Theoretical Physics 1
Answers to Examination 2000

Warning — these answers have been completely retyped... Please report any typos/errors. Suggestions for improvement/more detail are welcome.
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Q1. Kinetic energy is $T = \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{2}I\dot\theta^2$, where the moment of inertia $I = ma^2/3$. We therefore get $T = 2ma^2\dot\theta^2/3$ as required.

For the second rod, the centre of mass is at

$$x = 2a\sin\theta + a\sin\phi \ ; \quad y = -2a\cos\theta - a\cos\phi \ , \tag{1}$$

$$\Rightarrow \dot x^2 + \dot y^2 = a^2(4\dot\theta^2 + \dot\phi^2 + 4a\cos(\theta-\phi)\dot\theta\dot\phi) \ . \tag{2}$$

Adding the rotational and potential terms, we get the Lagrangian $\mathcal{L} \equiv T - V$

$$\mathcal{L} = ma^2\left(\frac{8}{3}\dot\theta^2 + \frac{2}{3}\dot\phi^2 + 2\cos(\theta-\phi)\dot\theta\dot\phi\right) + mga(3\cos\theta + \cos\phi) \ . \tag{3}$$

Lagrange's equations are (after some cancelations):

$$\begin{aligned} ma^2\left(\frac{16}{3}\ddot\theta + 2\cos(\theta-\phi)\ddot\phi\right) &= 2ma^2\sin(\theta-\phi)\dot\phi^2 - 3mga\sin\theta \\ ma^2\left(\frac{4}{3}\ddot\phi + 2\cos(\theta-\phi)\ddot\theta\right) &= 2ma^2\sin(\theta-\phi)\dot\theta^2 - mga\sin\phi \end{aligned} \tag{4}$$

for small θ, ϕ we ignore the third-order $\dot\theta^2, \dot\phi^2$ terms, set $\cos(\theta-\phi) = 1$ and use $\sin\theta \approx \theta$ etc to get

$$4\ddot\phi + 6\ddot\theta = -3(g/a)\phi \ ; \quad 6\ddot\phi + 16\ddot\theta = -9(g/a)\phi \tag{5}$$

as required.

We look for normal modes of the form $\theta, \phi \propto \exp(-i\omega t)$, so that the eigenvalue equation becomes

$$\begin{vmatrix} 3g/a - 4\omega^2 & -6\omega^2 \\ -6\omega^2 & 9g/a - 16\omega^2 \end{vmatrix} = 0 \Rightarrow 27(g/a)^2 - 84\omega^2 g/a + 28\omega^4 = 0 \tag{6}$$

which has solution

$$\omega^2 = 3\frac{g}{a}\left(\frac{1}{2} \pm \frac{1}{\sqrt{7}}\right) \ . \tag{7}$$

Q2. The Lagrangian is

$$-m_0c^2(1-\dot q^2/c^2)^{1/2} - V(q) \tag{8}$$

so that the canonical momentum is

$$p \equiv \frac{\partial\mathcal{L}}{\partial\dot q} = m_0\dot q\gamma \tag{9}$$

as expected. The Hamiltonian is

$$\mathcal{H} = m_0\gamma(\dot{q}^2 - c^2(1 - \dot{q}^2/c^2)) + V(q) = m_0c^2\gamma + V(q) \tag{10}$$

and is a constant of the motion because the Lagrangian is independent of t. To show it explicitly, use Lagrange's equation in the form

$$m_0\gamma^3\ddot{q} = -\frac{\partial V}{\partial q} \tag{11}$$

and multiply by $\dot{q}$. Remember that $\mathrm{d}V/\mathrm{d}t = \dot{q}\partial V/\partial q$.

For the case $V = \frac{1}{2}kq^2$ we get

$$E = \tfrac{1}{2}kq^2 + m_0c^2(1 - \dot{q}^2/c^2)^{-1/2} \Rightarrow \dot{q} = c\sqrt{1 - \frac{m_0^2c^4}{(E - \frac{1}{2}kq^2)^2}} \ . \tag{12}$$

For periodic motion of amplitude b the period is therefore

$$\tau = \frac{4}{c}\int_0^b \frac{\mathrm{d}q}{\sqrt{1 - \frac{m_0^2c^4}{(E - \frac{1}{2}kq^2)^2}}} \tag{13}$$

At $q = b$ the mass is stationary so that $E = m_0c^2 + \frac{1}{2}kb^2$. Subtracting $\frac{1}{2}kq^2$ from both sides we have

$$\frac{E - \frac{1}{2}kq^2}{m_0c^2} = 1 + \alpha(b^2 - q^2) \ , \tag{14}$$

where $\alpha \equiv k/2m_0c^2$.

The next bit is too difficult, and not surprisingly the answer given is wrong! The substitution $q = b\sin\theta$ yields

$$\tau = \frac{4b}{c}\int_0^{\pi/2} \mathrm{d}\theta \ \frac{\cos\theta(1 + \alpha b^2\cos^2\theta)}{\sqrt{(1 - \alpha b^2\cos^2\theta)^2 - 1}} \ . \tag{15}$$

Expansion in powers of α yields

$$\tau = \frac{4}{c\sqrt{2\alpha}}\int_0^{\pi/2} \mathrm{d}\theta \ \left(1 + \frac{3\alpha b^2}{4}\cos^2\theta + \mathcal{O}((\alpha b^2)^2)\right) \tag{16}$$

Integrating, we get

$$\tau = \frac{2\pi}{c\sqrt{2\alpha}}\left(1 + \frac{3\alpha b^2}{8} + \mathcal{O}((\alpha b^2)^2)\right) \ . \tag{17}$$

The non-relativistic limit is $\tau = 2\pi\sqrt{m_0/k}$ as expected.

Q3. Hamilton's equations are

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \; ; \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \; . \tag{18}$$

For a phase-space volume V with surface S the $6N$-dimensional flux $\dot{\boldsymbol{r}} \equiv (\dot{\boldsymbol{q}}, \dot{\mathbf{p}})$ satisfies

$$\oint \mathrm{d}S \cdot \dot{\boldsymbol{r}} = 0 \; . \tag{19}$$

Proof:

$$\oint \mathrm{d}S \cdot \dot{\boldsymbol{r}} = \int \mathrm{d}V \left(\frac{\partial}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) = 0 \tag{20}$$

Liouville's theorem implies that volumes in phase space evolve like an incompressable fluid. For an ensemble described by probability density $\rho(\boldsymbol{q}, \mathbf{p})$ the Gibbs' entropy $S \equiv -k \int \mathrm{d}V \;\; \rho \log \rho$ is a constant of the motion.

The principle of least time implies

$$\delta \int \mathrm{d}z \; \frac{n}{c} \sqrt{1 + (q')^2} = 0 \; . \tag{21}$$

The canonical momentum

$$p = \frac{n}{c} \frac{q'}{\sqrt{1 + (q')^2}} = \frac{n}{c} \sin\theta \; , \tag{22}$$

so that p, q will satisfy Liouville's theorem.

The visualisation of the phase space volume is always a bit mind-boggling, but here we are helped by setting $n = 1$, so that the rays remain straight. If $\theta \ll 1$ as well then we can find a simple expression for the boundary after propagation by length b.

Start with a parametric form for the ellipse:

$$q(0) = a \cos u \; ; \quad p(0) = b \sin u \; , \tag{23}$$

then after length l we have

$$q(l) = a \cos u + bl \sin u \; ; \quad p(l) = b \sin u \; . \tag{24}$$

The equation of the distorted ellipse is

$$(a^2 + b^2 l^2) p^2 - 2 l b^2 p q + b^2 q^2 = 1 \; , \tag{25}$$

which has a determinant $\alpha\gamma - \beta^2 = a^2 b^2$ which doesn't depend on l, so that the area is constant.

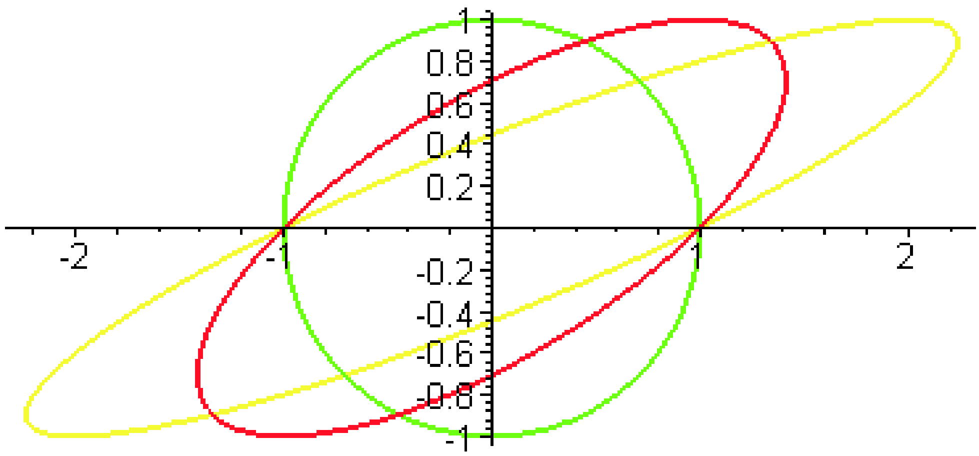

Figure 1: Phase-space volume for $t = 0, 1, 2$ showing distortion but no change of volume.

Q4. Noether's theorem is extremely general and powerful — this is just a simple example of it. Doing exactly what it says in the question we get

$$0 = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\alpha} = \frac{\partial \psi}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \psi} + \frac{\partial \dot{\psi}}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial \nabla \psi}{\partial \alpha}\cdot\frac{\partial \mathcal{L}}{\partial \nabla \psi} + \text{ terms in } \psi^* \ . \tag{26}$$

We now see that, for $\psi(\alpha) = \psi e^{i\alpha}$

$$\frac{\partial \psi}{\partial \alpha} = i\psi; \ \frac{\partial \dot{\psi}}{\partial \alpha} = i\dot{\psi}; \ \frac{\partial \nabla \psi}{\partial \alpha} = i\nabla \psi; \ \frac{\partial \psi^*}{\partial \alpha} = -i\psi^* \quad \text{etc} \ . \tag{27}$$

The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla \psi} \tag{28}$$

and the same for ψ^* Substituting this in (26) we assemble total derivatives; for example the first couple of terms give

$$i\psi\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} + i\dot{\psi}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\partial}{\partial t}\left(\psi\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \ . \tag{29}$$

The ψ^* terms come in with the opposite sign, so that we finally have Noether's theorem in the form

$$\frac{\partial}{\partial t}\left(\psi\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \psi^*\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*}\right) + \nabla\cdot\left(\psi\frac{\partial \mathcal{L}}{\partial \nabla \psi} - \psi^*\frac{\partial \mathcal{L}}{\partial \nabla \psi^*}\right) = 0 \ . \tag{30}$$

The momentum density for the free quantum particle is

$$\pi(t, \boldsymbol{r}) = i\hbar|\psi|^2 \tag{31}$$

and the current is

$$j(t, \boldsymbol{r}) = \frac{\hbar^2}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \ . \tag{32}$$

Q5. I would have preferred the condition in the form $\partial_\mu A^\mu = 0\ldots$ Explictly

$$A^\mu = (\varphi, \boldsymbol{A}) \ ; \quad \partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla) \tag{33}$$

so that, as required,

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A} \ . \tag{34}$$

We want to find the Green's function that satisfies

$$\varphi(t, \boldsymbol{r}) = \int \mathrm{d}t'\mathrm{d}\boldsymbol{r}' \ G(t, \boldsymbol{r}; t', \boldsymbol{r}')\frac{\rho(t', \boldsymbol{r}')}{\epsilon_0} \ . \tag{35}$$

The Green's function itself satisfies the equation

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(t - t')\delta^3(\boldsymbol{r} - \boldsymbol{r}') \ . \tag{36}$$

The Green's function depends only on $|\boldsymbol{r} - \boldsymbol{r}'|$.

Perform a 4-dimensional Fourier transform on $G(t, \boldsymbol{r})$ to $\tilde{G}(\omega, \boldsymbol{k})$ to get

$$\left(\frac{\omega^2}{c^2} - k^2\right)\tilde{G} = 1 \ . \tag{37}$$

Now back transform

$$G(t, \boldsymbol{r}) = \frac{c^2}{(2\pi)^4}\int \mathrm{d}\omega\mathrm{d}\boldsymbol{k} \ \frac{\exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t))}{\omega^2 - c^2k^2} \tag{38}$$

There are two poles in the ω plane at $\pm ck$; separating them by partial fractions we get

$$G(t, \boldsymbol{r}) = \frac{c}{(2\pi)^4}\int \mathrm{d}\omega\mathrm{d}\boldsymbol{k} \ \left(\frac{1}{\omega - ck} - \frac{1}{\omega + ck}\right)\frac{\exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t))}{2k} \ . \tag{39}$$

Taking the residues at these poles, we get the causal Green's function for $t > 0$

$$G(t, \boldsymbol{r}) = \frac{c}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k} \ \frac{\sin kr}{k}\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) \ . \tag{40}$$

Now do the angular part of the $\boldsymbol{k}$ integration, taking the $\theta = 0$ along the direction of $\boldsymbol{r}$. The ϕ integral is easy and we get

$$G(t, \boldsymbol{r}) = \frac{c}{(2\pi)^2}\int \mathrm{d}k\mathrm{d}\theta \ k^2\sin\theta\frac{\sin kct}{k}\exp(ikr\cos\theta) \ . \tag{41}$$

The θ integral is next, leaving

$$G(t, \boldsymbol{r}) = \frac{c}{2\pi^2 r} \int \mathrm{d}k \ \sin \mathrm{kct} \sin \mathrm{kr} \ . \tag{42}$$

The final integral over k yields

$$G(\boldsymbol{r}, t > 0) = \frac{c}{4\pi r} \delta(r - ct) \ . \tag{43}$$

This last step is rather subtle... to see it, write

$$\int_0^\infty \mathrm{d}k \sin kr \sin kct = -\frac{1}{8} \int_{-\infty}^\infty \left(e^{ikr} - e^{-ikr}\right)\left(e^{ikct} - e^{-ikct}\right) \ . \tag{44}$$

We now use the golden rule

$$\int_{-\infty}^\infty \mathrm{d}k \ e^{ikr} = 2\pi\delta(r) \ , \tag{45}$$

which generates 4 terms, but the $\delta(r + ct)$ ones are killed by the causal Heaviside function. This explains the final factor of 2.

Q6. I'll do this later... (perhaps).