NATURAL SCIENCES TRIPOS   Part II

Wednesday 16 January 2008   10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A thin hollow cylinder of circular cross-section and radius R can roll on a rough horizontal table. A particle is suspended from its axis by a light rod attached to a frictionless bearing. If M and m are the masses of the cylinder and particle, respectively, and c is the length of the rod, supposed less than R, construct the Lagrangian of the system. [8]

Show that small oscillations about the position of stable equilibrium have the same period as those of a simple pendulum of length $2Mc/(2M + m)$. [12]

The system is set in motion from rest by giving the cylinder a velocity V in the direction in which it can roll. By using the constants of the motion, show that the subsequent angular motion of the rod is given by

$$\frac{1}{2}mc^2\left(1 - \frac{m\cos^2\phi}{m+2M}\right)\dot{\phi}^2 = \frac{mM}{m+2M}V^2 - mgc(1-\cos\phi)$$

where ϕ is the angle it makes with the downward vertical. [13]

2 What are the main advantages of the Hamiltonian formulation of mechanics over the Lagrangian formulation? [4]

Derive Hamilton's equations of motion for the one-dimensional Hamiltonian

$$H(q,p) = p\dot{q} - L(q,\dot{q}) \ .$$

(You should assume the symbols have their usual meanings.) [5]

If the Hamiltonian is written in terms of transformed position and momentum coordinates

$$Q = Q(q,p) \ , \quad P = P(q,p)$$

show that the transformed Hamiltonian $H(Q,P)$ obeys Hamilton's equations of motion in the new coordinates if [8]

$$\{Q,P\}_{q,p} = 1$$

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where the Poisson bracket is defined by

$$\{U, V\}_{q,p} = \frac{\partial U}{\partial q}\frac{\partial V}{\partial p} - \frac{\partial U}{\partial p}\frac{\partial V}{\partial q}$$

Show that for the coordinate transformations

$$\begin{aligned} Q &= \arctan\left(\frac{m\omega q}{p}\right) \\ P &= \frac{1}{2m\omega}(p^2 + m^2\omega^2 q^2) \end{aligned}$$

the above Poisson bracket holds. [8]

Rewrite the one-dimensional harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

in terms of Q, P and solve Hamilton's equations in these coordinates. Show that your solutions are consistent with the solutions to the harmonic oscillator solved using q, p. [8]

3 The Lagrangian density for the Schrödinger wave function Ψ is

$$\mathcal{L} = \frac{\hbar}{2i}\left(\Psi\frac{\partial \Psi^*}{\partial t} - \Psi^*\frac{\partial \Psi}{\partial t}\right) - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\Psi \cdot \boldsymbol{\nabla}\Psi^* - V(\boldsymbol{r})\Psi\Psi^*$$

(a) Show that $\mathcal{L}$ has a global phase symmetry. [3]

(b) Derive the associated Noether density and current. Show explicitly that they satisfy the expected conservation equation. [10]

(c) Derive the stress-energy tensor components T^{00}, T^{0k}, T^{j0} and T^{jk}, where $j, k = 1, 2$ or 3. (Use units such that $c = 1$.) [10]

(d) Show explicitly that these components satisfy the expected conservation equations. [10]

4 The action for a system consisting of a relativistic charged particle and an electromagnetic field is

$$S = -\int mc^2 d\tau - \int eA_\mu dx^\mu(t) - \frac{1}{4\mu_0}\int F_{\alpha\beta}F^{\alpha\beta}d^4x$$

(a) Explain the meaning of the terms in this equation. [5]

(b) Show how the action leads to the equation of motion for the particle and the inhomogeneous Maxwell equations for the field. [15]

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(c) Explain what is meant by a gauge transformation and show that it does not affect the dynamics of the system. [7]

(d) Show how to include a charged scalar field φ in such a way that the gauge symmetry is preserved. [6]

[*You may assume that* $F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$]

5 A real scalar field φ has Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu\varphi)(\partial_\mu\varphi) + a\,\varphi^2 + b\,\varphi^4\right]$$

where a and b are real constants.

(a) Derive the equation of motion, the canonical momentum density and the Hamiltonian density. Find the conditions for the Hamiltonian density to be bounded from below. [10]

(b) The Lagrangian is symmetric under the transformation $\varphi \to -\varphi$. Find the conditions under which this symmetry is spontaneously broken, and the resulting minimum-energy solutions for the field and energy density. [10]

(c) Find the equation of motion and dispersion relation for small variations of the field around the minimum-energy solutions. [13]

6 The Green's function for a quantum-mechanical particle with Hamiltonian H is defined by

$$\left(i\hbar\frac{\partial}{\partial t} - H\right) G(\boldsymbol{r}, \boldsymbol{r}'; t, t') = \delta^3(\boldsymbol{r} - \boldsymbol{r}')\,\delta(t - t')$$

Use Fourier methods to derive the Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}'; z) = \int e^{iz(t-t')/\hbar} G(\boldsymbol{r}, \boldsymbol{r}'; t, t')\,dt$$

for a non-relativistic free particle in three dimensions ($H = -\hbar^2\boldsymbol{\nabla}^2/2m$), with $z = E + i\epsilon$ for the four cases [8]

(*i*) $E > 0,\ \epsilon > 0$; (*ii*) $E > 0,\ \epsilon < 0$; (*iii*) $E < 0,\ \epsilon > 0$; (*iv*) $E < 0,\ \epsilon < 0$.

The parameter ϵ should be assumed to be real and small.

Use your results to show that [6]

$$\Delta G(\boldsymbol{r}, \boldsymbol{r}'; E) = -2\pi i \frac{2m}{\hbar^2} \frac{\sin\left(\sqrt{2mE}|\boldsymbol{r} - \boldsymbol{r}'|/\hbar\right)}{4\pi^2|\boldsymbol{r} - \boldsymbol{r}'|} \Theta(E)$$

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where

$$\mathit{\Delta} G(\boldsymbol{r},\boldsymbol{r}';E) = \lim_{\epsilon\to 0}\left[G(\boldsymbol{r},\boldsymbol{r}';E+i|\epsilon|) - G(\boldsymbol{r},\boldsymbol{r}';E-i|\epsilon|)\right]$$

Calculate the density of states (the number of quantum states per unit energy per unit volume) $\rho(E)$ for a non-relativistic free particle in three dimensions using a simple phase-space argument. Hence show that for this case [6]

$$\rho(E) = \lim_{r\to r'} \frac{\mathit{\Delta} G(\boldsymbol{r},\boldsymbol{r}';E)}{-2\pi i}$$

Now consider the general case. For a system with Hamiltonian H, energy eigenvalues E_n and corresponding eigenfunctions $\phi_n(\boldsymbol{r})$, show that [6]

$$G(\boldsymbol{r},\boldsymbol{r}';z) = \sum_n \frac{\phi_n(\boldsymbol{r})\phi_n^*(\boldsymbol{r}')}{z-E_n}$$

Use this expression and the identity

$$\lim_{y\to 0^+}\frac{1}{x\pm iy} = P\frac{1}{x} \mp i\pi\delta(x)$$

to show in general that [7]

$$\rho(\boldsymbol{r};E) = \lim_{r\to r'} \frac{\mathit{\Delta} G(\boldsymbol{r},\boldsymbol{r}';E)}{-2\pi i}$$

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