NATURAL SCIENCES TRIPOS Part II

Wednesday 19 January 2000 10.30am to 12.30pm

THEORETICAL PHYSICS I

Answer **three** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains* **4** *sides and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

1 Uniform rod of length $2a$ and mass m is suspended at one end and swings freely in the vertical plane. Show that the kinetic energy of the rod is

$$T = \frac{2}{3} m a^2 \dot{\theta}^2$$

where θ is the angle the rod makes with the vertical. [5]

A second identical rod is suspended from the end of the first one and makes an angle ϕ with the vertical. For such a coupled system swinging in the vertical plane, write down the Lagrangian and obtain the corresponding equations of motion. [15]

Show that the approximate equations of motion for small oscillations ($|\theta|, |\phi| \ll 1$) are

$$\begin{aligned} 4\ddot{\phi} + 6\ddot{\theta} &= -3(g/a)\phi \\ 6\ddot{\phi} + 16\ddot{\theta} &= -9(g/a)\theta \end{aligned}$$

[7]

Thus show that there are two normal modes of oscillations with frequencies

$$\omega^2 = 3\frac{g}{a}\left\{\frac{1}{2} \pm \frac{1}{\sqrt{7}}\right\}.$$

[6]

2 The relativistic Lagrangian for a particle of rest mass m_0 moving in 1 dimension with the spatial coordinate $q(t)$ is

$$L(q, \dot{q}) = -\frac{m_0 c^2}{\gamma} - V, \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - \dot{q}^2/c^2}}$$

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and $V = V(q)$ is the potential energy.

Find the associated canonical momentum p. Show explicitly that Lagrange's equations imply that the total energy $E = p\,\dot{q} - L$ is a constant of the motion. [5] [7]

Consider the case $V = \frac{1}{2}k\,q^2$ and assume the motion is oscillatory between the limits $-b \le q \le b$. Show that the period of the oscillations is

$$\tau = \frac{4}{c}\int_0^b \frac{dq}{\sqrt{1 - \frac{m_0^2 c^4}{(E-\frac{1}{2}k\,q^2)^2}}}.$$

[7]

Show that the conserved total energy can be written in the form

$$\frac{E - \frac{1}{2}k\,q^2}{m_0 c^2} = 1 + \alpha(b^2 - q^2) \qquad \text{with} \quad \alpha = \frac{k}{2m_0 c^2}.$$

[7]

Hence show that the period

$$\tau \sim \frac{2\pi}{c}\frac{1}{\sqrt{\alpha}}\left[1 - \tfrac{3}{8}\alpha b^2\right] + \mathcal{O}(\alpha b^2)^3.$$

How does this result compare with the non-relativistic simple harmonic oscillator? [7]

3 A dynamical system is described by generalised coordinates $\{q_1, ...q_n\}$ and canonical momenta $\{p_1, ...p_n\}$. Write down the Hamilton equations for the motion of the system. What is the physical significance of the Hamiltonian $\mathcal{H}$? [5]

State and prove the Liouville's Theorem for the evolution of an ensemble of such systems. [10]

Show that a beam of light travelling in the $x - z$ plane forms a system to which the Liouville's Theorem applies, when for each ray:

$$q = \text{displacement from } z \text{ axis} \;\text{ and }\; p = n\,\sin\theta,$$

where n is the corresponding refractive index and θ is the angle of the ray with respect to the z axis. [You may assume Fermat's principle of least time for the propagation of the rays and then treat the z coordinate as if it were the time in the Lagrangian formalism.] [10]

Consider a light beam which has its phase space bounded by the ellipse

$$\frac{q^2}{a^2} + \frac{p^2}{b^2} = 1 \qquad \text{at} \;\; z = 0.$$

Find an equation for the phase space boundary after the beam has travelled a distance ℓ along the z axis and verify that the phase area is converved. (You may assume $n = 1$ and that θ is small). [8]

[For an ellipse described by a general quadratic form $\alpha x^2 + 2\beta xy + \gamma y^2 = \epsilon$ with a constraint $\alpha\gamma - \beta^2 = 1$, the area is equal to $\pi\epsilon$]

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4 The Noether's Theorem states that if the Lagrangian density $\mathcal{L}(\psi, \dot{\psi})$ (and not just its action $S = \int \mathcal{L}\, d\boldsymbol{r} dt$) is invariant under a phase transformation $\psi \to \psi_0\, e^{i\alpha}$, where α is any real constant, as well as S being stationary, then

$$\frac{\partial}{\partial t}\left(\psi\, \frac{\partial \mathcal{L}}{\partial[\dot{\psi}]} - \psi^*\, \frac{\partial \mathcal{L}}{\partial[\dot{\psi}^*]}\right) + \nabla \cdot \left(\psi\, \frac{\partial \mathcal{L}}{\partial[\nabla\psi]} - \psi^*\, \frac{\partial \mathcal{L}}{\partial[\nabla\psi^*]}\right) = 0.$$

Prove the theorem by starting with the invariance condition $d\mathcal{L}/d\alpha = 0$ and expanding it in terms of the partial derivatives $\partial\mathcal{L}/\partial\psi$, $\partial\mathcal{L}/\partial[\dot{\psi}]$, $\partial\mathcal{L}/\partial[\nabla\psi]$, etc. [10]
Then use the corresponding Euler-Lagrange equations for ψ and ψ^* and the expression for $\partial\psi(\alpha)/\partial\alpha$. [10]

If the Noether's condition is regarded as an equation of conservation of canonical momentum density for $\pi(\boldsymbol{r}, t) = \psi\, \partial\mathcal{L}/\partial\dot{\psi} - \psi^*\, \partial\mathcal{L}/\partial\dot{\psi}^*$, then the corresponding current is $j(\boldsymbol{r}, t) = \psi\, \partial\mathcal{L}/\partial[\nabla\psi] - \psi^*\, \partial\mathcal{L}/\partial[\nabla\psi^*]$. Find the momentum density and the current for the Lagrangian density of a free quantum particle [13]

$$\mathcal{L} = i\hbar\psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\nabla\psi \cdot \nabla\psi^*.$$

5 In the Lorentz gauge of electromagnetism, where the potentials are chosen to obey the condition $\partial^\mu A_\mu = 0$. Show that this corresponds, in a 3-dimensional frame, to

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0.$$

[5]

In the general case of moving charges, the scalar potential φ is given in terms of charge density ρ by the equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{1}{\varepsilon_0}\rho(\boldsymbol{r}, t).$$

Show how the problem of finding $\varphi(\boldsymbol{r}, t)$ can be reduced to the Green's function formalism and derive the partial differential equation for the linear response [7]
function $G(\boldsymbol{r} - \boldsymbol{r}', t - t')$. Discuss the necessary 3-d spherical symmetry of $G(\boldsymbol{r}, t)$. [3]

Find $G(\boldsymbol{r}, t)$ through the formal Fourier transform solution and integration on the complex plane. It is easiest to regard the wave-vector $\boldsymbol{k}$ as strictly real and [8]
focus on the ω integration. Illustrate the pole-moving technique in terms of introducing a small "dissipation" $\epsilon\, \partial\varphi/\partial t$ and thus obtain the causal $G(\boldsymbol{k}, t) = -\frac{c}{|k|}\sin(|k|ct)$.

Show that in the causal domain, $t > t'$, the Green's function is [10]

$$G(\boldsymbol{r}, t > 0) = \frac{c}{4\pi|\boldsymbol{r}|}\delta(|\boldsymbol{r}| - ct)$$

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and the corresponding "retarded" potential

$$\varphi(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}'.$$

6 For a 1-dimensional Brownian motion, outline the derivation of the generalised transition probability $G(x_b,t_b;\,x_a,t_a)$ in terms of a path integral [15]

$$G = \int_{(ab)} \mathcal{D}[x(t)]\,\exp\left[-\frac{1}{2D}\int_{t_a}^{t_b}\dot{x}^2 dt\right]$$

and illustrate the role of thermal fluctuations by using the Fluctuation-dissipation relationship for the diffusion constant. [7]

Discuss the relation between the Lagrangian dynamics and the Statistical mechanics, in terms of a most probable path for the system evolution and deviations from it driven by thermal noise. [11]