NATURAL SCIENCES TRIPOS Part II

Wednesday 14 January 2009 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains ??? sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A bead of mass m slides freely on a light wire of parabolic shape, which is forced to rotate with angular velocity ω about a vertical axis. The equation of the parabola is

$$z = \frac{1}{2}ar^2$$

where z is the height and r is the distance from the axis of rotation.

(a) Show that the Lagrangian for this system is [5]

$$L = \frac{1}{2}m\left[\left(1 + a^2r^2\right)\dot{r}^2 + \left(\omega^2 - ag\right)r^2\right]$$

(b) Find a constant of the motion. [6]

(c) The bead is released at $r = 1/a$ with $\dot{r} = v$. Show that if $\omega^2 \geq ag$ the bead escapes to infinity. Show that if $\omega^2 < ag$ it oscillates about $r = 0$, and find the maximum value of r. [9]

(d) Now suppose the wire is not forced but rotates freely about the vertical axis with angular velocity $\dot{\phi}$. Find the new Lagrangian and constants of the motion. [7]

(e) If the bead is released with the same initial conditions as before, i.e. $r = 1/a$, $\dot{r} = v$, $\dot{\phi} = \omega$, show that in this case it cannot escape to infinity for any value of ω, and find the maximum and minimum values of r. [6]

Answer: (a) The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2 + r^2\omega^2\right)$$

where $\dot{z} = ar\dot{r}$. The potential energy is $mgz = magr^2/2$. Hence

$$L = T - V = \frac{1}{2}m\left[\left(1 + a^2r^2\right)\dot{r}^2 + \left(\omega^2 - ag\right)r^2\right]$$

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(b) The Lagrangian is not explicitly time-dependent and therefore the Hamiltonian is conserved. Now $H = p_r\dot{r} - L$ *where*

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\left(1 + a^2r^2\right)\dot{r}$$

Hence

$$H = \frac{1}{2}m\left[\left(1 + a^2r^2\right)\dot{r}^2 - \left(\omega^2 - ag\right)r^2\right]$$

N.B. $H \neq T + V$.

(c) Initially

$$H = \frac{1}{2}m\left[2v^2 - \left(\omega^2 - ag\right)/a^2\right]$$

Hence the motion is such that

$$\left(1 + a^2r^2\right)\dot{r}^2 = \left(\omega^2 - ag\right)\left(r^2 - 1/a^2\right) + 2v^2$$

i.e.

$$\dot{r}^2 = \frac{(\omega^2 - ag)(r^2 - 1/a^2) + 2v^2}{1 + a^2r^2}$$

Thus $\dot{r} = 0$ *when*

$$r^2 = \frac{1}{a^2} - \frac{2v^2}{\omega^2 - ag} \equiv r_0^2$$

For $\omega^2 > ag$, *this means* $r_0 < 1/a$. *If* $v > 0$, *the bead moves outwards,* $\dot{r}$ *is never zero, and so the bead moves to* $r = \infty$. *If* $v < 0$, *the bead moves inwards to* $r = r_0$, *then* $\dot{r}$ *becomes and remains positive, so again the bead moves to* $r = \infty$.

For $\omega^2 < ag$, *this means* $r_0 > 1/a$. *If* $v > 0$, *the bead moves outwards to* $r = r_0$, *then* $\dot{r}$ *becomes negative and the bead moves inwards to* $r = 0$ *then out to* $r = r_0$ *again on the other side of the parabola, i.e. it oscillates about* $r = 0$ *with amplitude* r_0. *If* $v < 0$, *the bead moves inwards to* $r = 0$, *then out to* $r = r_0$ *on the other side of the parabola, i.e. again it oscillates about* $r = 0$ *with amplitude* r_0. *Thus the maximum value of* r *is*

$$r_{max} = r_0 = \sqrt{\frac{1}{a^2} + \frac{2v^2}{ag - \omega^2}}$$

(d) When the rotation is not forced, the angle ϕ *is another generalized coordinate. The Lagrangian becomes*

$$L = T - V = \frac{1}{2}m\left[\left(1 + a^2r^2\right)\dot{r}^2 + r^2\dot{\phi}^2 - agr^2\right]$$

The Hamiltonian is now $H = p_r\dot{r} + p_\phi\dot{\phi} - L$ *where*

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

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is a constant of the motion, since L does not depend explicitly on ϕ. *Hence*

$$H = \frac{1}{2}m\left[\left(1 + a^2r^2\right)\dot{r}^2 + r^2\dot{\phi}^2 + agr^2\right] ,$$

which is also constant of the motion, since L does not depend explicitly on time. N.B. $H = T + V$ *now.*

(e) The intial conditions give

$$H = \frac{1}{2}m\left[2v^2 + \left(\omega^2 + ag\right)/a^2\right]$$

as before, and $p_\phi = m\omega/a^2$, *so that* $\dot{\phi} = \omega/(ar)^2$ *and*

$$\left(1 + a^2r^2\right)\dot{r}^2 + \omega^2/(a^4r^2) + agr^2 = 2v^2 + \left(\omega^2 + ag\right)/a^2$$

Hence

$$\left(1 + a^2r^2\right)\dot{r}^2 = 2v^2 + \frac{1}{a^2}\left[\omega^2\left(1 - \frac{1}{a^2r^2}\right) + ag\left(1 - a^2r^2\right)\right]$$

If we let $r \to \infty$ *then the terms involving* r^2 *dominate on each side and we get* $\dot{r}^2 \to -ag$, *which is impossible. Hence the bead cannot escape to* $r = \infty$. *The points where* $\dot{r} = 0$ *are given by*

$$2a^2v^2 + \omega^2\left(1 - \frac{1}{a^2r^2}\right) + ag\left(1 - a^2r^2\right) = 0$$

which is a quadratic equation for $u = a^2r^2$:

$$\omega^2 - \left(2a^2v^2 + \omega^2 + ag\right)u + agu^2 = 0$$

therefore the maximum and minimum values of r are $r_\pm$ *where*

$$a^2r_\pm^2 = \frac{1}{ag}\left(2a^2v^2 + \omega^2 + ag \pm \sqrt{\left(2a^2v^2 + \omega^2 + ag\right)^2 - 4ag\omega^2}\right) .$$

2 Define the Hamiltonian of a dynamical system with a finite number of degrees of freedom. [3]

Explain briefly the concept of a canonical transformation. [3]

Show by means of a canonical transformation that the Hamiltonian

$$H = \frac{p^2}{2m} + pf(q) + \frac{1}{2}kq^2$$

describes the motion of a particle of mass m in some potential $U(q)$, and express $U(q)$ in terms of $f(q)$. [8]

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Show that the following transformation is canonical:

$$Q = \tan^{-1}\frac{\lambda q}{p}\,, \quad P = \frac{p^2+\lambda^2 q^2}{2\lambda} + g\left(\frac{p}{q}\right)\,,$$

where λ is an arbitrary constant and $g(x)$ is an arbitrary function. [8]

Hence, or otherwise, calculate the motion of the particle in the potential

$$U(q) = \frac{1}{2}kq^2 - \frac{1}{2}\frac{A^2}{q^2}$$

where $k > 0$. [8]

Discuss your answer. [3]

Answer: The Hamiltonian is defined as

$$H(\{q_j\},\{p_j\}) = \sum_j p_j\dot{q}_j - L$$

where q_j are the generalized coordinates and p_j are the corresponding canonical momenta, defined from the Lagrangian $L(\{q_j\},\{\dot{q}_j\})$ by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

A canonical transformation is a transformation of the generalized coordinates and canonical momenta to new variables $Q_k(\{q_j\},\{p_j\})$ and $P_k(\{q_j\},\{p_j\})$ that preserve the canonical Poisson brackets

$$\{Q_k, P_k\} = \{q_k, p_k\} = 1$$

(all other Poisson brackets vanishing), where

$$\{f,g\} \equiv \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)$$

It then follows that Q_k and P_k satisfy Hamilton's equations of motion:

$$\dot{Q}_k = \frac{\partial H}{\partial P_k}\,, \quad \dot{P}_k = -\frac{\partial H}{\partial Q_k}$$

We have

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} + f(q)$$

Hence $p = m[\dot{q} - f(q)]$ and

$$L = p\dot{q} - H = \frac{p^2}{2m} - \frac{1}{2}kq^2 = \frac{1}{2}m\{\dot{q}^2 - 2\dot{q}f(q) + [f(q)]^2\} - \frac{1}{2}kq^2$$

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The transformation $Q = q,\ P = p + mf(q)$ is canonical since $\{Q, P\} = 1 - 0 = 1$. Then

$$H = \frac{P^2}{2m} + \frac{1}{2}kQ^2 - \frac{1}{2}m[f(Q)]^2 = T + U$$

where

$$U(Q) = \frac{1}{2}kQ^2 - \frac{1}{2}m[f(Q)]^2$$

The transformation given,

$$Q = \tan^{-1}\frac{\lambda q}{p}\ , \quad P = \frac{p^2 + \lambda^2 q^2}{2\lambda} + g\left(\frac{p}{q}\right)\ ,$$

has

$$\begin{aligned} dQ &= \frac{\lambda q p}{p^2 + \lambda^2 q^2}\left(\frac{dq}{q} - \frac{dp}{p}\right) \\ dP &= \left[\lambda q - \frac{p}{q^2}g'\left(\frac{p}{q}\right)\right]dq + \left[\frac{p}{\lambda} + \frac{1}{q}g'\left(\frac{p}{q}\right)\right]dp \end{aligned}$$

Thus

$$\{Q, P\} = \frac{\lambda q p}{p^2 + \lambda^2 q^2}\left(\frac{1}{q}\left[\frac{p}{\lambda} + \frac{1}{q}g'\left(\frac{p}{q}\right)\right] + \frac{1}{p}\left[\lambda q - \frac{p}{q^2}g'\left(\frac{p}{q}\right)\right]\right) = 1$$

We can write

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 - \frac{1}{2}\frac{A^2}{q^2}$$

and go back, defining $p' = p - A\sqrt{m}/q,\ q' = q$, to give

$$H = \frac{p'^2}{2m} + \frac{1}{2}kq'^2 + \frac{A}{\sqrt{m}}\frac{p'}{q'}$$

Hence $H = \omega P$ where

$$\frac{\omega}{\lambda} = \frac{1}{m}\ , \quad \omega\lambda = k\ , \quad \omega g(x) = \frac{A}{\sqrt{m}}x$$

Hence

$$\omega = \sqrt{\frac{k}{m}}\ , \quad \lambda = \sqrt{km}\ , \quad g(x) = \frac{A}{\sqrt{k}}x$$

The equations of motion are

$$\dot{Q} = \omega\ , \quad \dot{P} = 0$$

so that $Q = \omega t + \phi$ and P is constant (which is obvious since $P = H/\omega$). Hence

$$p' = q\sqrt{km}\cot(\omega t + \phi)\ , \quad H = \frac{1}{2}kq^2[\cot^2(\omega t + \phi) + 1] + A\sqrt{k}\cot(\omega t + \phi)$$

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and so

$$q^2 = \frac{2}{k}\sin(\omega t+\phi)\left[H\sin(\omega t+\phi) - A\sqrt{k}\cos(\omega t+\phi)\right]$$

Given H and ϕ from the initial conditions, this solves for the motion. When $A = 0$ we get SHM as expected. For $A \neq 0$ the r.h.s. is an oscillating function. Therefore q^2 always evolves to 0 and thereafter there is no real solution, i.e. the particle gets trapped at the origin by the strong attractive $1/q^2$ potential.

3 Show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}$$

leads to Maxwell's equations for a free electromagnetic field. [6]

Given that the electromagnetic stress-energy tensor is

$$T^{\mu\nu} = -\frac{1}{\mu_0}F^{\mu}{}_{\lambda}F^{\nu\lambda} - g^{\mu\nu}\mathcal{L}\ ,$$

show explicitly that this tensor is conserved. [6]

An electromagnetic wave is represented by the 4-vector potential

$$A^{\mu} = (0, A\cos(kz-\omega t), A\sin(kz-\omega t), 0)\ .$$

(a) Evaluate the electric and magnetic fields. [6]

(b) Evaluate the Lagrangian density. [6]

(c) Evaluate the stress-energy tensor and interpret its components. [9]

$$\left[\text{You may assume that } F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\right]$$

Answer: The equation of motion for A_ν is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial\mathcal{L}}{\partial A_\nu}$$

The r.h.s. is zero and the l.h.s. gets four equal terms. Hence

$$\partial_\mu F^{\mu\nu} = 0$$

Using the expression given for $F^{\mu\nu}$, the first column ($\nu = 0$) gives $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$. The second column gives

$$-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = 0$$

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i.e.

$$(\boldsymbol{\nabla} \times \boldsymbol{B})_x = \mu_0 \epsilon_0 \dot{E}_x$$

Similarly the other columns give the corresponding components of this equation. The other Maxwell equations follow from the definitions

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}\,, \quad E_j = cF^{j0} = c(\partial^j A^0 - \partial^0 A^j) = (c\boldsymbol{\nabla} A^0 - \dot{\boldsymbol{A}})_j$$

so that $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ *and* $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$

We have to show that $\partial_\mu T^{\mu\nu} = 0$, *i.e. that*

$$-\frac{1}{\mu_0}\partial_\mu (F^\mu{}_\lambda F^{\nu\lambda}) + \frac{1}{4\mu_0}\partial^\nu (F_{\alpha\beta} F^{\alpha\beta}) = 0$$

Now

$$\partial_\mu (F^\mu{}_\lambda F^{\nu\lambda}) = (\partial_\mu F^\mu{}_\lambda) F^{\nu\lambda} + F^\mu{}_\lambda \partial_\mu F^{\nu\lambda}$$

From the equation of motion above, the first term on the r.h.s. is zero. Changing the summed labels μ, λ *to* α, β, *the second term is*

$$F_{\alpha\beta}\partial^\alpha F^{\nu\beta} = F_{\alpha\beta}(\partial^\alpha \partial^\nu A^\beta - \partial^\alpha \partial^\beta A^\nu)$$

The second term on the r.h.s. is symmetric in α, β *while* $F_{\alpha\beta}$ *is antisymmetric, so that term sums to zero. Furthermore*

$$\partial^\nu (F_{\alpha\beta} F^{\alpha\beta}) = 2F_{\alpha\beta}\partial^\nu F^{\alpha\beta} = 2F_{\alpha\beta}(\partial^\nu \partial^\alpha A^\beta - \partial^\nu \partial^\beta A^\alpha)$$

Therefore

$$2\mu_0 \partial_\mu T^{\mu\nu} = F_{\alpha\beta}(-2\partial^\alpha \partial^\nu A^\beta + \partial^\nu \partial^\alpha A^\beta - \partial^\nu \partial^\beta A^\alpha) = 0$$

since the expression in brackets is symmetric in α, β *while* $F_{\alpha\beta}$ *is antisymmetric.*

(a) The electric field is

$$\boldsymbol{E} = -\dot{\boldsymbol{A}} = A\omega\,(-\sin(kz - \omega t), \cos(kz - \omega t), 0)$$

The magnetic field is

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = Ak\,(-\cos(kz - \omega t), -\sin(kz - \omega t), 0)$$

(b) From the expression given,

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Hence

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{2\mu_0}(\boldsymbol{E}^2/c^2 - \boldsymbol{B}^2) = \frac{1}{2\mu_0}(-A^2\omega^2/c^2 + A^2 k^2) = 0$$

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(c) Since $\mathcal{L} = 0$ *for this field, we have*

$$T^{\mu\nu} = -\frac{1}{\mu_0} F^{\mu}{}_{\lambda} F^{\nu\lambda}$$

where, writing $\phi = kz - \omega t$,

$$F^{\mu}{}_{\lambda} = Ak \begin{pmatrix} 0 & -\sin\phi & \cos\phi & 0 \\ -\sin\phi & 0 & 0 & \sin\phi \\ \cos\phi & 0 & 0 & -\cos\phi \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

and

$$F^{\nu\lambda} = Ak \begin{pmatrix} 0 & \sin\phi & -\cos\phi & 0 \\ -\sin\phi & 0 & 0 & -\sin\phi \\ \cos\phi & 0 & 0 & \cos\phi \\ 0 & \sin\phi & -\cos\phi & 0 \end{pmatrix}$$

Hence

$$T^{\mu\nu} = \frac{A^2k^2}{\mu_0} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

T^{00} *is the energy density:*

$$T^{00} = \boldsymbol{B}^2/\mu_0 = \epsilon_0 \boldsymbol{E}^2$$

In units where $c = 1$, T^{03} *is the density of* p_z, T^{30} *is the energy flux in the* z*-direction, and* T^{33} *is the flux of* p_z *in the* z*-direction. All of these are equal in a plane wave travelling in the* z*-direction.*

4 A real scalar field $\varphi(x)$ has Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi)(\partial_\mu \varphi)$$

(a) Derive the equation of motion, the canonical momentum density and the Hamiltonian density. [6]

(b) Write a Fourier representation of the field and find the dispersion relation between the frequency and wave vector. [5]

(c) Derive the stress-energy tensor and show that it is conserved. [6]

(d) The system has a shift symmetry under $\varphi \to \varphi' = \varphi + c$ where c is a constant. Derive the associated Noether current and show that it is conserved. [6]

(e) Discuss whether you would expect the shift symmetry to be spontaneously broken. [5]

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(f) State Goldstone's theorem and discuss its applicability to this case. [5]

Answer: (a) The equation of motion is given by

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi}$$

which gives the wave equation

$$\partial_\mu \partial^\mu \varphi = 0$$

The canonical momentum density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\dot{\varphi}}{c^2}$$

and the Hamiltonian density is thus

$$\mathcal{H} = \pi \dot{\varphi} - \mathcal{L} = \frac{1}{2}\left[c^2 \pi^2 + (\boldsymbol{\nabla}\varphi)^2\right]$$

(b) The Fourier representation of a real field takes the form

$$\varphi = \int d^3\boldsymbol{k} N(\boldsymbol{k}) \left(a(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + a^*(\boldsymbol{k}) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \right)$$

where $N(\boldsymbol{k})$ is a normalization factor. Applying the equation of motion,

$$\partial_\mu \partial^\mu \varphi = -\int d^3\boldsymbol{k} N(\boldsymbol{k}) \left(\omega^2/c^2 - \boldsymbol{k}^2 \right) \left(a(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + a^*(\boldsymbol{k}) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \right) = 0$$

and hence the dispersion relation is $\omega^2/c^2 - \boldsymbol{k}^2 = 0$, i.e. $\omega = c|\boldsymbol{k}|$.

(c) The stress-energy tensor is

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$

Hence

$$T^{\mu\nu} = (\partial^\mu \varphi)(\partial^\nu \varphi) - \frac{1}{2} g^{\mu\nu} (\partial^\lambda \varphi)(\partial_\lambda \varphi)$$

This is conserved if $\partial_\mu T^{\mu\nu} = 0$. Now

$$\partial_\mu T^{\mu\nu} = (\partial_\mu \partial^\mu \varphi)(\partial^\nu \varphi) + (\partial^\mu \varphi)(\partial_\mu \partial^\nu \varphi) - \frac{1}{2} \partial^\nu [(\partial^\lambda \varphi)(\partial_\lambda \varphi)]$$

The first term vanishes due to the equation of motion. The third term involves

$$\partial^\nu [(\partial^\lambda \varphi)(\partial_\lambda \varphi)] = (\partial^\lambda \partial^\nu \varphi)(\partial_\lambda \varphi) + (\partial^\lambda \varphi)(\partial_\lambda \partial^\nu \varphi) = 2(\partial^\mu \varphi)(\partial_\mu \partial^\nu \varphi)$$

since λ and μ are just summed indices. Thus the second and third terms cancel, giving $\partial_\mu T^{\mu\nu} = 0$ as required.

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(d) The Noether current associated with symmetry under $\varphi \to \varphi + \delta\varphi$ is

$$J^\mu \propto \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi$$

In this case $\delta\varphi =$ constant, so we can take

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi$$

By the equation of motion, this is conserved:

$$\partial_\mu J^\mu = \partial_\mu \partial^\mu \varphi = 0$$

(e) We see from the Hamiltonian density in (a) that the minimum-energy form of the field is simply $\varphi =$ constant. The system must choose some particular value, so we do expect the symmetry to be spontaneously broken.

(f) Goldstone's theorem states that for every spontaneously broken continuous global symmetry there is a field with massless quanta.

In this case the field φ itself satifies the Klein-Gordon equation $\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$ with $m = 0$ and dispersion relation $\omega = c|\boldsymbol{k}|$. The quanta will have energy $E = \hbar\omega$ and momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$, so $E = c|\boldsymbol{p}|$ and the quanta are indeed massless, so Goldstone's theorem is obeyed.

5 In the Nambu-Jona-Lasinio model, a Dirac field ψ has Lagrangian density

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi + \frac{\lambda}{4}\left[(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)^2\right]$$

where $\overline{\psi} = \psi^\dagger \gamma^0$ and λ is a real, positive constant.

(a) Derive the equations of motion for ψ and $\overline{\psi}$ and show that they are consistent. [8]

(b) Express $\mathcal{L}$ in terms of the left- and right-handed fields

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi\ , \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi\ ,$$

and derive the equations of motion for ψ_L and ψ_R. [8]

(c) Show that there is a global symmetry with respect to independent phase changes in these fields, i.e.

$$\psi_L \to e^{i\alpha}\psi_L\ , \quad \psi_R \to e^{i\beta}\psi_R$$

where α and β are real constants. [8]

(d) Show that this symmetry is spontaneously broken but there remains a global symmetry with respect to identical phase changes in these fields, i.e. $\alpha = \beta$. [9]

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[*You may assume that* $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0$, $\gamma^5\gamma^5 = 1$, $\gamma^{5\dagger} = \gamma^5$ *and* $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.]

Answer: (a) The equation of motion for $\overline{\psi}$ *is*

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial\overline{\psi}}$$

The left-hand side is zero, so

$$0 = i\gamma^\mu\partial_\mu\psi + \frac{\lambda}{2}\left[(\overline{\psi}\psi)\psi - (\overline{\psi}\gamma^5\psi)\gamma^5\psi\right]$$

The equation of motion for ψ *is*

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi}$$

Hence

$$i\partial_\mu\overline{\psi}\gamma^\mu = \frac{\lambda}{2}\left[(\overline{\psi}\psi)\overline{\psi} - (\overline{\psi}\gamma^5\psi)\overline{\psi}\gamma^5\right]$$

Now $\overline{\psi} = \psi^\dagger\gamma^0$ *and* $\gamma^{0\dagger} = \gamma^0$*. Therefore, hermitian conjugating the whole equation*

$$-i\gamma^{\mu\dagger}\gamma^0\partial_\mu\psi = \frac{\lambda}{2}\left[(\overline{\psi}\psi)^*\gamma^0\psi - (\overline{\psi}\gamma^5\psi)^*\gamma^5\gamma^0\psi\right]$$

Multiplying on the left by γ^0 *and using the results given*

$$-i\gamma^\mu\partial_\mu\psi = \frac{\lambda}{2}\left[(\overline{\psi}\psi)^*\psi + (\overline{\psi}\gamma^5\psi)^*\gamma^5\psi\right]$$

Now

$$(\overline{\psi}\psi)^* = (\psi^\dagger\gamma^0\psi)^*$$

and since this is just a number we may transpose the whole expression to obtain

$$(\overline{\psi}\psi)^* = \psi^\dagger\gamma^0\psi = \overline{\psi}\psi$$

On the other hand

$$(\overline{\psi}\gamma^5\psi)^* = (\psi^\dagger\gamma^0\gamma^5\psi)^* = \psi^\dagger\gamma^5\gamma^0\psi = -\overline{\psi}\gamma^5\psi$$

so that

$$-i\gamma^\mu\partial_\mu\psi = \frac{\lambda}{2}\left[(\overline{\psi}\psi)\psi - (\overline{\psi}\gamma^5\psi)\gamma^5\psi\right]$$

which is indeed consistent with the above equation of motion for ψ*.*

(b) Clearly

$$\psi = \psi_R + \psi_L\ , \quad \gamma^5\psi = \psi_R - \psi_L$$

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Now

$$\overline{\psi}_R = \frac{1}{2}\psi^\dagger(1+\gamma^5)\gamma^0 = \frac{1}{2}\overline{\psi}(1-\gamma^5)$$

and similarly

$$\overline{\psi}_L = \frac{1}{2}\overline{\psi}(1+\gamma^5)$$

Therefore $\overline{\psi}_R\psi_R = \overline{\psi}_L\psi_L = 0$ *and*

$$\overline{\psi}_L\psi = \overline{\psi}_L\psi_R\ ,\ \ \overline{\psi}_R\psi = \overline{\psi}_R\psi_L\ ,\ \ \overline{\psi}\psi = \overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R$$

while

$$\overline{\psi}_L\gamma^5\psi = \overline{\psi}_L\psi_R\ ,\ \ \overline{\psi}_R\gamma^5\psi = -\overline{\psi}_R\psi_L\ ,\ \ \overline{\psi}\gamma^5\psi = \overline{\psi}_R\psi_L - \overline{\psi}_L\psi_R$$

so that

$$(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)^2 = 4(\overline{\psi}_R\psi_L)(\overline{\psi}_L\psi_R)$$

On the other hand

$$\overline{\psi}_L\gamma^\mu\partial_\mu\psi = \overline{\psi}_L\gamma^\mu\partial_\mu\psi_L\ ,\ \ \overline{\psi}_R\gamma^\mu\partial_\mu\psi = \overline{\psi}_R\gamma^\mu\partial_\mu\psi_R$$

Therefore

$$\mathcal{L} = i\overline{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L + \lambda(\overline{\psi}_R\psi_L)(\overline{\psi}_L\psi_R)$$

The fields ψ_L *and* ψ_R *can be treated as independent (or check this explicitly). Then, as above,*

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}_R)}\right) = 0 = \frac{\partial\mathcal{L}}{\partial\overline{\psi}_R}$$

gives

$$i\gamma^\mu\partial_\mu\psi_R + \lambda(\overline{\psi}_L\psi_R)\psi_L = 0$$

Similarly

$$i\gamma^\mu\partial_\mu\psi_L + \lambda(\overline{\psi}_R\psi_L)\psi_R = 0$$

(c) When

$$\psi_L \to e^{i\alpha}\psi_L\ ,\quad \psi_R \to e^{i\beta}\psi_R$$

we have

$$\overline{\psi}_L \to e^{-i\alpha}\overline{\psi}_L\ ,\quad \overline{\psi}_R \to e^{-i\beta}\overline{\psi}_R$$

Hence in the Lagrangian density

$$\mathcal{L} = i\overline{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L + \lambda(\overline{\psi}_R\psi_L)(\overline{\psi}_L\psi_R)$$

the first two terms do not change, while

$$\overline{\psi}_R\psi_L \to e^{i(\alpha-\beta)}\overline{\psi}_R\psi_L\ ,\ \ \overline{\psi}_L\psi_R \to e^{-i(\alpha-\beta)}\overline{\psi}_L\psi_R\ ,$$

so the last term doesn't change either.

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(d) The minimum-energy configuration of the field will have ψ constant, and then the Hamiltonian density will be

$$\mathcal{H}_{min} = -\lambda(\overline{\psi}_R\psi_L)(\overline{\psi}_L\psi_R)$$

Noticing that $\overline{\psi}_R\psi_L = (\overline{\psi}_L\psi_R)^*$, we see that this is

$$\mathcal{H}_{min} = -\lambda|\overline{\psi}_L\psi_R|^2 < 0$$

so that the field ψ will assume some large value, limited only by higher-order terms not shown. Once this has happened, $\psi \to e^{i\alpha}\psi$ implies $\gamma^5\psi \to e^{i\alpha}\gamma^5\psi$ and hence $\psi_R \to e^{i\alpha}\psi_R$ and $\psi_L \to e^{i\alpha}\psi_L$. Thus the only remaining symmetry is a change of ψ_R and ψ_L by the same phase.

6 The current density $j(t)$ in a conductor due to an applied electric field $\mathcal{E}(t)$ is given by

$$j(t) = \int \sigma(t-t')\mathcal{E}(t')\,dt'$$

where the linear response function $\sigma(t-t')$ vanishes for $t < t'$ and its Fourier transform gives the conductivity as a function of the frequency ω:

$$\sigma(\omega) = \int_0^\infty \sigma(\tau)e^{i\omega\tau}\,d\tau$$

(a) For a real electric field

$$\mathcal{E}(t) = Fe^{-i\omega t} + F^*e^{i\omega t}$$

show that the current density is

$$j(t) = \sigma(\omega)Fe^{-i\omega t} + \sigma(-\omega)F^*e^{i\omega t}$$

and hence that the real and imaginary parts of σ are even and odd functions of ω, respectively. [7]

(b) At high frequencies the conductor can be treated as a free electron gas. By considering the motion of an electron in the above electric field, show that this implies

$$\sigma(\omega) \underset{\omega\to\infty}{\longrightarrow} i\frac{ne^2}{m\omega}$$

where n is the electron number density and e and m are the electron charge and mass. [7]

(c) At low frequencies the conductivity has the form

$$\sigma(\omega) \underset{\omega\to 0}{\longrightarrow} i\frac{A}{\omega}$$

where A is a real constant.

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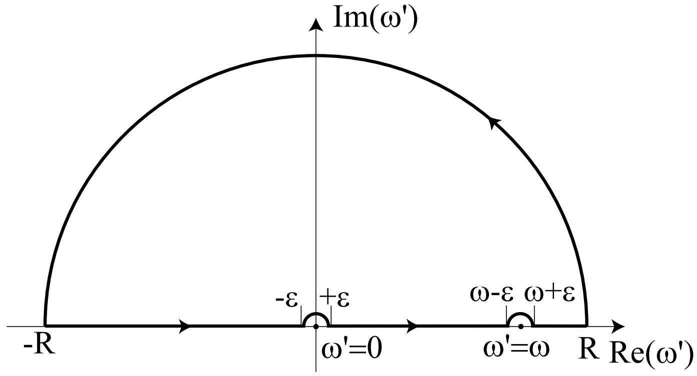

By considering the integral

$$\oint \frac{\sigma(\omega')}{\omega' - \omega}\, d\omega'$$

on the contour shown in the figure and taking the limits $R \to \infty$ and $\epsilon \to 0$, show that the real and imaginary parts of the conductivity, $\sigma_1(\omega)$ and $\sigma_2(\omega)$ respectively, satisfy the Kramers-Kronig relations [8]

$$\begin{aligned} \sigma_1(\omega) &= \frac{1}{\pi} P \int \frac{\sigma_2(\omega')}{\omega' - \omega} d\omega' \\ \sigma_2(\omega) &= -\frac{1}{\pi} P \int \frac{\sigma_1(\omega')}{\omega' - \omega} d\omega' + \frac{A}{\omega} \end{aligned}$$

(d) Show also that [3]

$$A = \frac{ne^2}{m} - \frac{1}{\pi} \int \sigma_1(\omega')\, d\omega'$$

(e) Given that the real part of the conductivity has the form

$$\sigma_1(\omega) = \sum_{\alpha,\beta} |\mathcal{M}_{\alpha\beta}|^2 \frac{f_\alpha - f_\beta}{\omega_{\beta\alpha}} \delta(\omega - \omega_{\beta\alpha})$$

where $\mathcal{M}_{\alpha\beta}$ is a quantum-mechanical matrix element between states α and β with energies E_α and E_β, $f_\alpha = f(E_\alpha)$ where $f(E)$ is the Fermi-Dirac distribution function, and $\omega_{\beta\alpha} = (E_\beta - E_\alpha)/\hbar$, show that [8]

$$\sigma(\omega) = i\frac{ne^2}{m\omega} - \lim_{\epsilon \to 0^+} \frac{i}{\pi\omega} \sum_{\alpha,\beta} |\mathcal{M}_{\alpha\beta}|^2 \frac{f_\alpha - f_\beta}{\omega_{\beta\alpha} - \omega - i\epsilon}$$

[*You may assume that* $\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x)$]

Answer: (a) Changing variable to $\tau = t - t'$,

$$\begin{aligned} j(t) &= \int_0^\infty \sigma(\tau)\mathcal{E}(t-\tau)\, dt' \\ &= \int_0^\infty \sigma(\tau)[Fe^{-i\omega(t-\tau)} + F^* e^{i\omega(t-\tau)}] \\ &= \sigma(\omega)Fe^{-i\omega t} + \sigma(-\omega)F^* e^{i\omega t} \end{aligned}$$

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This must be real, so $\sigma(-\omega) = \sigma(\omega)^*$. *Hence, writing* $\sigma(\omega) = \sigma_1(\omega) + i\sigma_2(\omega)$ *we have*

$$\begin{aligned} \sigma_1(\omega) &= \frac{1}{2}[\sigma(\omega) + \sigma(\omega)^*] = \frac{1}{2}[\sigma(\omega) + \sigma(-\omega)] = \text{ even} \\ \sigma_2(\omega) &= \frac{1}{2i}[\sigma(\omega) - \sigma(\omega)^*] = \frac{1}{2i}[\sigma(\omega) - \sigma(-\omega)] = \text{ odd} \end{aligned}$$

(b) The equation of motion of an electron is

$$m\ddot{x} = e\mathcal{E}$$

Writing

$$x = Ae^{-i\omega t} + A^* e^{i\omega t}$$

this gives

$$A = -\frac{eF}{m\omega^2}$$

The current density is

$$j(t) = ne\dot{x} = -ine\omega[Ae^{-i\omega t} - A^* e^{i\omega t}]$$

Hence

$$j(t) = i\frac{ne^2}{m\omega}[Fe^{-i\omega t} - F^* e^{i\omega t}]$$

which, comparing with the earlier expression, shows that

$$\sigma(\omega) = i\frac{ne^2}{m\omega}$$

for a free electron gas.

(c) Inverting the Fourier transform

$$\sigma(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sigma(\omega)e^{-i\omega\tau}\, d\omega$$

When $\tau < 0$ *we can close the contour in the upper half plane and this must give zero by causality. Hence all singularities of* $\sigma(\omega)$ *must lie in the lower half plane. It follows that the integrand in the given integral has no singularities in the upper half plane. Furthermore since the integrand falls off like* $(\omega')^{-2}$ *at infinity, the semicircle of radius* R *gives no contribution as* $R \to \infty$. *Therefore*

$$\oint \frac{\sigma(\omega')}{\omega' - \omega}\, d\omega' = 0$$

Now if we write $\sigma(\omega) = \phi(\omega)/\omega$ *then* $\phi(0) = iA$ *and the integrand becomes*

$$\frac{\phi(\omega')}{\omega'(\omega' - \omega)} = \frac{1}{\omega}\left(\frac{\phi(\omega')}{\omega' - \omega} - \frac{\phi(\omega')}{\omega'}\right)$$

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Using the formula given, we then have

$$\frac{P}{\omega}\int\left(\frac{\phi(\omega')}{\omega'-\omega}-\frac{\phi(\omega')}{\omega'}\right)d\omega'-\frac{i\pi}{\omega}[\phi(\omega)-\phi(0)]=0$$

Rearranging terms, this gives

$$\sigma(\omega)=-\frac{i}{\pi}P\int\frac{\sigma(\omega')}{\omega'-\omega}d\omega'+\frac{iA}{\omega}$$

Taking real and imaginary parts on each side gives the KK relations required.

(d) Taking $\omega\to\infty$ in the second KK relation and using

$$\frac{1}{\omega'-\omega}=-\frac{1}{\omega}\left(1-\frac{\omega'}{\omega}\right)=-\frac{1}{\omega}+\mathcal{O}\left(\frac{1}{\omega^2}\right)$$

gives

$$\sigma_2(\omega)\underset{\omega\to\infty}{\longrightarrow}\frac{1}{\pi\omega}\int\sigma(\omega')\,d\omega'+\frac{A}{\omega}=\frac{ne^2}{m\omega}$$

Hence

$$A=\frac{ne^2}{m}-\frac{1}{\pi}\int\sigma_1(\omega')\,d\omega'$$

(e) Substituting in the second KK relation

$$\sigma_2(\omega)=-\frac{1}{\pi}\sum_{\alpha,\beta}|\mathcal{M}_{\alpha\beta}|^2\frac{f_\alpha-f_\beta}{\omega_{\beta\alpha}}P\frac{1}{\omega_{\beta\alpha}-\omega}+\frac{A}{\omega}$$

Substituting the above expression for A

$$\begin{aligned}\sigma_2(\omega) &= -\frac{1}{\pi}\sum_{\alpha,\beta}|\mathcal{M}_{\alpha\beta}|^2\frac{f_\alpha-f_\beta}{\omega_{\beta\alpha}}\left(P\frac{1}{\omega_{\beta\alpha}-\omega}+\frac{1}{\omega}\right)+\frac{ne^2}{m\omega}\\ &= -\frac{1}{\pi}\sum_{\alpha,\beta}|\mathcal{M}_{\alpha\beta}|^2\frac{f_\alpha-f_\beta}{\omega}P\frac{1}{\omega_{\beta\alpha}-\omega}+\frac{ne^2}{m\omega}\end{aligned}$$

Hence

$$\begin{aligned}\sigma_1(\omega)+i\sigma_2(\omega) &= -\frac{i}{\pi\omega}\sum_{\alpha,\beta}|\mathcal{M}_{\alpha\beta}|^2(f_\alpha-f_\beta)\left(P\frac{1}{\omega_{\beta\alpha}-\omega}+i\delta(\omega_{\beta\alpha}-\omega)\right)+i\frac{ne^2}{m\omega}\\ &= i\frac{ne^2}{m\omega}-\lim_{\epsilon\to0^+}\frac{i}{\pi\omega}\sum_{\alpha,\beta}|\mathcal{M}_{\alpha\beta}|^2\frac{f_\alpha-f_\beta}{\omega_{\beta\alpha}-\omega-i\epsilon}\end{aligned}$$

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