NATURAL SCIENCES TRIPOS Part II

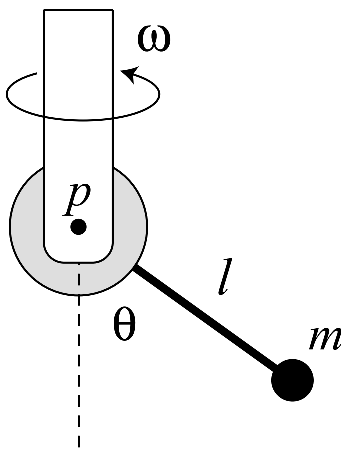
Wednesday 19 January 2005 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Describe briefly how Hamilton's principle of least action leads to Lagrange's equations of motion for a dynamical system having generalised coordinates q_i and velocities $\dot{q}_i$. [4]

A pendulum consisting of a light rod of length l with a mass m attached to its free end is attached to a frictionless bearing that rotates at angular frequency ω about a vertical axis. The bearing forces the pendulum to rotate about the vertical axis at frequency ω and allows the pendulum to swing to an arbitrary angle θ during the motion (see sketch).



Find the Lagrangian for this dynamical system. [5]

Derive the Euler-Lagrange equation of motion for θ. [5]

Find the rotation rate ω_C for which the stationary point with $\theta = 0$ becomes unstable. [6]

For $\omega > \omega_C$ what is the stable equilibrium value of θ? [6]

Calculate the frequency Ω of small oscillations about this point. [8]

(TURN OVER

2 A dynamical system is described by n independent generalised coordinates q_i and velocities $\dot{q}_i$, where $i = 1, 2...n$. The system has a Lagrangian

$$L = \frac{1}{2}\dot{\boldsymbol{q}}^T \cdot \boldsymbol{A} \cdot \dot{\boldsymbol{q}} - V$$

where $\boldsymbol{A}$ is a symmetric positive definite matrix of coefficients a_{ij} which may be functions of q_i but are independent of $\dot{q}_i$ and t. A potential function V is also independent of the $\dot{q}_i$.

Find an expression for the canonical momenta p_i. [8]

Hence show that the system has a Hamiltonian given by

$$H = \frac{1}{2}\boldsymbol{p}^T \cdot \boldsymbol{A}^{-1} \cdot \boldsymbol{p} + V$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are column vectors with components $\{q_i\}$ and $\{p_i\}$ and ^{T} denotes transpose. [6]

Show that p_2 and p_3 are constants of motion for the case where [10]

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2q_1^2 & -q_1 \\ 0 & -q_1 & 1 \end{pmatrix} \qquad \text{and} \quad V = -\frac{1}{2}\log q_1 \ .$$

For motion with $p_1 = 0$ and q_1 fixed, find the condition that must be satisfied by q_1, p_2 and p_3. Show that this condition implies a minimum value for p_3^2. [10]

3 Consider the following generalisation of the Lagrangian for a simple relativistic particle:

$$L = -m_0\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \ ,$$

where $(dx^0, dx^1, dx^2, dx^3) = (cdt, dx, dy, dz)$, and $g_{\mu\nu}$ is a symmetric tensor which varies with position and time, and m_0 is a constant.

For the simplifying case of only time plus one dimension of space, with $g_{00} = -g_{11} = g(x)$, independent of time, and also $g_{10} = g_{01} = 0$, show that the Euler-Lagrange equations reduce to the form [8]

$$\frac{d}{dt}(\Gamma m_0 v) = -\frac{m_0}{\Gamma}\frac{\partial\phi}{\partial x}$$

and give an explicit expression for Γ and the potential ϕ in terms of $g(x)$ and $v = dx/dt$. [8]

For the general case show that the equations of motion are given by [10]

$$\frac{d}{dt}\left(\gamma g_{k\nu}\frac{dx^\nu}{dt}\right) = \frac{1}{2}\gamma\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{\partial g_{\mu\nu}}{\partial x^k}$$

giving the explicit form for γ and indicating carefully what values are taken by the indices k, μ, ν. [8]

4 Describe how the Cauchy integral theorem can be used to evaluate contour integrals in the complex plane. Illustrate your answer by showing that

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{x^4 + 2\,x^2 + 1} = \frac{\pi}{2}\,.$$ [10]

Calculate the integrals

$$\int_{0}^{2\pi} \mathrm{d}x\ \frac{1}{\cos^2\ x + \cos\ x + 1}$$ [12]

and

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{1}{(1+x)^2 + (1+1/x)^2 + 1}\,.$$ [12]

5 Describe the conditions for potential flow of an incompressible fluid? [8]

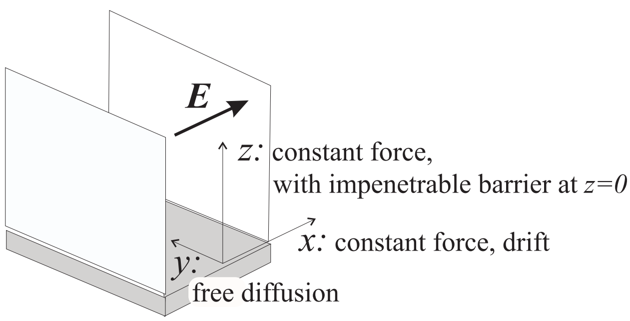
Consider a potential flow of a wide river flowing steadily over a large flat circular sandbank in the centre of the river, where its depth is half as deep as elsewhere. (The depth of river is uniform, except where it changes at the edge of sandbank, and is much less than the sandbank diameter, so you can consider an effectively 2-dimensional polar problem with the flow velocity $v_r = v_0 \cos\theta$ at $r \to \infty$.)

What are the appropriate boundary conditions, connecting the shallow circular region above the sandbank and the region outside it, and respecting the conservation of mass of water flowing into and over the bank? [8]

Show, by solving the Laplace equation for the flow potential $\phi(r,\theta)$, subject to these boundary conditions, that above the sandbank the current is uniform. [8]

Show that the flow over the bank is $\frac{4}{3}$ times faster than at large distances from it (where the velocity is also uniform and equal to v_0). [10]

6 A charged Brownian particle of mass m, carrying the charge q, diffuses between the vertical parallel plates of a capacitor, under the action of constant electric field $E = E_x$. Assuming the constant of particle friction is γ, write down the full Langevin equation describing the stochastic motion of the particle in all three dimension, x, y, z. [8]



(TURN OVER

Explain its limit describing the overdamped motion. [6]

In the overdamped limit, find the average particle position and the mean square displacement along the horizontal y-axis, and the average drift velocity along the x-axis. [10]

If the vertical motion of the particle is only allowed for $z \geq 0$ (restricted by the barrier below $z = 0$), find the equilibrium probability of finding the particle at a certain height z. [10]