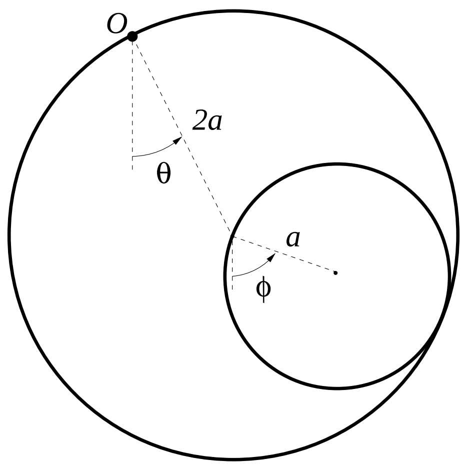
NATURAL SCIENCES TRIPOS Part II

Wednesday 15 January 2014 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains six sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A thin hollow cylinder of mass M and radius $2a$ can swing freely about a fixed horizontal axis passing through the point O, as illustrated in the figure. A second thin hollow cylinder of mass M and radius a rests on the inner surface of the first cylinder, lying parallel to its length.



(a) Assuming that no friction acts between the two cylinders and that the angular velocity of the smaller cylinder about its own axis is fixed to zero throughout the motion, derive the Lagrangian of the system as a function of the angles θ and ϕ. Expand it to second-order assuming both angles as well as their time derivatives are small and show that the result can be written as

$$L = \tfrac{1}{2}Ma^2(12\dot{\theta}^2 + \dot{\phi}^2 + 4\dot{\theta}\dot{\phi}) - \tfrac{1}{2}Mga(4\theta^2 + \phi^2),$$

up to irrelevant additive constants. [9]

(TURN OVER for continuation of question 1

(b) From the Euler–Lagrange equations, derive the equations of motion of the system. Show that the natural frequencies of small oscillations are given by

$$\omega_\pm^2 = \frac{(2 \pm \sqrt{2})g}{2a},$$

and describe the corresponding normal modes. [10]

(c) Assume now that friction acts between the two cylinders such that the smaller cylinder rolls without slipping on the inner surface of the larger one. Show that to second-order in the angles θ and ϕ and their time derivatives the Lagrangian of the system is

$$L = \tfrac{1}{2}Ma^2(16\dot{\theta}^2 + 2\dot{\phi}^2) - \tfrac{1}{2}Mga(4\theta^2 + \phi^2),$$

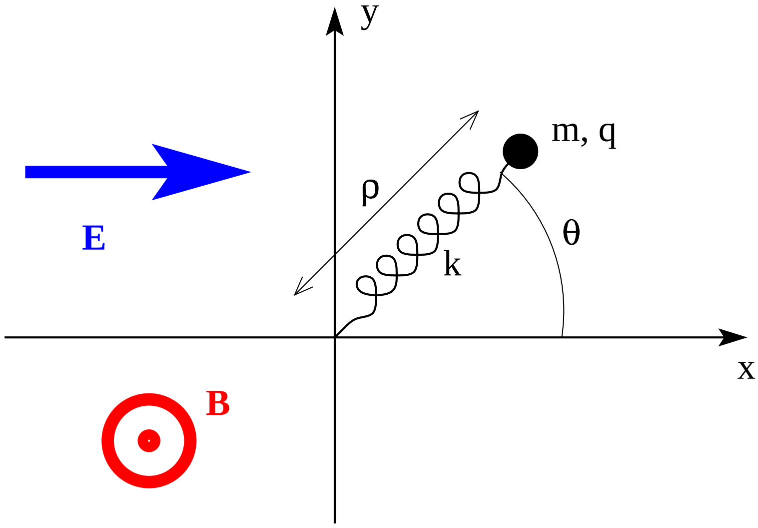
up to irrelevant additive constants. [7]

(d) Show that the equations of motion in this case have the form

$$\ddot{\theta} + \omega_1^2\theta = 0, \qquad \ddot{\phi} + \omega_2^2\phi = 0,$$

and obtain expressions for the natural frequencies of oscillation ω_1^2 and ω_2^2. Describe the corresponding normal modes. [7]

2 Consider a particle of mass m and charge $q > 0$ moving in two dimensions, in presence of a uniform static electric field $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, and a uniform static magnetic field perpendicular to the plane, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ ($E > 0$, $B > 0$). The particle is attached to a fixed point on the plane (say, the origin of the reference frame) by an ideal spring of zero natural length and constant k (see figure.)



(a) Using polar coordinates ρ, θ, show that the Lagrangian for the particle can be written as [9]

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right) - \frac{1}{2}k\rho^2 + q\left(E\rho\cos\theta + \frac{B}{2}\rho^2\dot{\theta}\right)$$

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Hint: it is helpful to choose a vector potential of the form

$$\boldsymbol{A} = \frac{B}{2}\Big(- \rho \sin\theta, \rho\cos\theta, 0\Big) .$$

(b) Derive the conjugate momenta p_ρ and p_θ, and show that the Hamiltonian for the particle takes the form [5]

$$H = \frac{p_\rho^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{\rho} - \frac{qB}{2}\rho\right)^2 + \frac{1}{2}k\rho^2 - qE\rho\cos\theta$$

Obtain Hamilton's equations of motion. [4]

(c) Using the Lagrangian from part (a), write the Euler-Lagrange equations of motion. Show that the equilibrium solution (i.e., when all time derivatives vanish) corresponds to $\rho_0 = qE/k$ and $\theta_0 = 0$. [5]

(d) Consider the Euler-Lagrange equations of motion expanded to first order around the equilibrium solution (you do *not* need to derive these equations):

$$m\frac{d^2\tilde{\rho}}{dt^2} + k\tilde{\rho} - qB\rho_0\frac{d\tilde{\theta}}{dt} = 0$$

$$m\rho_0\frac{d^2\tilde{\theta}}{dt^2} + qB\frac{d\tilde{\rho}}{dt} + qE\tilde{\theta} = 0$$

where $\theta = \theta_0 + \tilde{\theta}$ and $\rho = \rho_0 + \tilde{\rho}$, and both $\tilde{\rho}$ and $\tilde{\theta}$ as well as their derivatives are small. Show that these equations admit a solution of the form $\tilde{\rho} = \varepsilon\cos\omega t$ and $\tilde{\theta} = (\varepsilon/\rho_0)\sin\omega t$, where $\varepsilon \ll 1$, provided that ω takes the values: [7]

$$\omega_{1,2} = -\frac{qB}{2m}\left[1 \mp \sqrt{1 + \frac{4mk}{q^2B^2}}\,\right]$$

Draw the trajectory of the particle in the xy plane corresponding to this special solution (recall that we expanded to first order in ε and higher order terms ought to be disregarded); use arrows to indicate the directions of motion along the trajectory corresponding to the allowed values of ω. Compare the directions of motion with respect to that which you would expect from the motion of a positively charged particle in the magnetic field alone. [3]

3 A dynamical system with Hamiltonian $H(q_i, p_i, t)$ is described by independent coordinates q_i $(i = 1, ..., n)$ and corresponding generalised (canonical) momenta p_i.

(a) Show that Hamilton's equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t},$$

(TURN OVER for continuation of question 3

where $L = L(q_i, \dot{q}_i, t)$ is the Lagrangian of the system. [6]

(b) A new set of coordinates and momenta (Q_i, P_i) is defined by

$$Q_i = Q_i(q_j, p_j, t), \qquad P_i = P_i(q_j, p_j, t), \qquad i = 1, ..., n.$$

and a new Hamiltonian by $\mathcal{H}(Q_i, P_i, t) = \sum_i P_i \dot{Q}_i - \mathcal{L}(Q_i, \dot{Q}_i, t)$, where $\mathcal{L}(Q_i, \dot{Q}_i, t)$ is the Lagrangian in the new coordinates. Assuming that the variations in q_i and Q_i vanish at the end-points, use Hamilton's principle of least action to verify that the coordinate transformation is *canonical*, i.e. it preserves the form of Hamilton's equations of motion, if there exists a function $G = G(q_i, Q_i, t)$ such that

$$\frac{\mathrm{d}G}{\mathrm{d}t} = L(q_i, \dot{q}_i, t) - \mathcal{L}(Q_i, \dot{Q}_i, t).$$

[6]

(c) By using the above expression for $\mathrm{d}G/\mathrm{d}t$, or otherwise, show that if $\mathcal{H}(Q_i, P_i, t) = H(q_i, p_i, t)$ then the coordinate transformation is canonical if

$$\sum_i p_i \, \mathrm{d}q_i - \sum_i P_i \, \mathrm{d}Q_i$$

is an exact differential. [5]

(d) In the general case in which the relationship $\mathcal{H}(Q_i, P_i, t) = H(q_i, p_i, t)$ may not hold, show that

$$p_i = \frac{\partial G}{\partial q_i}, \qquad P_i = -\frac{\partial G}{\partial Q_i}, \qquad \mathcal{H} - H = \frac{\partial G}{\partial t}.$$

Hence show that Hamilton's equations of motion in the new coordinates may be brought into the trivial form $\dot{Q}_i = 0$ and $\dot{P}_i = 0$, if G is chosen to satisfy

$$\frac{\partial G}{\partial t} + H\left(q_i, \frac{\partial G}{\partial q_i}, t\right) = 0. \qquad (*)$$

[6]

(e) The Hamiltonian for a one-dimensional harmonic oscillator of mass m and natural frequency ω has the form

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

By assuming that $G = G_1(q) + G_2(t)$ in Eq. $(*)$ above, show that

$$G = \int \sqrt{2m(\beta - \tfrac{1}{2}m\omega^2 q^2)} \, \mathrm{d}q - \beta t,$$

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where β is a constant. By identifying the new coordinate $Q = \beta$, obtain the form of the new generalised momentum coordinate P and hence show that

$$q = \sqrt{\frac{2\beta}{m\omega^2}} \sin \omega(t - \gamma),$$

where γ is a constant. [10]

4 The Lagrangian density for a free complex scalar field ϕ of mass m is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi,$$

where ϕ and ϕ^* are considered as independent fields.

(a) Obtain the equations of motion for the fields ϕ and ϕ^* and use them to show that $\partial_\mu j^\mu = 0$, where

$$j^\mu = iq(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

and q is a real constant. [4]

(b) Use Noether's theorem to verify that the invariance of $\mathcal{L}$ under the global phase transformation $\phi'(x) = e^{-iq\alpha}\phi(x)$, with α a real constant, implies that $\partial_\mu j^\mu = 0$. [4]

(c) The Lagrangian density for the interaction of a complex scalar field ϕ of mass m with the electromagnetic field A_μ is

$$\hat{\mathcal{L}} = (\bar{D}_\mu \phi^*)(D^\mu \phi) - m^2 \phi^* \phi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where $D_\mu \phi = (\partial_\mu + iqA_\mu)\phi$, $\bar{D}_\mu \phi^* = (\partial_\mu - iqA_\mu)\phi^*$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Show that the equation of motion for the electromagnetic field A_μ is $\partial_\mu F^{\mu\nu} = J^\nu$, where

$$J^\mu = iq(\phi^* D^\mu \phi - \phi \bar{D}^\mu \phi^*),$$

and hence show that $\partial_\mu J^\mu = 0$. [10]

(d) Show that $\hat{\mathcal{L}}$ is invariant under the local phase transformation $\phi'(x) = e^{-iq\alpha(x)}\phi(x)$, provided $A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$. [7]

(e) Hence use Noether's theorem to verify that $\partial_\mu J^\mu = 0$. [8]

5 Consider the Klein-Gordon Lagrangian density for a complex scalar field in Minkowski space, coupled to an external (static) electromagnetic field and to a time-dependent driving force

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi + ieA_\mu [\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi] + f(t)(\phi + \phi^*)$$

where A_μ is a function of the space coordinates $\boldsymbol{r}$ but is independent of time.

(TURN OVER for continuation of question 5

(a) Show that the Euler-Lagrange equations in the Lorenz gauge ($\partial_\mu A^\mu = 0$) can be written as

$$\partial_\mu \partial^\mu \phi + 2ieA^\mu(\boldsymbol{r})\partial_\mu \phi + m^2 \phi = f(t)$$

and equivalently for ϕ^*. [8]

(b) The Green's function $\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t')$ is a solution of the above equation of motion when the right hand side is replaced by $\delta(t-t')\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$. Using the following sign convention for the Fourier transform,

$$\mathcal{G}(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} G(\boldsymbol{k}; \omega)\, e^{-i\omega(t-t') + i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} ,$$

show that $G(\boldsymbol{k}; \omega)$ satisfies the equation

$$\left[-\omega^2 + k^2 + m^2\right] G(\boldsymbol{k}; \omega) + 2e \int \frac{d^3k'}{(2\pi)^3} \left[A^0(\boldsymbol{k}-\boldsymbol{k}')\omega - \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{k}'\right] G(\boldsymbol{k}'; \omega) = 1$$

where $A^\mu(\boldsymbol{k}) = \int A^\mu(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r$. [10]

(c) Inverting the above equation to find $G(\boldsymbol{k}; \omega)$ is in general difficult. Consider the simplified (although unphysical) case where $\boldsymbol{A}(\boldsymbol{k}-\boldsymbol{k}') = 0$ and $A^0(\boldsymbol{k}-\boldsymbol{k}') = (2\pi)^3\, i\gamma\, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$. Show that one can then obtain $G(\boldsymbol{k}; t, t')$ from the integral [3]

$$G(\boldsymbol{k}; t, t') = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-\omega^2 + 2e\gamma i\omega + k^2 + m^2}$$

Discuss the location of the poles as a function of $\boldsymbol{k}$, for fixed m, e, and γ. Draw schematically where they appear in the complex ω plane for $k^2 + m^2 > e^2\gamma^2$ and for $k^2 + m^2 < e^2\gamma^2$. What happens when $k^2 + m^2 = e^2\gamma^2$? [8]

(d) Assume that $k^2 + m^2 > e^2\gamma^2$. Using contour integration and Cauchy's theorem, compute $G(\boldsymbol{k}; t, t')$ for $t > t'$ as well as $t < t'$. Justify your choice of contour in each case. [4]

6 Consider the Landau free energy expansion of a system with complex order parameter $\phi(x)$ in 1D:

$$\beta H = \int f\, dx = \int \left[a\phi^*\phi + \frac{1}{2}(\phi^*\phi)^2 + c\,(\partial_x \phi^*)(\partial_x \phi) + d\left(\partial_x^2 \phi^*\right)\left(\partial_x^2 \phi\right)\right] dx$$

with the coefficients a, c, d real.

(a) When $c > 0$ (and you may as well set $d = 0$), this is equivalent to the free energy expected for an Ising ferromagnet, except that the order parameter is now complex. Find the physical state of the system as a function of the coefficients in the free energy and discuss the nature of the phase transition. [6]

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What type of symmetry is spontaneously broken at this transition? Compute the dependence of the order parameter on the coefficient a close to the transition, in the ordered phase. [6]

(b) Compute the behaviour of the zero-field magnetic susceptibility $\chi = (\partial\phi/\partial B)\big|_{B=0}$ close to the transition (both above and below). Once again, assume $c > 0$ and $d = 0$. Consider the case of a magnetic field B pointing along the real axis in the complex ϕ plane, i.e., add the term $-B(\phi + \phi^*)/2$ to the free energy given above, with B real. [6]

(c) Let us then set $d = 1$ and consider the general case where c is allowed to take on negative as well as positive values. Assume that the order parameter takes the form $\phi(x) = \phi_0 e^{i(kx+\delta)}$, where $\phi_0 > 0$, k and δ are real constants. Find the values that these constants need to take in order to minimize the free energy, as a function of the coefficients a and c. [12]
Hint: substituting the given form for $\phi(x)$ into the free energy:

$$f\Big|_{\phi(x)=\phi_0 e^{i(kx+\delta)}} = a\phi_0^2 + \frac{1}{2}\phi_0^4 + ck^2\phi_0^2 + k^4\phi_0^2,$$

obtain the location of the extrema using partial derivatives with respect to ϕ_0 and k; then, compare the values of the free energy at these extrema to find which is the absolute minimum.
What happens to the dependence on δ? How does it relate to the type of symmetry that is spontaneously broken across the transition considered in part (b)? [3]

END OF PAPER