Correlated Phases of Atomic Bose Gases on a Rotating Lattice: Composite Fermion Theory for Bosonic Atoms

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GM & N. R. Cooper, arXiv:0904.3097

ICAM-I2CAM Institute for Complex Adaptive Matter

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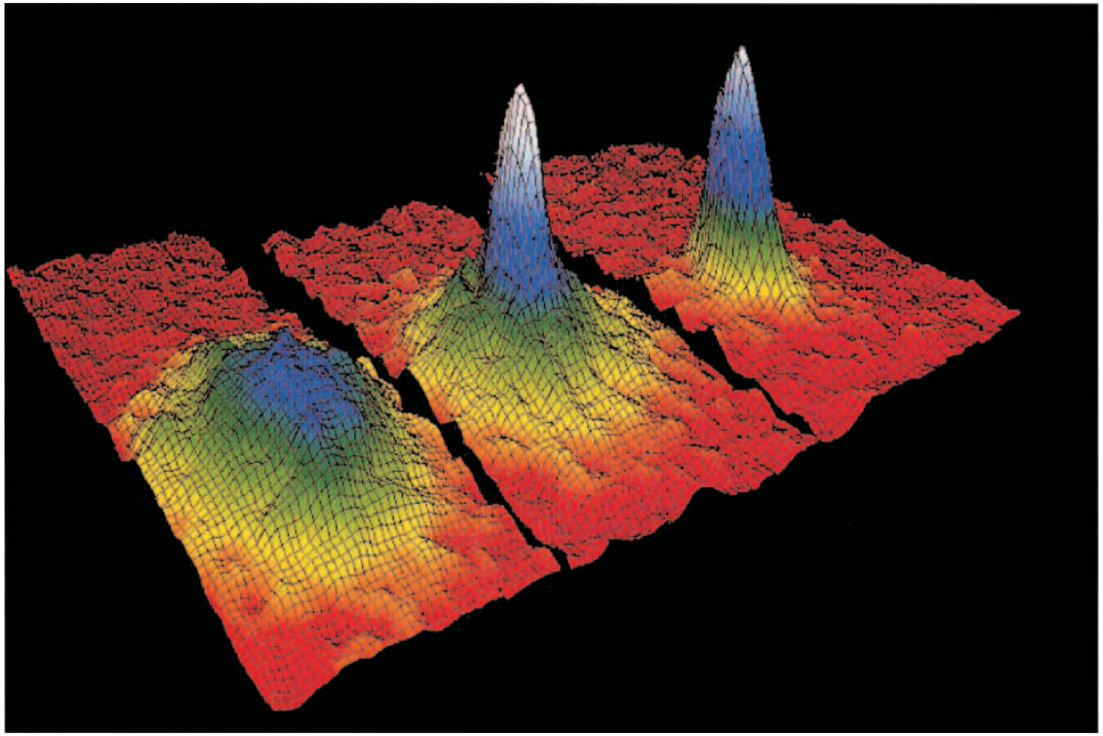
Overview

- Strongly Correlated Phases of Ultracold Atomic Bose Gases
- Atomic Bose Gases on a “Rotating Lattice”
- Strongly Correlated Phases: Numerical Evidence
- Summary

Atomic Bose Einstein Condensates



[Anderson *et. al.* [JILA], Science **269**, 198 (1995).]

s-wave scattering length $a_{\mathrm{s}} \simeq 5\text{nm} \ll \bar{a} \simeq 100\text{nm}$

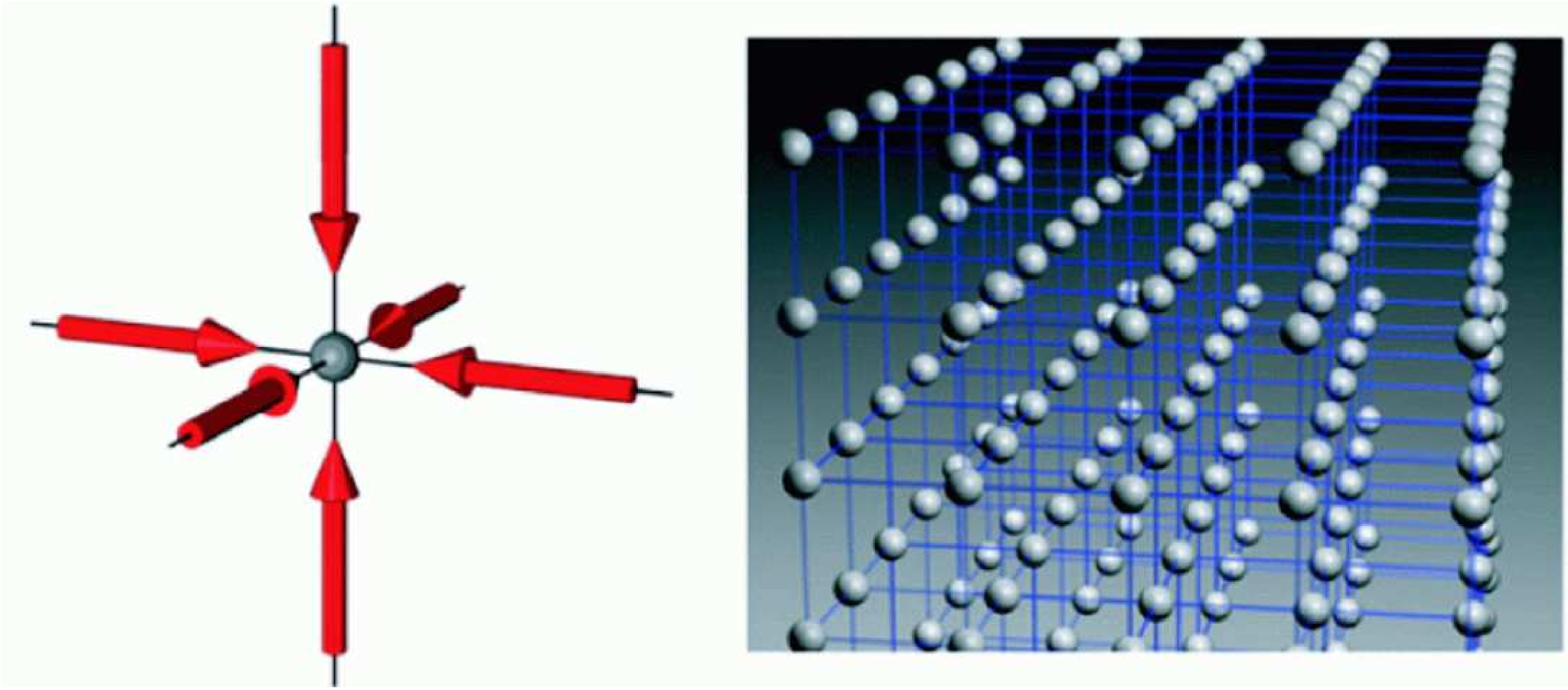
$\Rightarrow$ *Weakly* interacting

Groundstate wavefunction $$\Psi(\{\boldsymbol{r}_i\}) \simeq \prod_{i=1}^{N} \psi_c(\boldsymbol{r}_i)$$

Strongly Correlated Phases of Atomic Bose Gases

(1) Optical Lattice

[Bloch, Dalibard & Zwerger, RMP **80**, 885 (2008)]



Bose-Hubbard model

[Jaksch *et al.*, PRL **81**, 3108 (1998)]

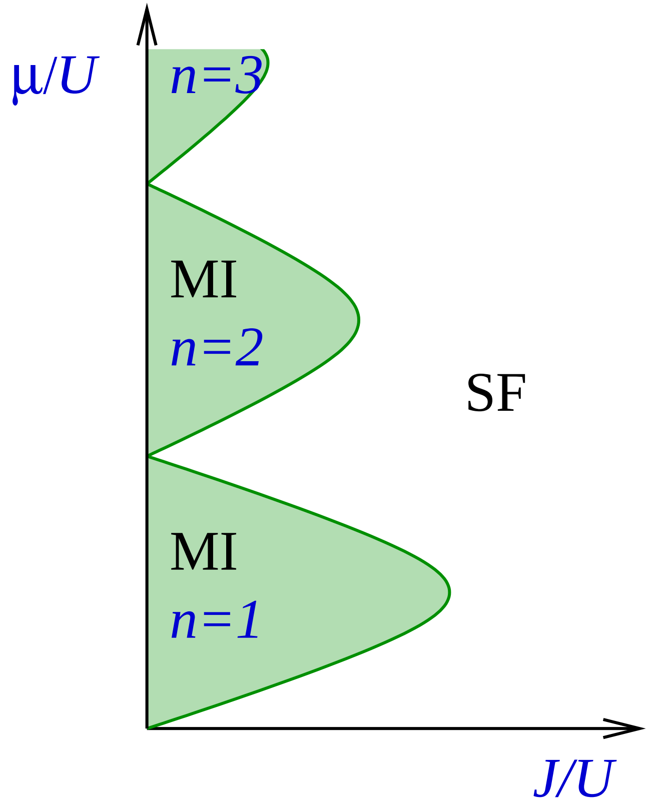
$$H = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}_\alpha^\dagger \hat{b}_\beta + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

Strongly correlated regime for $U/J \gg 1$
at particle density $n \sim 1$.

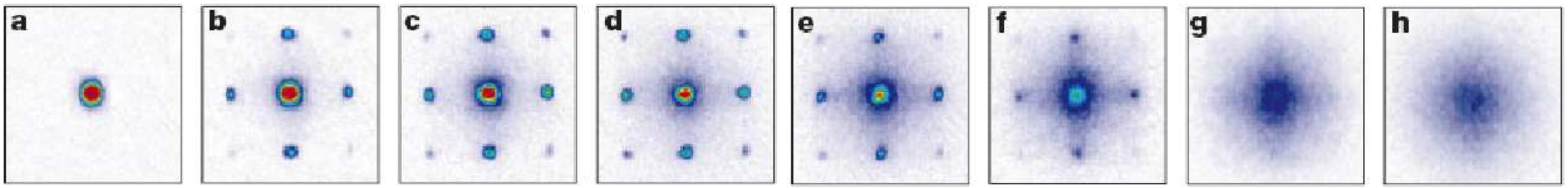
$T = 0$: competition between

- superfluid (BEC)
- Mott insulators, at $n = 1, 2, \ldots$

[Fisher *et al.*, PRB **40**, 546 (1989)]



Transition to Mott insulator observed in experiment [Greiner *et al.*, Nature **415**, 39 (2002)]



[a) no lattice; b) SF (weak lattice potential) — h) MI (strong lattice potential)]

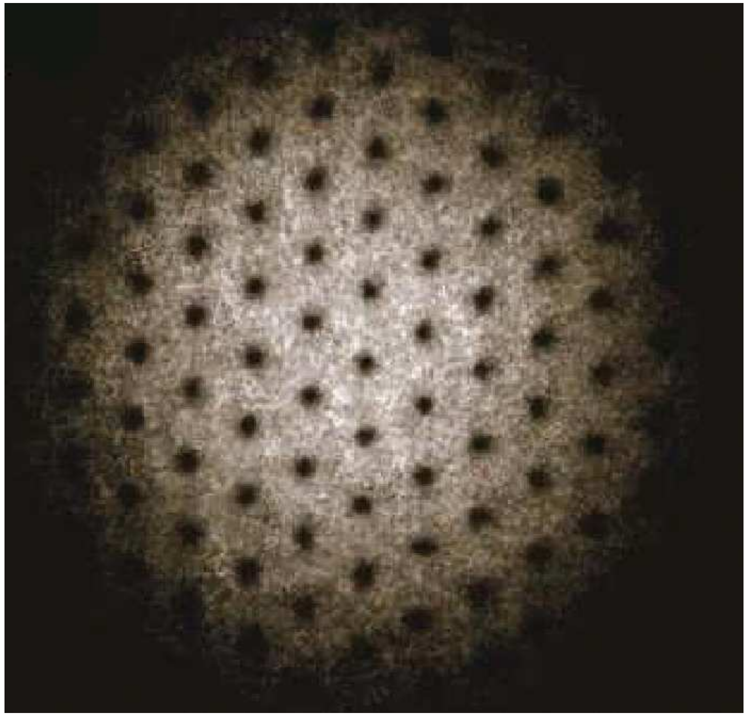
Strongly Correlated Phases of Atomic Bose Gases

(2) Rapid Rotation

Rotation frequency, Ω

Quantized vortices

Vortex density $n_{\mathrm{v}} = \frac{2M\Omega}{h}$



[Coddington *et al.* [JILA], PRA **70**, 063607 (2004)]

Harmonic confinement frequency $\omega_\perp$.

$\Omega \simeq \omega_\perp$: $\Rightarrow$quasi-2D Landau level spectrum

[Wilkin, Gunn & Smith, PRL **80**, 2265 (1998)]

Filling Factor $\nu \equiv \frac{n_{2\mathrm{d}}}{n_{\mathrm{v}}}$ [N. R. Cooper, Wilkin & Gunn, PRL **87**, 120405 (2001)]

Critical filling factor $\nu_{\mathrm{c}} \simeq 6$

- $\nu > \nu_{\mathrm{c}}$: Vortex Lattice (BEC)

- $\nu < \nu_{\mathrm{c}}$: *Bosonic* versions of fractional quantum Hall states:

Laughlin, hierarchy/CF, Moore-Read & Read-Rezayi phases, smectic +...?

[For a review, see: N. R. Cooper, Adv. Phys. **57**, 539 (2008)]

e.g. Laughlin state, $\nu = \frac{1}{2}$

$$\Psi_L(\{\boldsymbol{r}_i\}) \propto \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i|^2/4} \qquad \left[z \equiv \frac{(x+iy)}{\ell}; \ell \equiv \sqrt{\frac{1}{2\pi n_{\mathrm{v}}}} \right]$$

Atomic Bose Gases on a "Rotating Lattice"

- Rotating lattice [Tung, Schweikhard, Cornell (2006); Williams *et al.* (2008)]
- Tunneling phases [Jaksch & Zoller (2003); Mueller (2004); Sørensen, Demler & Lukin (2005)]

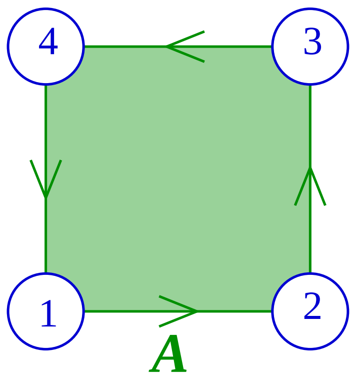
Bose-Hubbard model with "magnetic field" (2D square lattice)

$$H = -J \sum_{\langle \alpha,\beta \rangle} \left[\hat{b}^\dagger_\alpha \hat{b}_\beta e^{iA_{\alpha\beta}} + h.c. \right] + \frac{1}{2} U \sum_\alpha \hat{n}_\alpha (\hat{n}_\alpha - 1) - \mu \sum_\alpha \hat{n}_\alpha$$

Particle density, n

Interaction strength, U/J

Vortex density, n_v



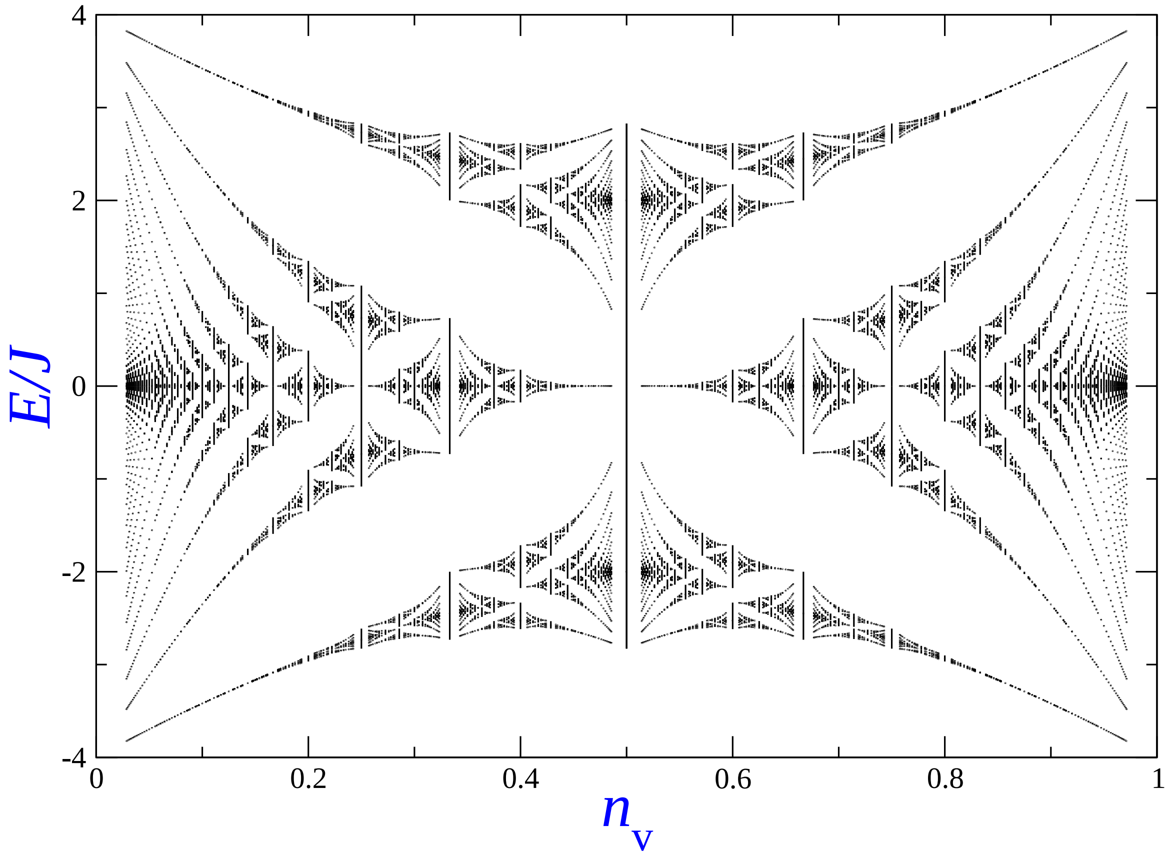
$\sum_{\text{plaquette}} A_{\alpha\beta} = 2\pi n_v$

$(0 \leq n_v < 1)$

What are the groundstates of bosons on a "rotating lattice"?

Single particle spectrum is the "Hofstadter butterfly"

[Harper, Proc. Phys. Soc. Lond. A **68**, 874 (1955); Hofstadter, PRB **14**, 2239 (1976)]



$n, n_v \ll 1 \Rightarrow$ continuum limit [Sørensen, Demler & Lukin, PRL (2005); Hafezi *et al.*, PRA (2007)]

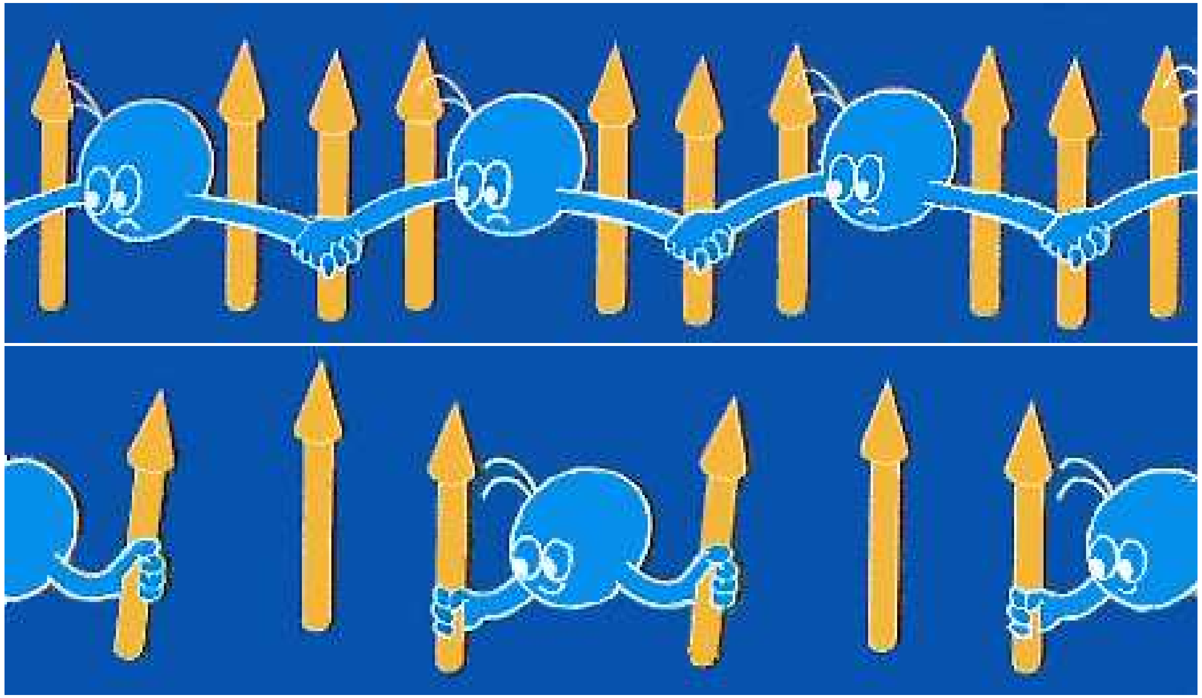
Are there new strongly correlated phases on the lattice for $n \sim n_v \sim 1$?

Hard-core limit $U \gg J \Rightarrow 0 \leq n_\alpha \leq 1$ [frustrated spin-1/2 quantum magnet]

Strongly Correlated States

Composite Fermions [Jain, Read, Girvin...]

Interacting electrons in magnetic field $\Rightarrow$non-interacting *composite fermions*.



[Illustration by Kwon Park]

Composite fermion = bound state of an electron with two flux quanta.

Rapidly rotating bosons in the continuum

Composite fermion = a bound state of a boson with *one vortex*.

[N. R. Cooper & Wilkin, PRB **80**, 16279 (1999)]

$$\Psi_{\mathrm{B}}(\{\boldsymbol{r}_i\}) \propto \mathcal{P}_{LLL} \prod_{i<j} (z_i - z_j)\, \psi_{\mathrm{CF}}(\{\boldsymbol{r}_i\})$$

$$n_{\mathrm{v}}^{\mathrm{CF}} = n_{\mathrm{v}} - n$$

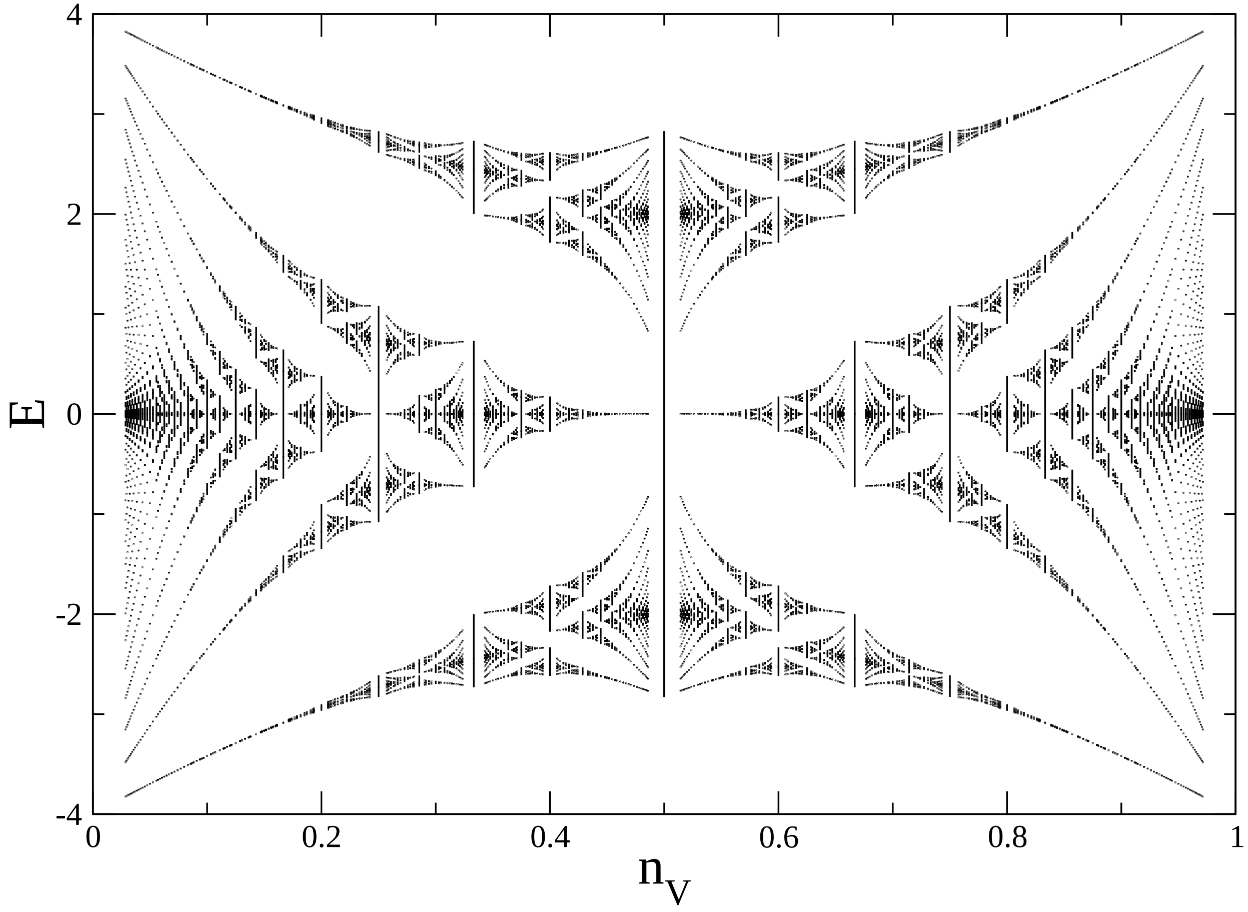
CFs fill p Landau levels for

$$\frac{n}{n_{\mathrm{v}}^{\mathrm{CF}}} = \pm p \qquad \Rightarrow \qquad \nu = \frac{n}{n_{\mathrm{v}}} = \frac{p}{p \pm 1}$$

⇒(trial) incompressible states of interacting bosons,
describe exact groundstates well for $\nu = 1/2, 2/3, (3/4)$

[Regnault & Jolicoeur, PRL **91**, 030402 (2003); . . .]

Lattice: CF spectrum is the "Hofstadter butterfly" [Kol & Read, PRB **48**, 8890 (1993)]



Filled band of CFs at $(n, n_{\rm v}^{\rm CF})$ $\Rightarrow$ trial incompressible state of bosons at $(n, n_{\rm v})$

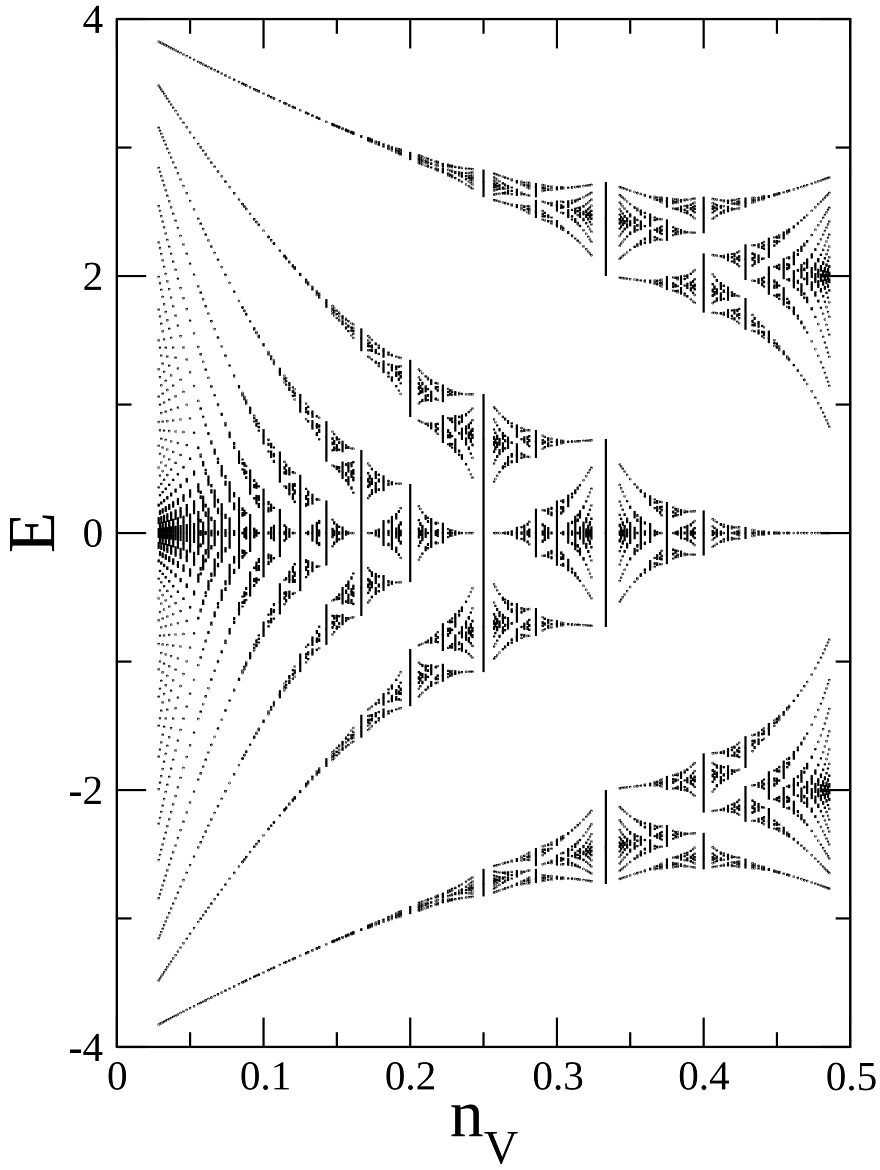
There can exist incompressible states with no counterpart in the continuum

Calculating fillings in the Hofstadter diagram

[GM & N. R. Cooper, arXiv:0904.3097]

Recursive structure:
[Hofstadter, PRB **14**, 2239 (1976)]

- Series of subcells which resemble unit-cell via rectantularization



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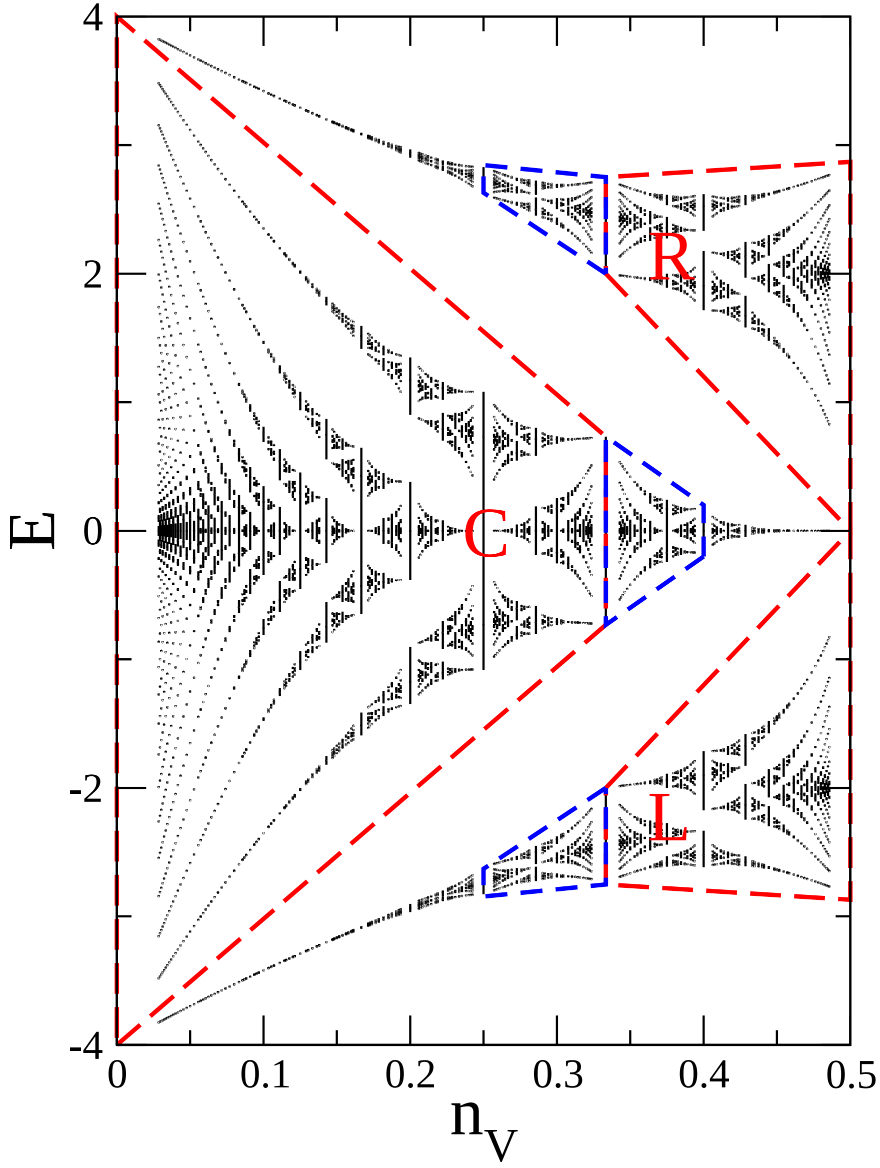
- Three 'trains' of subcells, named **L**eft, **R**ight and **C**enter.

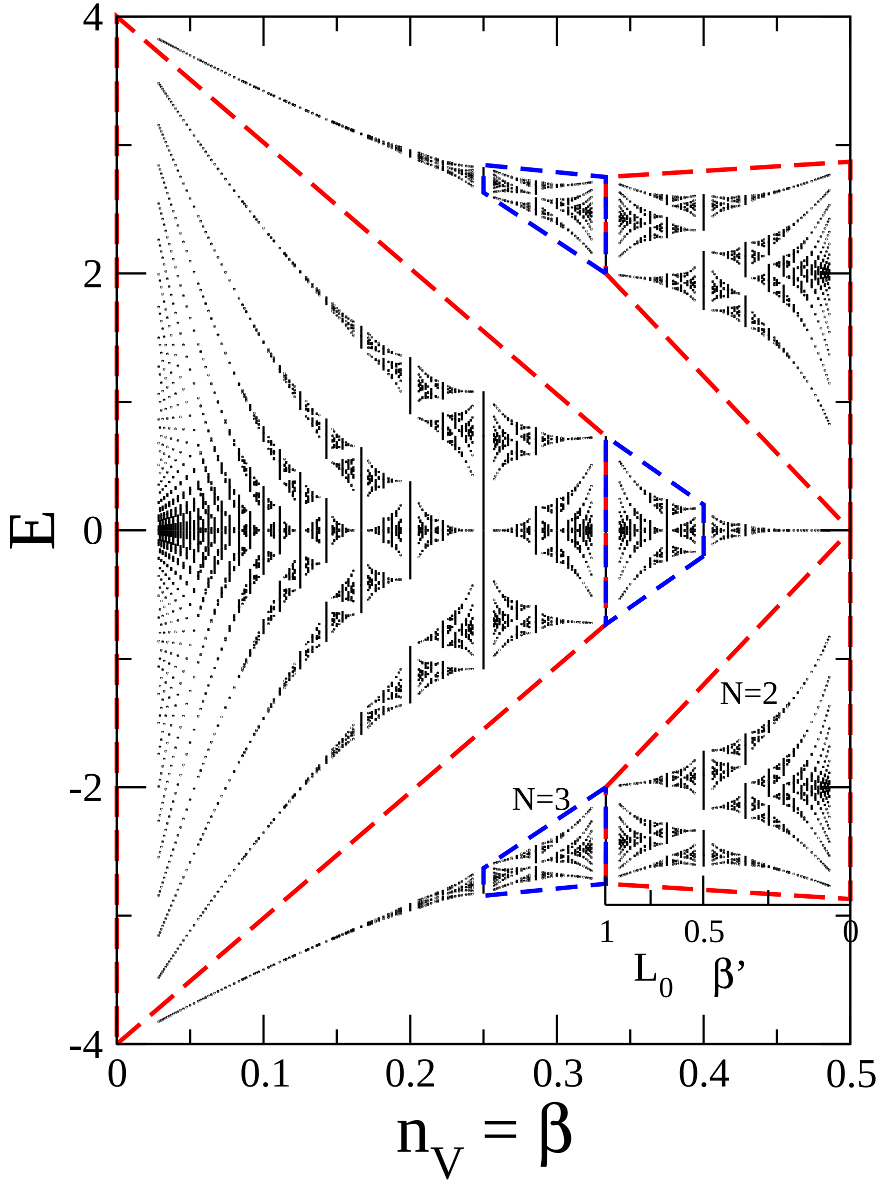


Calculating fillings in the Hofstadter diagram

[GM & N. R. Cooper, arXiv:0904.3097]

Recursive structure:
[Hofstadter, PRB **14**, 2239 (1976)]

- Series of subcells which resemble unit-cell via rectantularization
- Three 'trains' of subcells, named **L**eft, **R**ight and **C**enter.
- Consecutive subcells in each train characterized by $N = \lfloor \beta^{-1} \rfloor$
- subcell variable β', defined by $\beta = [N + \beta']^{-1}$
- denominator t of cell variable $\beta^{(n)} = r/t$ indicates number of bands in cell



$\Rightarrow$Obtain filling $n = \frac{\#\text{bands filled}}{\#\text{bands total}}$ by counting bands

I) Counting bands in unit cell [Hofstadter, PRB **14**, 2239 (1976), GM & N. R. Cooper, arXiv:0904.3097]

- Total number of bands q follows from

$$\beta = n_v = \frac{p}{q}.$$

- Bands up to the first gap of the unit-cell

⇒ *All* bands of the subcell in the L-train.

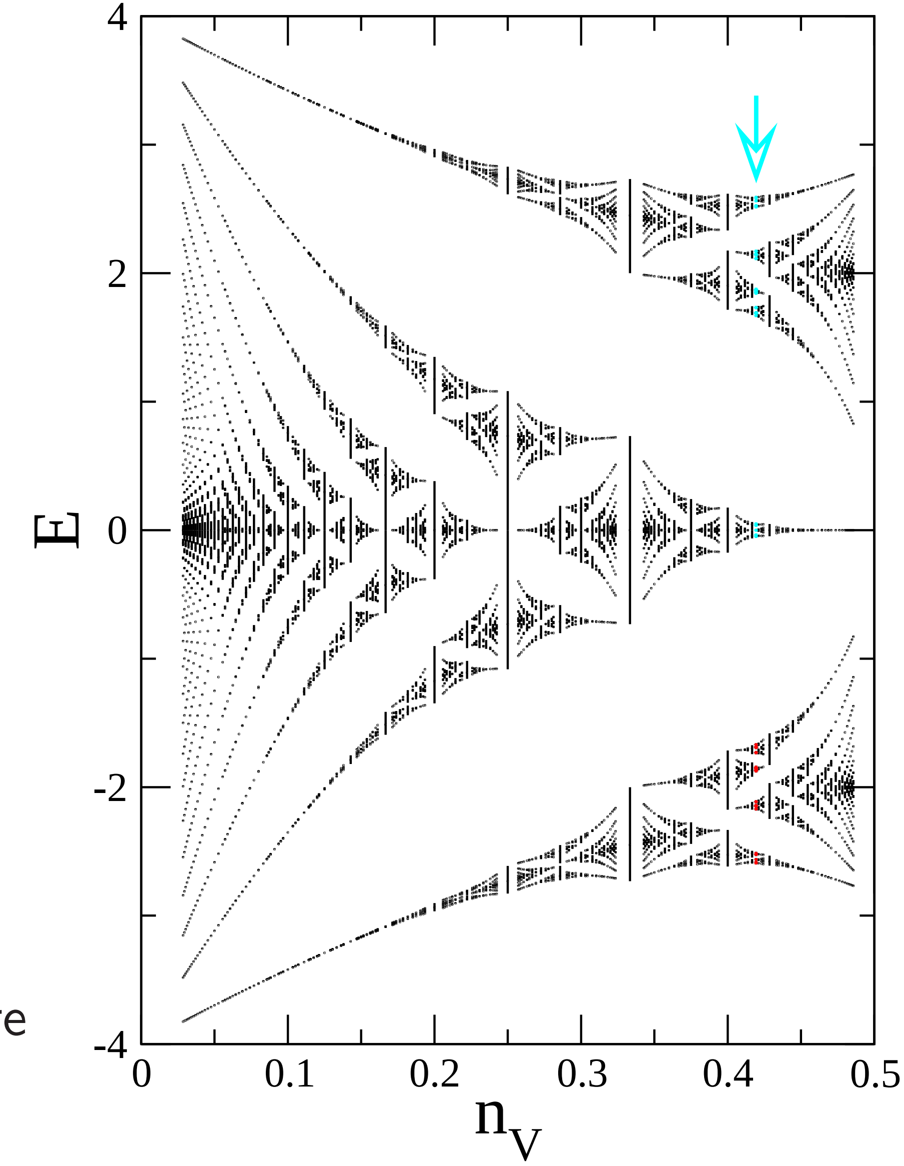
- Evaluate local variable of subcell:

$$\beta' = \beta^{-1} - \lfloor \beta^{-1} \rfloor = \frac{q - Np}{p}$$

- The subcell contains p bands

⇒ Therefore, $n = \frac{p}{q} = \beta = n_v$

- Result in line with continuum limit, where degeneracy of lowest Landau level is equal to flux the density: $n = n_v$



II) Counting bands in a subcell

[Hofstadter, PRB **14**, 2239 (1976), GM & N. R. Cooper, arXiv:0904.3097]

- Know local variable of subcell:

$$\beta' = \beta^{-1} - \lfloor \beta^{-1} \rfloor = \frac{q - Np}{p}$$

- Also need number of bands in sub-subcell:

$$\beta'' = \beta'^{-1} - \lfloor \beta'^{-1} \rfloor = \frac{p - (q - Np)p}{q - Np}$$

⇒Evaluate the filling:

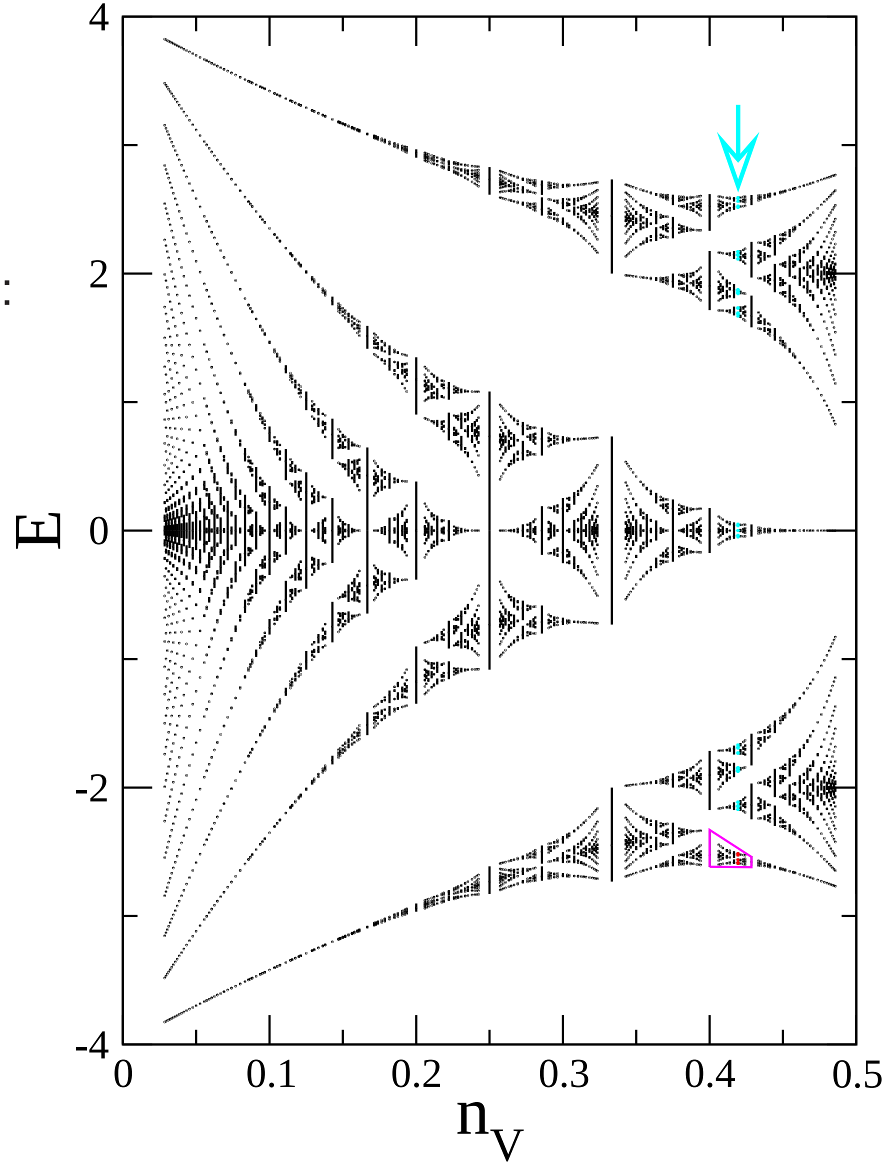
$$n = \frac{q - Np}{q} = 1 - N\beta$$

- Again, n depends linearly on β

Therefore, by induction:

$0)$ $n \propto n_v$ in unit-cell, and

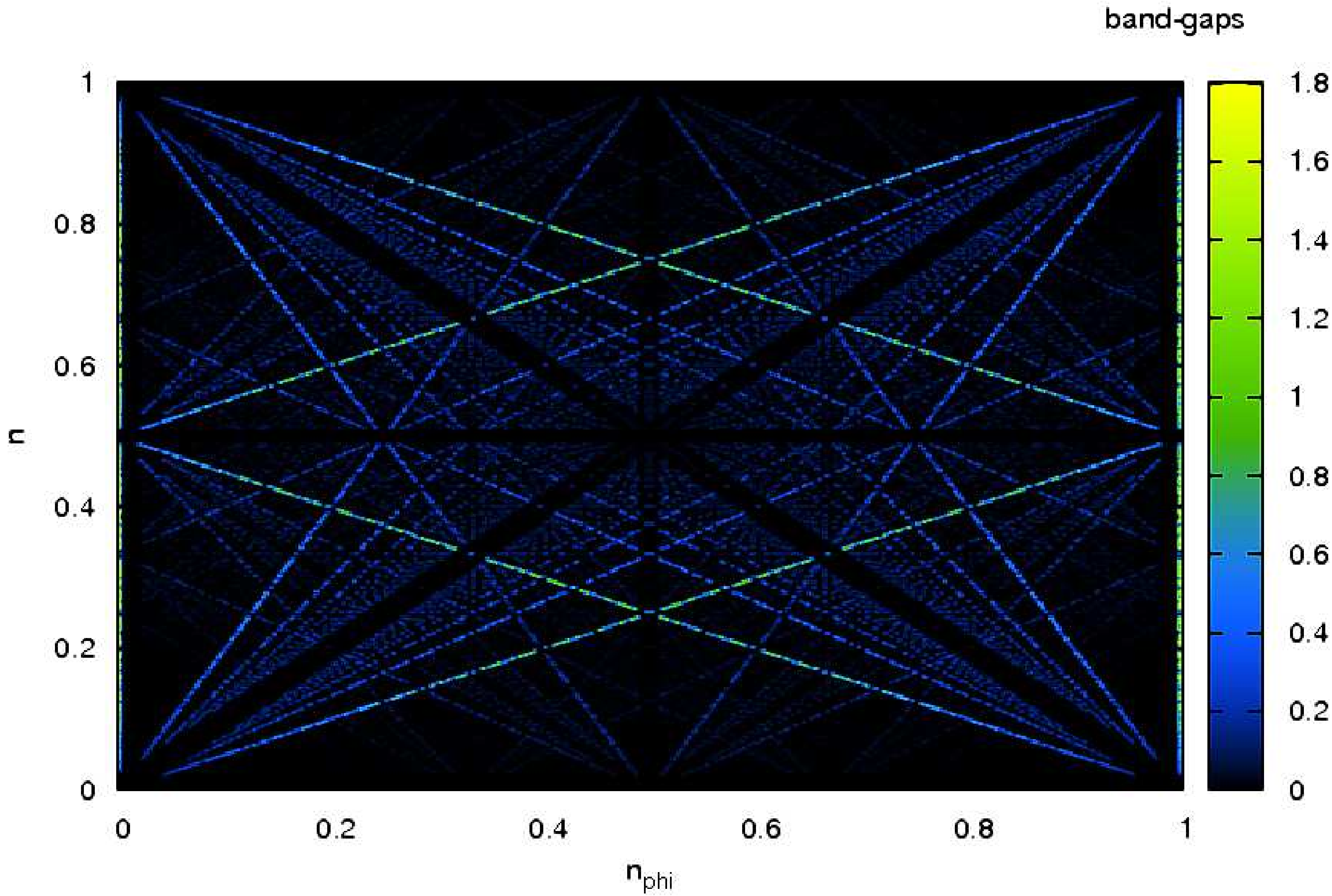
$+1)$ In a subcell, n depends linearly on the local variable of the mother-cell



⇒For all gaps of the Hofstadter spectrum, $n = \alpha n_v + \delta$ linear in n_v.

Gaps for non-interacting CFs

[GM & N. R. Cooper, arXiv:0904.3097]



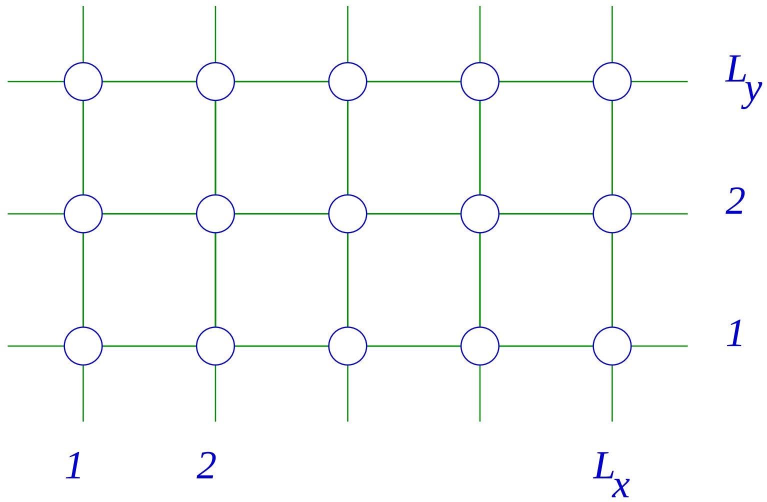
Do these new phases describe the exact groundstates?

Numerical Methods

- Exact Diagonalization

$L_x \times L_y$ square lattice, with periodic boundary conditions (torus).

$$N = nL_xL_y$$
$$N_{\mathrm{v}} = n_{\mathrm{v}}L_xL_y$$



- Low-energy spectrum (Lanczos) for hard-core interactions $U \gg J$.

- Limited by rapidly growing Hilbert-spaces, $N \leq 6$.

- Expect strong finite size effects.

Composite Fermion Wavefunction

Continuum

$$\Psi_{\rm B}(\{\boldsymbol{r}_i\}) \propto \mathcal{P}_{LLL} \underbrace{\prod_{i<j}(z_i - z_j)}\ \psi_{\rm CF}(\{\boldsymbol{r}_i\})$$

Slater det. of lowest Landau level wavefunctions:
$\nu = 1$ state of fermions.

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Slater det. of lowest Landau level wavefunctions:
$\nu = 1$ state of fermions.

Lattice

[GM & N. R. Cooper, arXiv:0904.3097]

$$\Psi_{\mathrm{B}}(\{\boldsymbol{r}_i\}) \propto \underbrace{\psi_{\mathrm{J}}^{(\phi_x,\phi_y)}(\{\boldsymbol{r}_i\})}\ \psi_{\mathrm{CF}}^{(-\phi_x,-\phi_y)}(\{\boldsymbol{r}_i\})$$

$\nu = 1$ state of fermions.

- Hard-core bosons.
- Generalized periodic boundary conditions: phases (ϕ_x, ϕ_y).

  - Recovers the two $\nu = 1/2$ Laughlin wavefunctions in continuum limit.

[Haldane & Rezayi, PRB **31**, 2529 (1985)]

Continuum CF States, $\nu \equiv \frac{n}{n_\text{v}} = \frac{p}{p\pm 1}$

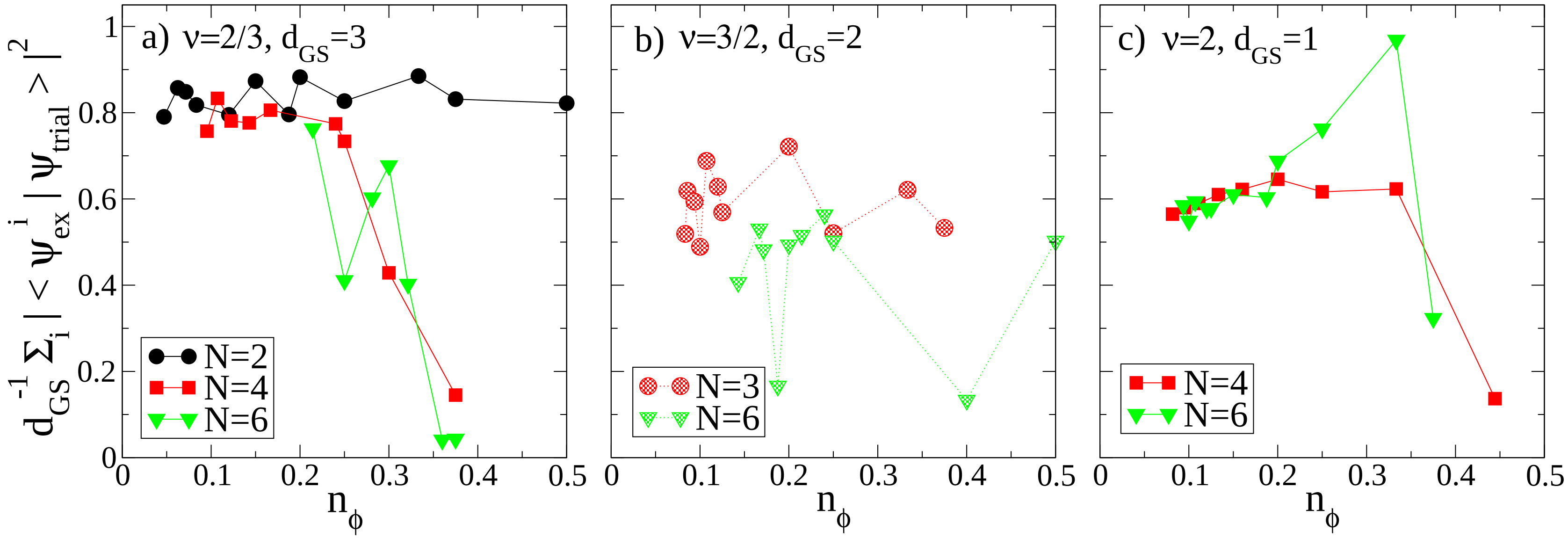
Laughlin State $\nu = 1/2$

[Sørensen, Demler & Lukin, PRL (2005); Hafezi *et al.*, PRA (2007)]

Describes the groundstate on the lattice up to $n_\text{v} \simeq 0.4$.

CF States $\nu = 2/3, 3/2, 2$

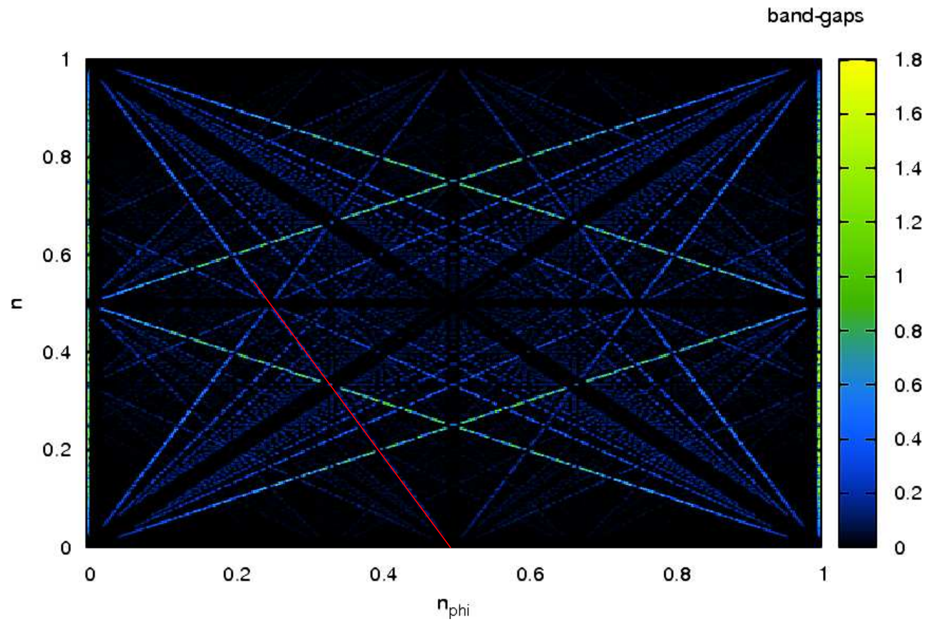
[GM & N. R. Cooper, arXiv:0904.3097]



- CF state at $\nu = 2/3$ applies also for *hard-core* interactions
- Competition with Read-Rezayi phases at $\nu = 3/2, 2$?

CF states stabilized by the lattice

Evidence for strongly correlated states at a series of these new cases.



Evidence for $n_{\mathrm{v}} = \frac{1}{2}(1 - n)$:

Groundstate is consistent with the CF state for $n \lesssim 1/6$

Overlap with trial CF state

n	N	$N_{\rm v}$	L_x	L_y	$\lvert\langle\Psi^{\rm CF}_{\rm trial}\lvert g.s.\rangle\rvert^2$	Hilbert spc dim.
1/7	2	6	2	7	0.437	91
1/7	3	9	3	7	0.745	1330
1/7	4	12	4	7	0 [0.275]	20.5k
1/7	5	15	5	7	0.563	324k
1/7	6	18	6	7	0.328	5.2M
1/9	2	8	2	9	0.360	153
1/9	3	12	3	9	0.841	2925
1/9	4	16	4	9	0 [0.152]	58.9k
1/9	4	16	6	6	0.306	58.9k
1/9	5	20	5	9	0.459	1.2M

- Sizeable overlap with CF state (no free parameters!)
- Correct groundstate degeneracy on the torus (1).
- Correct Chern number (2), tested for $N \leq 5$.

Evidence for wider applicability of CF ansatz.

Calculation of Chern Numbers

[Hatsugai, JPSJ **74**, 1374 (2005), Hafezi *et al.*, PRA **76** (2007)]

- Definition of Chern numbers:

$$\mathcal{C}_n = \frac{1}{2\pi i}\int_{T^2} d^2p[\partial_1 A_2(p) - \partial_2 A_1(p)] \equiv \frac{1}{2\pi i}\int_{T^2} d^2p \mathcal{F}_{12},$$

where A_μ is the Berry connection $A_\mu = \langle \Psi_n(p)|\partial_\mu|\Psi_n(p)\rangle$, and p a set of two periodic quantum numbers.

For $p = k$, one obtains the Hall voltage as $\sigma_{xy} = -\frac{e^2}{h}\sum_n \mathcal{C}_n$.

To calculate $\mathcal{C}$, we note the following:

- Integral over closed surface, can only be non-zero if A_μ becomes singular.
- Field strength is gauge invariant, can make use of gauge transformations

$$A_\mu(p) \rightarrow A_\mu - \partial_\mu \chi(p)$$
$$\Psi_n(p) \rightarrow e^{i\chi(p)}\Psi_n(p)$$

to define multiple patches where the vector potential is regular.

Can find gauge transform such that the transformed vector potential A'_μ becomes regular at singularities $\mathcal{S}_i$ in A_μ, i.e. the singular part was absorbed by the gauge $\partial_\mu \chi(p) = A_\mu(p) - A'_\mu$ and thus,

$$\mathcal{C}_n = \frac{1}{2\pi} \sum_i \oint_{\partial \mathcal{S}_i} \nabla \chi \cdot dp.$$

Generically, singularities will be at different locations in different gauges.
⇒Take two reference states Φ, Φ' to define two gauge choices such that Ψ_n has a real projection onto the reference states:

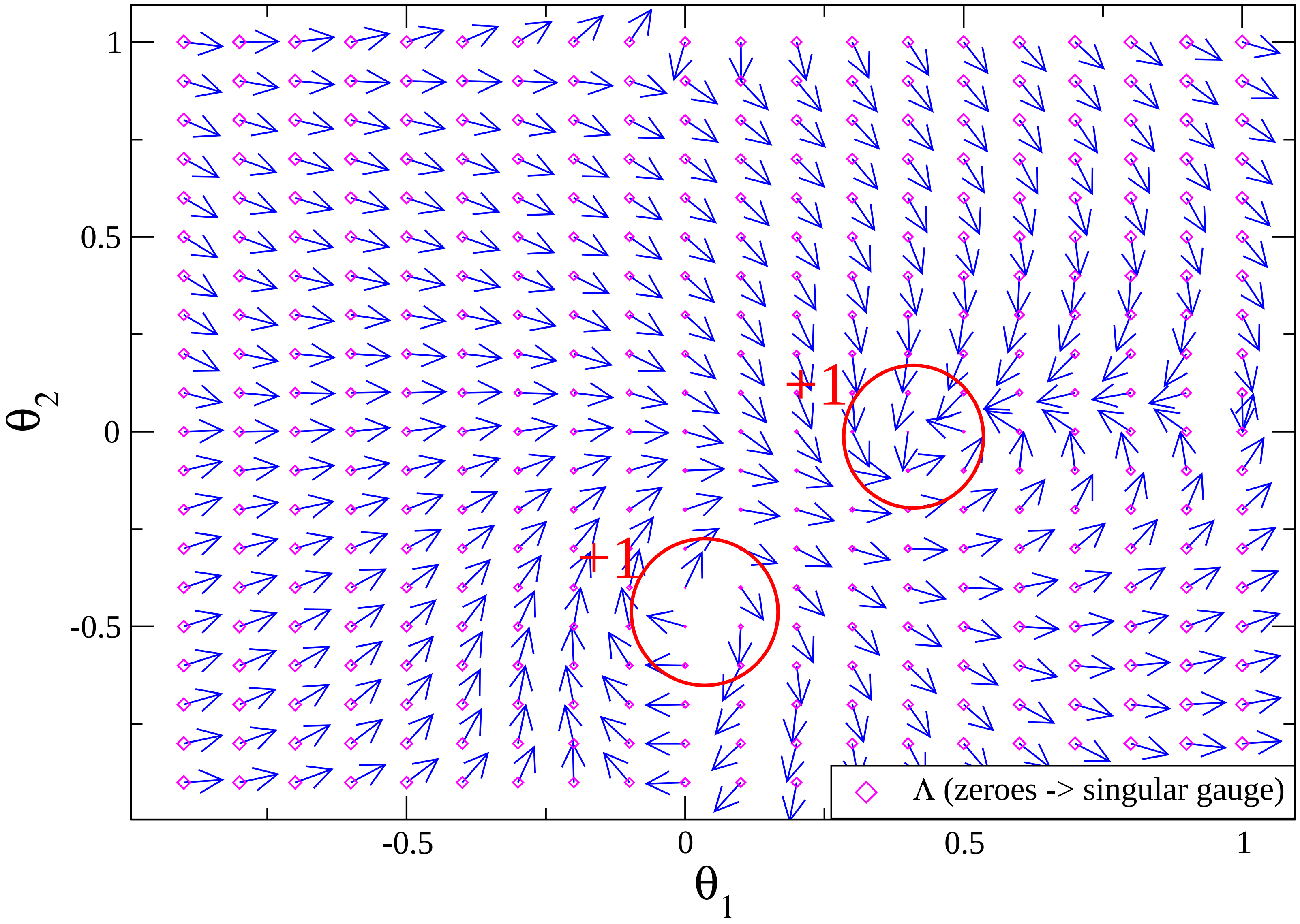
$$\Psi_\Phi = \Psi \left(\Psi^\dagger \Phi \right), \quad \Psi_{\Phi'} = \Psi \left(\Psi^\dagger \Phi' \right)$$

One can read off the gauge that transforms between Φ and Φ'

$$\Psi_\Phi = (\Phi^\dagger \Psi)(\Psi^\dagger \Phi') \Psi_{\Phi'} \equiv e^{i\chi} \Psi_{\Phi'}.$$

This gauge becomes singular wherever $\Lambda = |\Phi^\dagger \Psi|^2 = 0$.
⇒The integral above can be evaluated graphically!

Graphical evaluation of Chern Numbers



⇒Chern number of $\mathcal{C} = 2$, shown here for $n = 1/9$, $n_\phi = 4/9$ on 6×6 sites.

Summary

- Ultracold atomic Bose gases on a rotating lattice offer the possibility to explore novel aspects of the FQHE: the FQHE of bosons; the interplay of the FQHE and lattice periodicity.

- A generalized composite fermion construction leads to the prediction of strongly correlated incompressible phases of bosons at certain $(n, n_{\rm v})$, including states which are stabilized by the lattice.

- We find numerical evidence for uncondensed incompressible fluids for several of the predicted cases. This shows a wider applicability of the CF construction than its continuum formulation.

- There are many other cases $(n, n_{\rm v})$ to understand: CF states compete with other possible phases: condensed states / vortex lattice states, striped/smectic states, etc.