NATURAL SCIENCES TRIPOS Part II

Wednesday 14 January 2004 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 A system consists of one point-like particle with a mass M and position vector $\boldsymbol{R}$ and n particles with the same mass m and position vectors $\boldsymbol{R}_\alpha$, with $\alpha = 1, ..., n$. All interactions are described by the potential energy $U(\boldsymbol{R}, \boldsymbol{R}_\alpha)$. Write down and describe the terms in the Lagrangian for this system. [6]

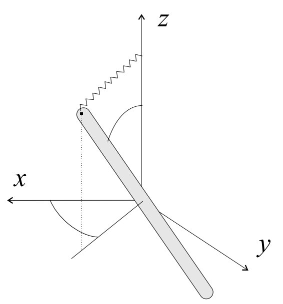
The system evolves such that its centre of mass remains fixed (take, for definiteness, that it remains at the origin of the coordinate system). Describe the type of constraint this is and reduce the number of degrees of freedom, writing the Lagrangian in terms of the relative coordinates $\boldsymbol{r}_\alpha = \boldsymbol{R}_\alpha - \boldsymbol{R}$. [10]

Find the canonical momenta $\boldsymbol{p}_\alpha$ for this system. [6]

Hence, or otherwise, prove that the Hamiltonian function for this system takes the form [12]

$$H = \frac{1}{2m}\sum_\alpha \boldsymbol{p}_\alpha^2 + \frac{1}{2M}\left(\sum_\alpha \boldsymbol{p}_\alpha\right)^2 + U.$$

2 A thin uniform rod of mass m and length ℓ is pivoted at its centre. Its top end is linked to a point with coordinates $(0, 0, \frac{1}{2}\ell)$ with an elastic spring of zero equilibrium length with spring constant κ, see sketch. (The transverse moment of inertia for a rod is $I_\perp = \frac{1}{12}m\ell^2$; you may regard $I_\parallel = 0$.)



Write the Lagrangian for this system in terms of angular coordinates θ and ϕ. [10]

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Obtain the canonical momenta for this system and write down the Hamiltonian. [4]

Find and discuss the canonical (Hamilton's) equations and the conservation laws that you can identify for this system. [4]

Determine the conditions for which it is possible to have a steady state motion consisting of a rotation about the z axis with an angular velocity $\dot{\phi} = \Omega$. [10]

Show that the frequency of small oscillations about the polar angle θ_0 of this steady-state is equal to [6]

$$\omega = \Omega\sqrt{1 - \frac{9\kappa^2}{m^2\Omega^4}}.$$

[Hint: In principal axes, the components of angular velocity of a solid body are expressed through Euler angles:
$\Omega_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$; $\Omega_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$; $\Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$.]

3 A relativistic particle of rest mass m_0 is acted on by a constant force F, along its line of motion. Write down the Lagrangian for this particle. [8]

Show that the Hamiltonian for this particle is [8]

$$\mathcal{H}(p,q) = \sqrt{p^2c^2 + \mathcal{E}_0^2} - Fq,$$

where $\mathcal{E}_0$ is the rest energy of the particle.

Prove that this Hamiltonian is a conserved quantity. [4]

Find the complete solution, $p(t)$ and $q(t)$, given the initial conditions $p(0) = 0$, $q(0) = 0$. [8]

Find the time dependence of the velocity $\dot{q}(t)$ and comment on its limits for small and large t. [6]

4 Consider a damped one-dimensional harmonic oscillator,

$$\ddot{x} + \gamma\dot{x} + \Omega^2 x = f(t)\ ,$$

excited by a top-hat pulse force: $f(t) = f_0$ for $t \in (0, a)$ and $f(t) = 0$ otherwise (γ, Ω and a are positive constants, $4\Omega^2 > \gamma$). Show that the Fourier transform x_ω takes the form [8]

$$x_\omega = -i\frac{f_0(1 - e^{i\omega a})}{\omega(\omega^2 + i\omega\gamma - \Omega^2)}.$$

Discuss the application of Green's functions and the role of causality in dynamical problems. [8]

Examine the possibility of integration on the complex plane of $[\mathrm{Re}\omega, \mathrm{Im}\omega]$ to perform the inverse Fourier transformation of x_ω. In particular compare the regimes where $t < a$ and $t > a$, and the structure of the singularities. [8]

Hence, or otherwise, find the asymptotic form of $x(t)$ at long times, $t \gg a$ and comment on your result. [10]

5 Using the principles of integration on the complex plane, prove the following property of Dirac delta function: [14]

$$\int_{-\infty}^{\infty} f(x)\delta(x-y)dx = f(y)$$

(for an arbitrary regular function $f(x)$, bound at $x \to \pm\infty$), using the following definition of delta function

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2} \; .$$

The complete gamma function $\Gamma(z)$ is defined to be an extension of the factorial function to real and complex number arguments. It is related to the factorial by $\Gamma(n) = (n-1)!$. It is analytic everywhere except at $z = 0, -1, -2, ...$ The integral representation is $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

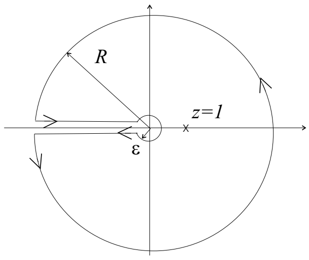
Show that the following relationship holds [6]

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{u^{x-1}}{1+u} du \; .$$

Hence, or otherwise, prove the famous Euler reflection formula [14]

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \; .$$

[Two hints: (1) When writing the product of two Gamma functions via a double integral, $\int_0^\infty t^{x-1}e^{-t}dt \int_0^\infty s^{-x}e^{-s}ds$, make a substitution $t = s\,u$. (2) On the plane of complex u, use the close contour sketched, with $R \to \infty$ and $\epsilon \to 0$, and examine how different parts of it contribute to the integral $\oint dz\, z^{x-1}/(1-z)$.]



6 Describe the standard form of the Langevin (stochastic) equation,

$$\dot{q}_\alpha = F_\alpha(\boldsymbol{q}) + G_\alpha^k(\boldsymbol{q})A_k(t) \; ,$$

and its relation to classical Brownian motion. [6]

Outline principles leading to the Fokker-Planck (kinetic) equation for the probability density $f(\boldsymbol{q}, t)$: [6]

$$\frac{\partial f(\boldsymbol{q},t)}{\partial t} = \left\{ -\frac{\partial K_\alpha(\boldsymbol{q})}{\partial q_\alpha} + \frac{1}{2}\frac{\partial^2}{\partial q_\alpha \partial q_\beta} Q_{\alpha\beta}(\boldsymbol{q}) \right\} f(\boldsymbol{q},t) \; ,$$

$$\text{with} \qquad K_\alpha = F_\alpha + \frac{1}{2}\frac{\partial G_\alpha^i}{\partial q_\beta} G_\beta^k \delta_{ik}; \quad Q_{\alpha\beta} = G_\alpha^i G_\beta^k \delta_{ik} \; .$$

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Obtain the Fokker-Planck equation for the free Brownian motion. [8]

Discuss the conditions under which the probability of a Brownian particle satisfies an ordinary diffusion equation. [6]

By finding the equilibrium probability density of a free Brownian particle, prove the fluctuation-dissipation relationship between the friction and the thermal noise constants. [8]