NATURAL SCIENCES TRIPOS Part II

Wednesday 16 January 2008 10.30am to 12.30pm

THEORETICAL PHYSICS I - Solutions

1 Question on Lagrangian mechanics – see handwritten solution.

2 Question on Hamiltonian mechanics – see handwritten solution.

3 (a) The given Lagrangian density is

$$\mathcal{L} = \frac{\hbar}{2i}\left(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}\right) - \frac{\hbar^2}{2m}\nabla\Psi\cdot\nabla\Psi^* - V(\boldsymbol{r})\Psi\Psi^*$$

Under $\Psi \to \Psi e^{-i\alpha}$, $\Psi^* \to \Psi^* e^{+i\alpha}$, where α is a real constant, each term is unchanged, so there is a global phase symmetry.

(b) The Noether 4-current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\delta\Psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)}\delta\Psi^*$$

where $\delta\Psi$ is the small change in Ψ. In this case $\delta\Psi = -i\alpha\Psi$, and so, dropping the overall factor of α,

$$J^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\Psi + i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)}\Psi^* = (c\rho, \boldsymbol{J})$$

where

$$\rho = -i\frac{\partial\mathcal{L}}{\partial\dot{\Psi}}\Psi + i\frac{\partial\mathcal{L}}{\partial\dot{\Psi}^*}\Psi^* = \hbar\Psi\Psi^*$$

and

$$\boldsymbol{J} = -i\frac{\partial\mathcal{L}}{\partial(\nabla\Psi)}\Psi + i\frac{\partial\mathcal{L}}{\partial(\nabla\Psi^*)}\Psi^* = \frac{i\hbar^2}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$$

The equation of motion (the Schrödinger equation) is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

so that

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \hbar\Psi^*\frac{\partial\Psi}{\partial t} + \hbar\Psi\frac{\partial\Psi^*}{\partial t}\\ &= \frac{i\hbar^2}{2m}\Psi^*\nabla^2\Psi - \frac{i\hbar^2}{2m}\Psi\nabla^2\Psi^*\\ &= -\nabla\cdot\boldsymbol{J}\end{aligned}$$

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which is the conservation equation for the probability density.

(c) The stress-energy tensor is

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\partial^\nu\Psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)}\partial^\nu\Psi^* - g^{\mu\nu}\mathcal{L}$$

i.e.

$$\begin{aligned} T^{00} &= \frac{\partial\mathcal{L}}{\partial\dot{\Psi}}\frac{\partial\Psi}{\partial t} + \frac{\partial\mathcal{L}}{\partial\dot{\Psi}^*}\frac{\partial\Psi^*}{\partial t} - \mathcal{L} \\ &= \frac{\hbar}{2i}\left(-\Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}\right) - \mathcal{L} \\ &= \frac{\hbar^2}{2m}\nabla\Psi\cdot\nabla\Psi^* + V(\boldsymbol{r})\Psi\Psi^* \quad (= \mathcal{H}) \end{aligned}$$

and

$$\begin{aligned} T^{j0} &= \frac{\partial\mathcal{L}}{\partial\nabla_j\Psi}\frac{\partial\Psi}{\partial t} + \frac{\partial\mathcal{L}}{\partial\nabla_j\Psi^*}\frac{\partial\Psi^*}{\partial t} \\ &= \frac{-\hbar^2}{2m}\left(\nabla_j\Psi^*\frac{\partial\Psi}{\partial t} + \nabla_j\Psi\frac{\partial\Psi^*}{\partial t}\right) \end{aligned}$$

Similarly (N.B. $\partial^k = -\nabla_k$)

$$T^{0k} = \frac{\hbar}{2i}\left(\Psi^*\nabla_k\Psi - \Psi\nabla_k\Psi^*\right)$$

and

$$T^{jk} = \frac{\hbar^2}{2m}\left(\nabla_j\Psi^*\nabla_k\Psi + \nabla_j\Psi\nabla_k\Psi^*\right) + \delta_{jk}\mathcal{L}$$

(d) From the above equations we have

$$\nabla_j T^{j0} = \frac{-\hbar^2}{2m}\left(\nabla^2\Psi^*\frac{\partial\Psi}{\partial t} + \nabla^2\Psi\frac{\partial\Psi^*}{\partial t} + \nabla\Psi^*\cdot\nabla\frac{\partial\Psi}{\partial t} + \nabla\Psi\cdot\nabla\frac{\partial\Psi^*}{\partial t}\right)$$

Using the Schrödinger equation to eliminate $\nabla^2\Psi$ and $\nabla^2\Psi^*$, this becomes

$$\nabla_j T^{j0} = \frac{-\hbar^2}{2m}\left(-V\Psi^*\frac{\partial\Psi}{\partial t} - V\Psi\frac{\partial\Psi^*}{\partial t} + \nabla\Psi^*\cdot\nabla\frac{\partial\Psi}{\partial t} + \nabla\Psi\cdot\nabla\frac{\partial\Psi^*}{\partial t}\right) = -\frac{\partial T^{00}}{\partial t}$$

which is the conservation equation for the energy density.

Similarly

$$\begin{aligned} \nabla_j T^{jk} &= \frac{\hbar^2}{2m}\left(\nabla^2\Psi^*\nabla_k\Psi + \nabla^2\Psi\nabla_k\Psi^* + \nabla\Psi^*\cdot\nabla(\nabla_k\Psi) + \nabla\Psi\cdot\nabla(\nabla_k\Psi^*)\right) + \nabla_k\mathcal{L} \\ &= \left(V\Psi^* + i\hbar\frac{\partial\Psi^*}{\partial t}\right)\nabla_k\Psi + \left(V\Psi - i\hbar\frac{\partial\Psi}{\partial t}\right)\nabla_k\Psi^* \end{aligned}$$

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\begin{aligned}
&+\frac{\hbar}{2i}\left(\nabla_k\Psi\frac{\partial\Psi^*}{\partial t}-\nabla_k\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\nabla_k\frac{\partial\Psi^*}{\partial t}-\Psi^*\nabla_k\frac{\partial\Psi}{\partial t}\right)\\
&-V\Psi\nabla_k\Psi^*-V\Psi^*\nabla_k\Psi-\Psi\Psi^*\nabla_kV\\
=\;&\frac{\hbar}{2i}\left(-\nabla_k\Psi\frac{\partial\Psi^*}{\partial t}+\nabla_k\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\nabla_k\frac{\partial\Psi^*}{\partial t}-\Psi^*\nabla_k\frac{\partial\Psi}{\partial t}\right)-\Psi\Psi^*\nabla_kV\\
=\;&-\frac{\partial T^{0k}}{\partial t}+\Psi\Psi^*F_k
\end{aligned}
$$

where $\boldsymbol{F}=-\nabla V$ is the applied force. Thus

$$\frac{\partial T^{0k}}{\partial t}=-\nabla_jT^{jk}+\Psi\Psi^*F_k$$

which is the expected conservation equation for the momentum density:
Rate of change of mom. density = – (mom. flux) + (prob. density)×force

4 (a) The first term is the free particle action, where m is the (rest-)mass and τ is the proper time. The second term is the interaction between the particle and the field, where e is the charge, A^μ is the 4-vector potential and dx^μ is the element of path taken by the particle. The final term is the free field action, where $F^{\alpha\beta}$ is the field-strength tensor, related to the 4-vector potential by

$$F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha\ .$$

(b) We have $S=\int L\,dt$, $d\tau=dt/\gamma(v)$ (where $\gamma(v)=1/\sqrt{1-v^2/c^2}$ and $dx^\mu=(dx^\mu/dt)dt$, so the Lagrangian for the charged particle is

$$L=-\frac{mc^2}{\gamma(v)}-e(\phi-\boldsymbol{A}\cdot\boldsymbol{v})$$

where ϕ is the scalar potential and $\boldsymbol{A}$ is the 3-vector potential. The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\boldsymbol{v}}\right)=\frac{\partial L}{\partial\boldsymbol{x}}=\nabla L$$

i.e.

$$\frac{d}{dt}(\gamma m\boldsymbol{v}+e\boldsymbol{A})=-e\nabla\phi+\nabla(\boldsymbol{A}\cdot\boldsymbol{v})$$

Now

$$\frac{d\boldsymbol{A}}{dt}=\frac{\partial\boldsymbol{A}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$$

and

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{v})-(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}=\boldsymbol{v}\times(\nabla\times\boldsymbol{A})=\boldsymbol{v}\times\boldsymbol{B}$$

so the equation of motion for the particle is

$$\frac{d}{dt}(\gamma m\boldsymbol{v}+e\boldsymbol{A})=-e\nabla\phi+e\,\boldsymbol{v}\times\boldsymbol{B}$$

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or, since $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$,

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \; .$$

Writing

$$\int A_\mu \, dx^\mu(t) = \int A_\mu \frac{dx^\mu}{dt}\delta^3(\boldsymbol{r} - \boldsymbol{x}(t))d^3\boldsymbol{r}\, dt$$

the part of $\mathcal{L}$ concerning the e.m. field is

$$\mathcal{L}_{em} = -eA_\mu \frac{dx^\mu}{dt}\delta^3(\boldsymbol{r} - \boldsymbol{x}(t)) - \frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}$$

Treating A_α as the field variables, the equation of motion for the e.m. field is

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\right) = \frac{\partial\mathcal{L}}{\partial A_\alpha}$$

Writing

$$F_{\alpha\beta}F^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

we see that the derivative on the l.h.s. has four equal terms, giving

$$\frac{1}{\mu_0}\partial_\mu F^{\mu\alpha} = e\frac{dx^\alpha}{dt}\delta^3(\boldsymbol{r} - \boldsymbol{x}(t))$$

The r.h.s. is the 4-current density J^α due to the moving charged particle. Writing $J^\mu = (c\rho, \boldsymbol{J})$, $c = 1/\sqrt{\mu_0\varepsilon_0}$ and using the expression given for $F^{\alpha\beta}$, this gives

$$\nabla\cdot\boldsymbol{E} = \rho/\varepsilon_0 \; , \quad -\varepsilon_0\partial\boldsymbol{E}/\partial t + (\nabla\times\boldsymbol{B}) = \mu_0\,\boldsymbol{J}$$

which are the inhomogeneous Maxwell equations.

(c) A gauge transformation is performed by adding the derivative of a scalar function to the 4-vector potential:

$$A_\alpha \to A_\alpha + \partial_\alpha f$$

This does not affect $F_{\alpha\beta}$ since $\partial_\alpha\partial_\beta f = \partial_\beta\partial_\alpha f$. The term involving the current density changes by

$$\delta\mathcal{L}_{em} = -(\partial_\alpha f)J^\alpha$$

corresponding to a change in the action

$$\delta S = -\int (\partial_\alpha f)J^\alpha \, d^4x = \int f\,\partial_\alpha J^\alpha \, d^4x - \int \partial_\alpha(fJ^\alpha)\, d^4x$$

The first term on the r.h.s. is zero since the electromagnetic current is conserved, $\partial_\alpha J^\alpha = 0$. The second term can be integrated to give a surface term, which contributes at most a constant to S, which cannot affect the equations of motion.

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(d) A charged scalar field is represented by a complex field φ. Its free-particle Lagrangian density is of the Klein-Gordon form

$$\mathcal{L}_{KG} = (\partial_\mu\varphi)^*(\partial^\mu\varphi) - m^2\varphi^*\varphi$$

The interaction with an electromagnetic field is included by replacing the derivative ∂_μ by the covariant derivative

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$$

Thus a scalar field is included by adding the term

$$\mathcal{L}_{KG} = (D_\mu\varphi)^*(D^\mu\varphi) - m^2\varphi^*\varphi$$

to the Lagrangian density. The symmetry under the gauge transformation $A_\alpha \rightarrow A_\alpha + \partial_\alpha f$ is preserved by making a compensating phase change in the scalar field:

$$\varphi \rightarrow \varphi\, e^{-ief}\ .$$

5 (a) The given Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left[(\partial^\mu\varphi)(\partial_\mu\varphi) + a\,\varphi^2 + b\,\varphi^4\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 - (\nabla\varphi)^2 + a\,\varphi^2 + b\,\varphi^4\right] \end{aligned}$$

The equation of motion is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

i.e.

$$\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi - a\,\varphi - 2b\,\varphi^3 = 0$$

The momentum density is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{\partial\varphi}{\partial t}$$

The Hamiltonial density is

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 + (\nabla\varphi)^2 - a\,\varphi^2 - b\,\varphi^4\right]$$

For this to be bounded from below we require *either* $b < 0$ *or* ($b = 0$ and $a < 0$).

(b) For constant φ we have $\mathcal{H} = -(a\,\varphi^2 + b\varphi^4)/2$. When $b < 0$, this has a minimum at φ_0 where

$$\frac{\partial\mathcal{H}}{\partial\varphi^2} = -\frac{1}{2}(a + 2b\,\varphi_0^2) = 0$$

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i.e. when $a > 0$ and

$$\varphi_0 = \pm\sqrt{\frac{-a}{2b}}$$

The system must choose either the + or − root, so the symmetry is spontaneously broken when $a > 0$ and $b < 0$. The minimum energy density is then

$$\mathcal{H}_0 = \frac{1}{2}\frac{a^2}{2b} - \frac{b}{2}\left(\frac{-a}{2b}\right)^2 = \frac{a^2}{8b} < 0$$

(c) Write $\varphi = \varphi_0 + \chi$. Then

$$\begin{aligned} a\,\varphi^2 + b\,\varphi^4 &= a\,(\varphi_0^2 + 2\varphi_0\chi + \chi^2) + b\,(\varphi_0^4 + 4\varphi_0^3\chi + 6\varphi_0^2\chi^2 + 4\varphi_0\chi^3 + \chi^4) \\ &= a\,\varphi_0^2 + b\,\varphi_0^4 + 2\varphi_0(\chi(a + 2b\varphi_0^2) + \chi^2(a + 6b\varphi_0^2) + \mathcal{O}(\chi^3) \\ &= a\,\varphi_0^2 + b\,\varphi_0^4 - 2a\,\chi^2 + \mathcal{O}(\chi^3) \end{aligned}$$

Hence the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu\chi)(\partial_\mu\chi) - 2a\,\chi^2 + \mathcal{O}(\chi^3)\right] - \mathcal{H}_0$$

and the equation of motion for χ is

$$\frac{\partial^2\chi}{\partial t^2} - \nabla^2\chi + 2a\,\chi = \mathcal{O}(\chi^2)$$

For χ small the r.h.s. is negligible and the dispersion relation, obtained by substituting e.g. $\chi = \cos(kx - \omega t)$, is $-\omega^2 + k^2 + 2a = 0$, i.e.

$$\omega = \sqrt{k^2 + 2a}$$

which corresponds to a real mass $m = \sqrt{2a}$ (in natural units) for the quanta of the field.

6 Question on Green's functions – see handwritten solution.

END OF PAPER

The co-ordinate system we will use is as follows: the mass m is at (x, y), the pendulum makes an angle ϕ to the downward vertical, the centre of the cylinder is at (X, R) and the cylinder rotates at angular velocity $\dot{\theta}$ such that $\dot{X} = R\dot{\theta}$. Then, $x = X + c\sin\phi$, $y = R - c\cos\phi$ and $\dot{x} = R\dot{\theta} + c\dot{\phi}\cos\phi$, $\dot{y} = c\dot{\phi}\sin\phi$. The kinetic and rotational energies of the cylinder are both $MR^2\dot{\theta}^2/2$, and the potential energy of the mass m is $-mga\cos\phi$. The Lagrangian is therefore

$$L = \frac{m}{2}\left(R^2\dot{\theta}^2 + c^2\dot{\phi}^2 + 2cR\dot{\theta}\dot{\phi}\cos\phi\right) + MR^2\dot{\theta}^2 + mga\cos\phi$$

and the Euler-Lagrange equations are

$$\begin{aligned} mc^2\ddot{\phi} + mcR\ddot{\theta}\cos\phi - mcR\dot{\theta}\dot{\phi}\sin\phi + mgc\sin\phi &= 0 \\ (m + 2M)R^2\dot{\theta} + mcR\dot{\phi}\cos\phi &= \kappa \end{aligned}$$

for some constant κ (as $\frac{\partial L}{\partial \theta} = 0$ so $\frac{\partial L}{\partial \dot{\theta}}$ is constant). Rearranging the second equation and differentiating,

$$\ddot{\theta} = \frac{mcR}{R^2(m + 2M)}\left(-\ddot{\phi}\cos\phi + \dot{\phi}^2\sin\phi\right).$$

Substituting this into the equation of motion for θ,

$$\begin{aligned} \left(mc^2 - \frac{(mcR)^2}{R^2(m + 2M)}\cos^2\phi\right)\ddot{\phi} &+ \frac{2(mcR)^2}{R^2(m + 2M)}\dot{\phi}^2\cos\phi\sin\phi \\ &- \frac{mcR\kappa}{R^2(m + 2M)}\dot{\phi}\sin\phi + mgc\sin\phi = 0 \end{aligned}$$

For small oscillations, $\phi \sim \epsilon\sin\omega t$, $\dot{\phi} \sim \epsilon\omega\cos\omega t$, $\cos\phi \sim 1$ and we can ignore terms proportional to $\dot{\phi}^2$ and $\dot{\phi}\sin\phi$ which are second order in ϵ. The above equation then becomes

$$\frac{2Mc}{m + 2M}\ddot{\phi} + g\sin\phi = 0$$

This is the equation of motion for a pendulum of length $\lambda = 2Mc/(m + 2M)$.

The Hamiltonian is

$$\begin{aligned} H &= p_\theta\dot{\theta} + p_\phi\dot{\phi} - L \\ &= \frac{1}{2}mc^2\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + MR^2\dot{\theta}^2 + mcR\dot{\theta}\dot{\phi}\cos\phi - mgc\cos\phi \end{aligned}$$

which is independent of time and therefore conserved. Initially $\phi = 0$, $\dot{X} = R\dot{\theta} = V$, and $\dot{x} = V + c\dot{\phi} = 0$ so $\dot{\phi} = -V/c$. Substituting this into the Hamiltonian, we see that the initial energy of the system is $E = MV^2 - mgc$. We also found earlier that the angular momentum is conserved; in terms of V, the initial angular momentum is $\kappa = 2MRV$. Equating the initial energy of the system to the Hamiltonian, substituting the previously derived expressions for $\dot{\theta}$ and κ, and rearranging we find

$$\frac{1}{2}mc^2\left(1 - \frac{m\cos^2\phi}{m + 2M}\right)\dot{\phi}^2 = \frac{mM}{m + 2M}V^2 - mgc(1 - \cos\phi)$$

Advantages of the Hamiltonian formulation over the Lagrangian formulation include:

- The Lagrangian formulation leads to N second order differential equations while the Hamiltonian formulation leads to $2N$ first order differential equations, which may be easier to solve.
- The q_i in the Lagrangian formulation must be position co-ordinates, whereas in the Hamiltonian formulation the q_i and p_i are on an equal footing and the q_i need not be position co-ordinates. This also means that canonical transformations with mixed momenta and postion can be used to greatly simplify Hamilton's equations. (This will be illustrated later in this question)
- The Hamiltonian formulation can lead to easily identifiable constants of motion (for example, if H is independent of t then energy is conserved). The presence of a conserved quantity immediately simplifies Hamilton's equations, whereas no immediate simplification would occur in the Lagrangian formulation.
- There is a simple relationship between the (classical) Hamiltonian formulation and quantum mechanics.

Starting from the Hamiltonian

$$H(q,p,t) = p\dot{q} - L(q,\dot{q},t)$$

we find the differential $\mathrm{d}H$ and substitute using the Euler-Lagrange equation:

$$\begin{aligned}
\mathrm{d}H &= \frac{\partial H}{\partial p}\mathrm{d}p + \frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial t}\mathrm{d}t \\
&= p\mathrm{d}\dot{q} + \dot{q}\mathrm{d}p - \frac{\partial L}{\partial q}\mathrm{d}q - \frac{\partial L}{\partial \dot{q}}\mathrm{d}\dot{q} - \frac{\partial L}{\partial t}\mathrm{d}t \\
&= \dot{q}\mathrm{d}p - \dot{p}\mathrm{d}q - \frac{\partial L}{\partial t}\mathrm{d}t.
\end{aligned}$$

Comparing the first and third lines therefore gives Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad -\dot{p} = \frac{\partial H}{\partial q}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

We then write the first two of Hamilton's equations in matrix form,

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

and note that our co-ordinate transformation means that

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{pmatrix}$$

and so

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{pmatrix}.$$

Performing one of the matrix multiplications, we then note that for Hamilton's equations to be obeyed in the Q, P co-ordinates we require

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \\ -\frac{\partial P}{\partial p} & \frac{\partial P}{\partial q} \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix}.$$

The 2,1 element of this matrix equation then tells us that

$$-1 = -\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = -\{Q, P\}_{q,p}$$

as required.

For the co-ordinate transformation given in the question,

$$\frac{\partial Q}{\partial q} = \frac{m\omega p}{p^2 + (m\omega q)^2}, \quad \frac{\partial Q}{\partial p} = \frac{-m\omega q}{p^2 + (m\omega q)^2}, \quad \frac{\partial P}{\partial q} = m\omega q, \quad \frac{\partial P}{\partial p} = \frac{p}{m\omega}$$

from which it follows that the above Poisson bracket holds. Now, the inverse co-ordinate transform is

$$\begin{aligned} p &= \sqrt{2m\omega P}\cos Q \\ q &= \frac{\sqrt{2m\omega P}}{m\omega}\sin Q \end{aligned}$$

so in the Q, P co-ordinate system the Hamiltonian for the simple harmonic oscillator is $H = \omega P$. Hamilton's equations are then

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega, \quad -\dot{P} = \frac{\partial H}{\partial Q} = 0$$

the solutions to which are

$$Q = \omega t + \alpha, \quad P = \beta$$

for some constants α, β. Finally, we rewrite this solution in the q, p co-ordinate system:

$$q = \frac{\sqrt{2m\omega\beta}}{m\omega}\cos(\omega t + \alpha), \quad p = \sqrt{2m\omega\beta}\sin(\omega t + \alpha)$$

which is the familiar solution for a simple harmonic oscillator.

The Green's function (by definition) satisfies the equation

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\right)G(\boldsymbol{r},\boldsymbol{r'};t,t')=\delta^3(\boldsymbol{r}-\boldsymbol{r'})\delta(t-t')$$

The Fourier transform of the right hand side is 1. To Fourier transform the left hand side, we will change variables to $\tau=t-t'$, so

$$i\hbar\int \mathrm{d}\tau\, e^{iz\tau/\hbar}\frac{\partial G(\tau)}{\partial t}=-i\hbar\int\frac{iz}{\hbar}e^{iz\tau/\hbar}G(\tau)\,\mathrm{d}\tau=zG(\boldsymbol{r},\boldsymbol{r'};z)$$

where we have integrated by parts, noting that the boundary term is zero. Thus,

$$\left(z+\frac{\hbar^2}{2m}\nabla^2\right)G(\boldsymbol{r},\boldsymbol{r'};z)=\delta^3(\boldsymbol{r}-\boldsymbol{r'})$$

Next, we note that $\nabla^2\leftrightarrow -k^2$ under Fourier transformation and let $\boldsymbol{p}=\boldsymbol{r}-\boldsymbol{r'}$ so that

$$\begin{aligned}
G(\boldsymbol{r},\boldsymbol{r'};z) &= \int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r'})}}{z-\hbar^2k^2/2m}\\
&= \frac{1}{(2\pi)^3}\int k^2\,\mathrm{d}k\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{p}}}{z-\hbar^2k^2/2m}\\
&= \frac{1}{(2\pi)^2}\int_0^\infty\frac{k^2\,\mathrm{d}k}{z-\hbar^2k^2/2m}\int_0^\pi \mathrm{d}\theta\sin\theta e^{ikp\cos\theta}\\
&= \frac{1}{(2\pi)^2}\int_0^\infty\frac{k^2\,\mathrm{d}k}{z-\hbar^2k^2/2m}\frac{e^{ikp}-e^{-ikp}}{ikp}\\
&= \frac{2m}{i(2\pi)^2p\hbar^2}\int_{-\infty}^\infty\frac{ke^{ikp}}{2mz/\hbar^2-k^2}\mathrm{d}k
\end{aligned}$$

where the symmetry properties of the k integrand were used to change the range of k integration. The integrand has poles at $k=\pm\sqrt{2mz}/\hbar$. We first consider the case $z=E+i\epsilon$, which has poles at $k_{1,2}=\pm(\sqrt{2m}/\hbar)E^{1/2}e^{i\epsilon/2}$ (because $(E^{1/2}e^{i\epsilon/2})^2=Ee^{i\epsilon}\simeq E+i\epsilon$ when ϵ is small). We will close the contour of integration with a semi-circle in the upper half plane; the contribution from the semi-circular arc vanishes due to Jordan's lemma. Only the pole at k_1 is enclosed which has residue $-e^{ik_1p}/2$, using l'Hôpital's rule. When $z=E-i\epsilon$ the poles move to $k_{1,2}=\pm(\sqrt{2m}/\hbar)E^{1/2}e^{-i\epsilon/2}$ and the pole enclosed by the contour is the one at $k_2=-(\sqrt{2m}/\hbar)E^{1/2}e^{-i\epsilon/2}$ with residue $-e^{ik_2p}$. (For the $E<0$ cases, we just need to replace $\sqrt{E}$ by $i\sqrt{E}$). We multiply the resiudes by $2\pi i$ to arrive at the required integrals, and then use the definition of ΔG given in the question along with the integrals for the $E>0$ cases to arrive at

$$\Delta G=-2\pi i\frac{2m}{\hbar^2}\frac{\sin\left(\sqrt{2mE}|\boldsymbol{r}-\boldsymbol{r'}|/\hbar\right)}{4\pi^2|\boldsymbol{r}-\boldsymbol{r'}|}\Theta(E)$$

For a particle in free space, $E = \hbar^2 k^2/2m$ so $\mathrm{d}k/\mathrm{d}E = \sqrt{2m}/2\hbar\sqrt{E}$. Each state in k-space occupies a volume $(2\pi)^3$, so the number in a sphere of radius k is $n = 4\pi k^3/3(2\pi)^3$. Therefore, the density of states is

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}E} &= \frac{4\pi k^2}{(2\pi)^3}\frac{\mathrm{d}k}{\mathrm{d}E} \\ &= \frac{m}{2\pi^2\hbar^3}\sqrt{2mE}.\end{aligned}$$

Noting that the limit of $\sin(x)/x$ as x tends to zero is 1, comparing this to the earlier equation for ΔG shows that in this case

$$\rho(E) = \lim_{\boldsymbol{r}\to\boldsymbol{r'}} \frac{\Delta G(\boldsymbol{r},\boldsymbol{r'};E)}{-2\pi i}.$$

For the last part we again use $G = 1/(z - E_n)$, so

$$G = \sum_n \frac{|n\rangle\langle n|}{z - E_n}$$

because the eigenstates $|n\rangle$ form a complete set, and

$$\begin{aligned}G(\boldsymbol{r},\boldsymbol{r'};z) \equiv \langle r|G|r'\rangle &= \sum_n \frac{\langle r|n\rangle\langle n|r'\rangle}{z - E_n} \\ &= \sum_n \frac{\phi_n(\boldsymbol{r})\phi_n^*(\boldsymbol{r'})}{z - E_n}\end{aligned}$$

Writing $z = E \pm is$,

$$G^{\pm}(\boldsymbol{r},\boldsymbol{r'};z) = \sum_n \frac{\phi_n(\boldsymbol{r})\phi_n^*(\boldsymbol{r'})}{(E - E_n) \pm is}.$$

Using the identity given in the question,

$$\Delta G(\boldsymbol{r},\boldsymbol{r};E) = -2\pi i \sum_n \phi_n(\boldsymbol{r})\phi_n^*(\boldsymbol{r'})\delta(E - E_n)$$

and so

$$\rho(E) = \lim_{\boldsymbol{r}\to\boldsymbol{r'}} \frac{\Delta G(\boldsymbol{r},\boldsymbol{r'};E)}{-2\pi i}$$

as required.