Theoretical Physics 1
Answers to Examination 2002

Warning — these answers have been completely retyped... Please report any typos/errors.
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Q1. Bookwork: the conjugate momenta are $p_i \equiv \partial L/\partial q_i$. The Hamiltonian is

$$H \equiv \sum_i p_i \dot{q}_i - L \;, \tag{1}$$

which is a function of (q_i, p_i) but not $\dot{q}_i$. Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \;; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \;, \tag{2}$$

i.e. a set of $2N$ first-order equations for the coordinates and momenta.

The Poisson bracket $\{f, g\}$ of functions $f(q_i p_i)$ and $g(q_i p_i)$ is defined as

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{3}$$

It is useful because the evolution equation of any quantity $f(\boldsymbol{q}, \boldsymbol{p}, t)$ can be expressed (using Hamilon's equations) as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\} \tag{4}$$

To see this write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \frac{\mathrm{d}q_i}{\mathrm{d}t}\frac{\partial f}{\partial q_i} + \sum_i \frac{\mathrm{d}p_i}{\mathrm{d}t}\frac{\partial f}{\partial p_i} \tag{5}$$

and use Hamilton's equations.

For a vector field $\boldsymbol{A}(\boldsymbol{x})$ function of position $\boldsymbol{x}$, we have $q_i = x_i$. Then, for the kth component of $\boldsymbol{A}$ and jth component of $\boldsymbol{p}$

$$\{p_j\, A_k\} = \sum_i \left(\frac{\partial p_j}{\partial x_i}\frac{\partial A_k}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial A_k}{\partial x_i} \right) \tag{6}$$

The first term is zero and the second evaluates to $-\partial A_k/\partial x_j$.

The Langrangian is $L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - e(\phi - \dot{\boldsymbol{x}}\cdot\boldsymbol{A})$. The canonical momentum is $\boldsymbol{p} = m\boldsymbol{v} + e\boldsymbol{A}$. The classical Hamiltonian is $(\boldsymbol{p} - e\boldsymbol{A})^2/2m + e\phi$. The corresponding quantum operator is

$$\begin{aligned} \hat{H} &= \tfrac{1}{2m}\left(\hat{\boldsymbol{p}}^2 - e\hat{\boldsymbol{A}}\cdot\hat{\boldsymbol{p}} - e\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{A}} + e^2\hat{\boldsymbol{A}}^2\right) + e\phi \\ &= \tfrac{1}{2m}\left(\hat{\boldsymbol{p}}^2 - 2e\hat{\boldsymbol{A}}\cdot\hat{\boldsymbol{p}} - e[\hat{p}_j, \hat{A}_j] + e^2\hat{\boldsymbol{A}}^2\right) + e\phi \end{aligned} \tag{7}$$

The commutator term evaluates to $[\hat{p_j^2}, \hat{A}_j] = -i\hbar\nabla\cdot\boldsymbol{A}$, which vanishes in the given gauge. After that it's plain sailing using $\hat{p} = -i\hbar\nabla$ to get

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar e}{m}\boldsymbol{A}\cdot\nabla + \frac{e^2\boldsymbol{A}^2}{2m} + e\phi \tag{8}$$

and the Schrödinger equation is just $i\hbar\partial\psi/\partial t = \hat{H}\psi$.

Q2. Bookwork: Hamilton's principle is $\delta\int \mathrm{d}t\ L(q_i, \dot{q}_i, t) = 0$ and leads (via the calculus of variations) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial q_i} \tag{9}$$

i.e. N 2nd-order equations for the coordinates q_i.

Using generalised coordinates θ, x, the kinetic energy is

$$\tfrac{1}{2}ma^2\dot{\theta}^2 + \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{4}m(\dot{x} + a\dot{\theta})^2 \tag{10}$$

where $I = ma^2$ for the disc of mass $2m$. The potential energy is $-mga\cos\theta - \frac{1}{2}mg(x + a\theta) + \frac{1}{2}kx^2$, so the Lagrangian is

$$L = \frac{5}{4}ma^2\dot{\theta}^2 + \frac{1}{2}ma\dot{\theta}\dot{x} + \frac{1}{4}m\dot{x}^2 + mga\cos\theta + \tfrac{1}{2}m(x + a\theta) - \tfrac{1}{2}kx^2 \tag{11}$$

and Lagrange's equations are

$$\begin{aligned} \tfrac{5}{2}ma^2\ddot{\theta} + \tfrac{1}{2}ma\ddot{x} &= -mga\sin\theta + \tfrac{1}{2}mga \\ \tfrac{1}{2}ma\ddot{\theta} + \tfrac{1}{2}m\ddot{x} &= \tfrac{1}{2}mg - kx \end{aligned} \tag{12}$$

The conditions for equilibrium are, for zero LHS,

$$x_0 = mg/2k\ ; \quad \sin\theta_0 = \tfrac{1}{2} \Rightarrow \theta_0 = \pi/6 \tag{13}$$

Expanding about this position $\delta\theta \equiv \theta - \theta_0$ we need the result $\sin\theta \approx \sin\theta_0 + \delta\theta\ \cos\theta_0$, where $\cos\theta_0 = \sqrt{3}/2$ For oscillations like $\exp(-i\omega t)$, we tidy up to get

$$\begin{pmatrix} \frac{\sqrt{3}g}{2a} - \frac{5m\omega^2}{2} & -\frac{m\omega^2}{2} \\ -\frac{m\omega^2}{2} & k - \frac{m\omega^2}{2} \end{pmatrix} \begin{pmatrix} \delta\theta \\ \delta x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

For non-trivial solutions, the determinant must vanish leading to

$$m^2\omega^4 - \left(\frac{5k}{2} + \frac{\sqrt{3}g}{4}\right)m\omega^2 + \frac{\sqrt{3}gk}{2a} = 0 \tag{15}$$

as required.

For the limit $k \gg g/a$ the upper root is about the same as the sum of roots, so $\omega^2 \approx 5k/2m$. The product of the roots is $\sqrt{3}gk/2am$, so the low frequency mode has $\omega^2 \approx \sqrt{3}g/5a$, as is appropriate for an effective spring constant of $\sqrt{3}g/2a$ and an effective mass of $5m/2$.

The other limit $k \ll g/a$ has a high mode of $\omega^2 \approx \sqrt{3}g/4a$ the $m/2$ mass has become uncoupled and a low mode of $\omega^2 \approx 2k/m$ (same reason).

Q3. The forces are viscosity (with $\gamma = 6\pi\eta\, av$), gravity and the stochastic force $A(t)$. This last one needs comment, being on average zero $\langle A(t)\rangle = 0$ and uncorrelated at different times. The strength of the stochastic force is Γ, where $\langle A(t)A(t')\rangle = \Gamma\delta(t-t')$.

Setting $g = 0$ for the moment, we have

$$v(t) = e^{\frac{-\gamma t}{m}} \int_0^t \mathrm{d}\xi\; e^{\frac{\gamma\xi}{m}} A(\xi)/m \tag{16}$$

Averaging, we get

$$\left\langle v^2(t)\right\rangle = e^{\frac{-2\gamma t}{m}} \frac{1}{m^2} \int_0^t \mathrm{d}\xi \int_0^t \mathrm{d}\eta\; e^{\frac{\gamma(\xi+\eta)}{m}} \langle A(\xi)A(\eta)\rangle \tag{17}$$

The δ-function sets $\xi = \eta$ and we have

$$\left\langle v^2(t)\right\rangle = \frac{e^{\frac{-2\gamma t}{m}}}{m^2} \int_0^t \mathrm{d}\xi\; e^{\frac{2\gamma\eta}{m}} \Gamma = \frac{\Gamma}{2m\gamma} e^{\frac{-2\gamma t}{m}} \left(e^{\frac{2\gamma t}{m}} - 1\right) \tag{18}$$

which tends to $\Gamma/2m\gamma = \Gamma/12\pi\eta am$ as $t \to \infty$. In equilibrium $\langle v^2\rangle = kT/m$, so $\Gamma = 2\gamma kT$ as required.

Diffusion equation for overdamped case drops the $\dot{v}$ term so

$$\gamma\dot{z} = -mg + A \tag{19}$$

which Eugene assures me implies that the kinetic equation for the probability distribution is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial z}\left(\frac{mg}{\gamma} + \frac{\Gamma}{2\gamma^2}\frac{\partial}{\partial z}\right) P \tag{20}$$

in equilibrium the LHS is zero so $\frac{\partial P}{\partial z} = -(2\gamma mg/\Gamma)P$ and $P \propto \exp(-2\gamma mgz/\Gamma) = \exp(-mgz/kT)$, the Boltzmann distribution, as expected.

The centre of mass $\langle z\rangle = kT/mg$.

Q4. (a) It's easiest, but not crucial, to write $\sin x \sin(x-\alpha)$ as $-\frac{1}{2}\Re\left(e^{i(2x-\alpha)} - e^{i\alpha}\right)$. Then closing over the top, we have

$$-\oint \mathrm{d}z\; \frac{e^{i(2x-\alpha)} - e^{i\alpha}}{2x(x-\alpha)} = -\oint \mathrm{d}z\; \frac{1}{2\alpha}\left(e^{i(2x-\alpha)} - e^{i\alpha}\right)\left(\frac{1}{x-\alpha} - \frac{1}{x}\right) \tag{21}$$

This is equal to πi times the residues, rather than $2\pi i$, because the contour goes through the poles (lenient treatment of offenders...) The residues are $e^{\pm i\alpha}$ (twice each), hence result using $2i\sin\alpha = e^{i\alpha} - e^{-i\alpha}$.

Alternative: use Fourier transform of sinc function is 1 in $[-\pi, \pi]$, 0 elsewhere. Integral given is *correlation* of two sinc functions – hence is FT(product) (FT real here, but it's really $\tilde{f}\tilde{g}^*$ in the correlation theorem, rather than the convolution theorem). So FT(answer) is just the same $\Rightarrow$ answer is $\sin(\alpha)/\alpha$.

(b) Nice key-hole integral: I is integral to be found — $\int_{\Gamma_1} = I$, $\int_{\Gamma_3} = -e^{-2ip\pi} I$, $\int_{\Gamma_2} = 0$ if $p > -1$ and $\int_{\Gamma_4} = 0$ if $p < 1$. (Bonus for proving conditions...)

Locate poles: $1 + 2z\cos\theta + z^2 = (z + e^{i\theta})(z + e^{-i\theta})$, so poles at $z = -e^{\pm i\theta}$ (note sign). Converting to partial fractions

$$\left(1 - e^{-2ip\pi}\right) I = \oint_{\Gamma} \mathrm{d}z \left(\frac{1}{z + e^{-i\theta}} - \frac{1}{z + e^{i\theta}}\right) \frac{z^{-p}}{e^{i\theta} - e^{-i\theta}} \tag{22}$$

The residue at $z = -e^{i\theta}$ is found by writing it as $z = e^{i\pi} e^{i\theta}$, we get $z^{-p} = e^{-ip\pi} e^{ip\theta}$. Then, using the residue theorem

$$\left(1 - e^{-2ip\pi}\right) I = 2\pi i e^{-ip\pi} \frac{e^{ip\theta} - e^{-ip\theta}}{e^{i\theta} - e^{-i\theta}} \tag{23}$$

which proves the result.

Q5. Preferred version of (t, ω) Fourier transform and its inverse is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \; f(t) e^{i\omega t} \; ; \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \; \tilde{f}(\omega) e^{-i\omega t} \tag{24}$$

For the (x, k) pair I prefer the opposite sign – the reason being the it is the convention in QM that $e^{i(kx - \omega t)}$ represents a positive energy wave travelling in the $+x$-direction (remember $i\hbar\dot{\psi} = E\psi$). No arguments, please, just do it my way...

Lagrangian fields: the action is

$$S = \int \mathrm{d}t\mathrm{d}x \; \left(\tfrac{1}{2} T \left(\frac{\partial u}{\partial x}\right)^2 - \tfrac{1}{2}\rho \left(\frac{\partial u}{\partial t}\right)^2\right) \tag{25}$$

the equations of motion are

$$\frac{\partial}{\partial x}\left(T \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho \frac{\partial u}{\partial t}\right) = 0 \tag{26}$$

With damping present add a term $\gamma \dot{u}$:

$$T \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} \tag{27}$$

Fourier transforming this equation we have

$$\left(\rho\omega^2 - i\gamma\omega - Tk^2\right) \tilde{u}(\omega, k) = 0 \tag{28}$$

The Green's function (for $t' = x' = 0$ for simplicity) is (sign not important)

$$G(x, t) = \int \frac{\mathrm{d}\omega \mathrm{d}k}{(2\pi^2)} \; \frac{e^{i(kx - \omega t)}}{\rho\omega^2 - i\gamma\omega - Tk^2} \tag{29}$$

The ω integral has poles at $\omega_{1,2} = \frac{i\gamma}{2\rho} \pm \sqrt{Tk^2/\rho + \gamma^2/4\rho^2}$, i.e. in the upper half-plane. For $t < 0$ we must close the contour in the lower half-plane to ensure that the integral over the arc is zero. The result is then zero, generating a $\theta(t)$ Heaviside function. The Green's function is therefore causal.

For $t > 0$ some tidying up leaves the final integral as

$$G(x,t) = \frac{\theta(t)e^{-\gamma t/2\rho}}{2\pi} \int \mathrm{d}k \, \frac{e^{ikx} \sin \sqrt{Tk^2/\rho + \gamma^2/4\rho^2}t}{Tk^2/\rho + \gamma^2/4\rho^2} \tag{30}$$

(not sufficiently checked…) You are not expected to know that off-hand, but this is a standard Bessel function…

Q6.

- Need to describe, for a discrete ID process the idea that

$$P_{N+1}(m) = w(m, m')P_N(m-1) \tag{31}$$

- Principle of detailed balance is $w(m, m')P(m') = w(m', m)P(m)$ for each pair m, m'.
- For a 1-dimensional free random walk ($w = 1/2$), derive the diffusion equation and discuss the evaluation of (for example) $G \propto \exp(-x^2/2Dt)$.
- Build the path integral for $G(a, b)$ ($\equiv w(a, b)$) as the product

$$\prod_{j=1}^{N-1} e^{-\frac{1}{2D\tau}(x_{j+1}-x_j)^2} \longrightarrow \int \mathcal{D}x \; e^{-\frac{1}{2D}\int_a^b \mathrm{d}t \; \dot{x}^2} \tag{32}$$

- Mention the general case

$$G(a, b) \propto \int \mathcal{D}x \; e^{-\frac{1}{Dm}\int_a^b \mathrm{d}t \; L(x,\dot{x})} \tag{33}$$

  Don't forget: $D = \frac{kT}{\gamma m}$.
- Identify the classical trajectory (extremum of S) with the most probable path as $\gamma/kT \to \infty$.