# Tale 9
# on Hidden Symmetry of Kepler's Problem and Accidental Degeneracy in Hydrogen Atom

## Classical Mechanics. Orbital Motion

Kepler's problem has the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} \tag{1}$$

It is known since Kepler that the orbits of the planets in the gravitational field of Sun are closed curves - the ellipses. Already this is quite an unusual thing. Indeed, two coordinates $x$ and $y$ and two components of momentum $p_x$ and $p_y$ form a four-dimensional phase space. The energy and angular momentum conservation laws $H(p, r) = E$ and $xp_y - yp_x = L$ provide two constraints, which reduce the number of independent degrees of freedom to two. Therefore, the invariant manifold is a two-dimensional surface in the four-dimensional phase space. One can see that this surface has no edges. A trajectory $\{\mathbf{r}(t), \mathbf{p}(t)\}$ corresponds to a one-parameter line which belongs to this surface. Under some initial conditions this line in the phase space may be closed, which means that periods of evolutions in $x$ and $y$ are equal. This would also mean that the orbit in the real space is a closed line as well. But this cannot not be a general case. Quite the contrary, general case corresponds to a line, winding around on invariant surface. This corresponds to a rosette in the real space. That very fact that the orbits in the Kepler's problem are closed independently from initial conditions may mean that there are more constraints which reduce the number of independent degrees of freedom to one. Show where this additional constraint comes from.

## Fock's sphere

Energy conservations gives for the negative energies:

$$H(\mathbf{p}, \mathbf{r}) = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} = E = -\frac{p_0^2}{2m}$$

which allows to express the radius $r$ through the momentum $\mathbf{p}$:

$$r = \frac{2m\alpha}{p_0^2 + \mathbf{p}^2}. \qquad (2)$$

Infinitesimal action

$$dS = -\mathbf{r}d\mathbf{p}$$

is minimal when $\mathbf{r}$ and $\dot{\mathbf{p}}$ are parallel (this also follows from equation of motion). Therefore,

$$dS^2 = (2m\alpha)^2 \frac{dp_x^2 + dp_y^2}{(p_0^2 + \mathbf{p}^2)^2}. \tag{3}$$

It is convenient at this stage to make a stereo-graphic projection:

$$p_x = p_0 \cos\phi \cot\frac{\theta}{2} \tag{4}$$

$$p_y = p_0 \sin\phi \cot\frac{\theta}{2} \tag{5}$$

which gives after substitution of (4) and (5) into Eq (3):

$$dS^2 = \left(\frac{m\alpha}{p_0}\right)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{6}$$

One can easy recognize a natural metric on 2D sphere in expression in the brackets in the right hand side of Eq (6). Principle of the less action reads, therefore, as a condition that all the trajectories of the Kepler's problem in the momentum space are the geodesic lines on the stereo-graphic sphere (4, 5). The geodesic lines on any sphere are great circles. Being projected to the plane $(p_x, p_y)$, these circles give the ellipses.
It follows from Eqs(2), (4) and (5) that

$$r = \frac{2m\alpha}{p_0^2} \sin^2\frac{\theta}{2}. \tag{7}$$

The equation of motions says that

$$\dot{\mathbf{p}} = -\ \alpha\ \frac{\mathbf{r}}{r^3} = -\frac{p_0^4}{4m^2\alpha}\ \frac{\mathbf{n}}{\sin^4\theta/2}, \qquad \mathbf{n} = \frac{\mathbf{r}}{r}; \tag{8}$$

$$\dot{\mathbf{p}}^2 = \frac{\alpha^2}{r^4} = \frac{p_0^8}{16m^4\alpha^2}\ \sin^{-8}\frac{\theta}{2}. \tag{9}$$

Using parameterizations (4), (5), calculating time derivatives of the $x$- and $y$-components of the momentum $\mathbf{p}$ and substituting them into Eq (9), we, finally, obtain:

$$\frac{1}{\sin\theta/2} = \left(\frac{2m^2\alpha}{p_0^3}\right)^{1/2}\ (\ \sin^2\theta\ \dot{\phi}^2 + \dot{\theta}^2\ )^{1/4}; \tag{10}$$

$$\dot{\mathbf{p}}^2 = \frac{m^4\alpha^2}{p_0^4}\ (\ \sin^2\theta\ \dot{\phi}^2 + \dot{\theta}^2\ )^2. \tag{11}$$

## Anti-Lagrangian and Conservation Laws

After the Fock sphere has been constructed, it easy to develop the relevant Lagrangian formalism. Since the Fock sphere has been constructed in the momentum space, it is convenient to introduce the "anti-Lagrangian" $\tilde{\mathcal{L}}(\mathbf{p}, \dot{\mathbf{p}})$:

$$\tilde{\mathcal{L}}(\mathbf{p}, \dot{\mathbf{p}}) = -\dot{\mathbf{p}}\mathbf{r} - \mathcal{H}(\mathbf{p}, \mathbf{r}), \qquad \dot{\mathbf{p}} = \frac{\partial \mathcal{H}}{\partial \mathbf{r}}. \tag{12}$$

The result of this substitution gives the anti-Lagrangian for the Kepler problem (1):

$$\tilde{\mathcal{L}} = -\ \frac{\mathbf{p}^2}{2m} + 2\ \alpha^{1/2}\ (\dot{\mathbf{p}}^2)^{1/4}. \tag{13}$$

Using the parameterizations (4), (5), we may express the anti-Lagrangian through the angles $\theta$ and $\phi$ on the stereo-graphic sphere and their time derivatives $\dot\theta$ and $\dot\phi$:

$$\tilde{\mathcal{L}} = -\,\frac{p_0^2}{2m}\ \cot^2\frac{\theta}{2} + \frac{2m\alpha}{p_0}\ \left[\ \sin^2\theta\ \dot\phi^2 + \dot\theta^2\ \right]^{1/2}. \qquad (14)$$

The anti-Lagrangian contains two parts: potential and kinetic energy. The former forms the first term in Eq (14), while the latter corresponds to the second term. The expression in square brackets is the square of the speed $v = (v_\phi^2 + v_\theta^2)^{1/2}$ of the point on the stereo-graphic sphere.
The anti-Lagrangian (14) does not depend on azimuthal angle $\phi$ ($\phi$ is a cyclic variable). Therefore, the canonically conjugated momentum $I_z = \partial\tilde{\mathcal{L}}/\partial\dot\phi$ is an integral of the motion:

$$I_z = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot\phi} = \frac{2m\alpha}{p_0}\ \frac{\sin^2\theta}{\sqrt{\ \sin^2\theta\ \dot\phi^2 + \dot\theta^2}}\ \dot\phi = \frac{2m\alpha}{p_0}\ \frac{v_\phi}{v}\ \sin\theta. \quad (15)$$

At pere-helium and affelium $v_\phi = v$ and, therefore,

$$\sin\theta_0 = \frac{p_0\ I_z}{2\ m\ \alpha}.$$

and Eq (15) reads

$$\sin\theta_0 = \frac{v_\phi}{v}\ \sin\theta. \qquad (16)$$

Eq (16) explicitely means that the trajectory on the stere-graphic sphere belongs to the plane which has the angle $\theta_0$ with the vertical axis. Thus, this trajectory is the line of the crossing of the mentioned plane with the sphere: the

great circle.
Using the inverse stereo-graphic projecton, we can prove that the conservation of the invarian $I_z$ corresponds in the real space to conservation of the angular momentum $L = xp_y - yp_x$.
It is also clear, that there are two more rotations of the stereo-graphic sphere (around axes $x$ and $y$):

$$\delta\phi = \cot\theta \ \cot\phi \ \delta\theta \tag{17}$$

$$\delta\phi = - \ \cot\theta \ \tan\phi \ \delta\theta \tag{18}$$

respectively. Both these two rotations map the sphere on itself and map any geodesic line on another geodesic line. Therefore, the anti-Lagrangian must get nothing after these rotations but a total time derivative (check this). This implies existence of two more integrals of the motion $I_x$ and $I_y$. Analogously to $I_z$ being proportional to the angular momentum $L$, these additional integrals of the moton $I_x$ and $I_y$ are proportianal to the compotents of the, so called, Runge-Lentz vectors:

$$\mathbf{A} = -\sqrt{m} \cdot \left( \frac{[\mathbf{p} \times \mathbf{L}]}{m} - \frac{\alpha \mathbf{r}}{r} \right) \tag{19}$$

## Quantum Mechanics. Accidental Degeneracy in 2D Hydrogen Atom

This section follows to ideas of the paper by Wolfgang Pauli (1926) in which he derived the Balmer serial law for spectra of the Hydrogen atom from the Matrix Quantum Mechanics. The difference with this original derivation, made deliberately, is that the space dimension is chosen to be equal to two.

First of all, it is necessary to generalize the expression for the Runge-Lentz vector to make it valid for non-commuting coordinate and momentum.

$$A_\beta = -\sqrt{m} \cdot \left( \frac{[pL]_\beta}{2m} - \frac{[Lp]_\beta}{2m} - \frac{\alpha\ r_\beta}{r} \right)$$

The Runge-Lentz vector for two-dimensional electron has two components: $A_x$ and $A_y$. Direct calculation gives that

$$A^2 = 2H \left( L^2 + \frac{1}{4} \right) + m\alpha^2 \tag{20}$$

and the commutation relations have the form

$$\begin{aligned} [A_x, A_y] &= -2iHL && (21) \\ [L, A_x] &= iA_y && (22) \\ [L, A_y] &= -iA_x && (23) \end{aligned}$$

In this subsection we discuss properties of the bound states, which means that the values of energy $E_n =< n|H|n >$ are supposed to be negative. Therefore, two hermitian operators

$$\hat{a}_{x,y} = \frac{A_{x,y}}{\sqrt{-2H}}$$

are well defined. As a result the commutation relations could be rewritten as

$$\begin{aligned} [a_x, a_y] &= iL && (24) \\ [L, a_x] &= ia_y && (25) \\ [L, a_y] &= -ia_x && (26) \end{aligned}$$

and the Eq(20) take the form

$$a^2 + L^2 = -\frac{1}{4} + \frac{m\alpha^2}{-2H} \tag{27}$$

After performing this routine job we are prepare to be surprised by its result. Indeed, we see that the commutation relations in the form of Eqs (24,25,26) forms the $su(2)$-algebra (or isomorphous algebra $so(3)$). This means that the values $a_{x,y}$ and $L$ form three components of the "spin" $\mathbf{j}$, which commutes with Hamiltonians $H$. Angular momentum $L$ is $z$ component of total momentum $j$. Since $L$ has only integer values, $j$ also has only integer values, which means that the dynamic symmetry of the Kepler's problem is rather $SO(3)$ than $SU(2)$. Therefore, all bound states of 2D electron in the Coulomb potential form multiplets with total integer "spin" $j$ . The values of energy $E_j$, corresponding to this "spins", may be found from Eq(27)

$$j(j+1) = \frac{m\alpha^2}{2|E_j|} - \frac{1}{4} \tag{28}$$

or, keeping in mind that $j = n - 1$ for $n = 1, 2, 3..$,

$$E_n = -\frac{m\alpha^2}{2(n - 1/2)^2} \tag{29}$$

Due to this condition the values of the angular momentum $L$ are limited by the integers $l$ which don't exceed $n-1$. The fact, that all states with different values of $l$, but the same $j$ have equal energies, is called "the accidental degeneracy". If the Coulomb potential is perturbed by a perturbation of a general form, the multiplets are split into doublets with

$L = \pm l$.
Finally, every multiplet of the egenstates of the hydrogen atom forms a basis of an irreducible representation of the the algebra $so(3)$. A natural connection between this algebra and the Fock sphere will be discussed in the next section.

## Wave functions

Pauli's finding was continued by Fock (see Fock's sphere), who has found explicit form the noticed symmetry and was able to show explicit way of finding the eigenstates.
Since the energy of the ground state is

$$E_1 = -2m\alpha^2, \qquad p_0 = 2m\alpha.$$

The ground state is a constant on the stereo-graphic sphere. Therefore, the wave function of the ground state in the momentum representation is

$$\psi_1(\mathbf{p}) \propto \frac{1}{p_0^2 + \mathbf{p}^2}. \tag{30}$$

In the coordinate representation, the wave function has the following form:

$$\begin{aligned} \psi_1(r) &\propto \int_0^\infty \frac{pdp}{p_0^2 + p^2} \int_0^{2\pi} d\phi e^{ipr\cos\phi} \\ &\propto \int_0^\infty p\ dp\ \frac{J_0(pr)}{p^2 + p_0^2} = K_0(p_0 r), \end{aligned} \tag{31}$$

where $K_0(x)$ is the MacDonald function. The first excited state corresponds to the energy

$$E_2 = \frac{2m\alpha^2}{9}, \qquad p_0 = \frac{m\alpha}{3}$$

and forms a triplet $\psi_{2,0}$ and $\psi_{2,\pm 1}$. On the stereo-graphic sphere, these functions corresponds to the $p$-triplet:

$$\psi_{2,0} \propto \cos\theta; \quad \psi_{2,\pm} \propto \ \sin\theta \ \exp\{\pm\, i\phi\}. \tag{32}$$

Inverting the stereo-graphic projection, we find the functions $\psi_{2,0}(\mathbf{p})$ and $\psi_{2,\pm}(\mathbf{p})$.

$$\psi_{2,0}(\mathbf{p}) \propto \frac{p_0^2 - p^2}{p_0^2 + p^2}; \tag{33}$$

$$\psi_{2,\pm}(\mathbf{p}) \propto \frac{p_0^2}{p_0^2 + p^2}\ e^{\pm i\phi}. \tag{34}$$

In order to find the functions in the coordinates representation, we need to perform the Furrier transform.
A remarkable quality of the proposed method is that we did not deal at any stage with an explicit solution of the Schrödinger equation, replacing all the troubles by the mentioned stereo-graphic projecting.

# Continuum

Let us begin with classical mechanics. The energy $E = p_0^2/2m$ is positive. Therefore

$$r = \frac{2m\alpha}{\mathbf{p}^2 - p_0^2} \tag{35}$$

$$dS^2 = \left(\frac{2m\alpha}{\mathbf{p}^2 - p_0^2}\right)^2 (dp_x^2 + dp_y^2) \tag{36}$$

The hyperbolic projection

$$p_x = p_0 \cos\phi \ \coth\frac{\theta}{2}; \quad p_x = p_0 \sin\phi \ \coth\frac{\theta}{2}$$

projects the outer part of the circle $p_x^2 + p_y^2 = p_0^2$ to hyperboloid, and the metric (41) takes the form

$$dS^2 = \left(\frac{m\alpha}{p_0}\right)^2 \left(d\theta^2 + \sinh^2\theta d\phi^2\right) \tag{37}$$

of the natural metric on this hyperboloid. The group of motion of the two-sheets Hyperboloid is $SO(2,1)$. Since this group is not compact, its representations are labled not by any discrete quantum numbers but a contineous quantum number $k$ $(0 < k < \infty)$.
The Laplace-Beltrami operator on hyperboloid has the following form:

$$\hat{\nabla}^2 = \frac{1}{\sinh\theta}\,\frac{\partial}{\partial\theta}\left(\sinh\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sinh^2\theta}\,\frac{\partial^2}{\partial\phi^2}. \tag{38}$$

Looking for its eigenfinctions in the form $\psi_{k,m}(\theta,\phi) = e^{im\phi}R_{km}(\cosh\theta)$, we get the equation for $R_{km}(x)$:

$$\frac{d^2R}{dx^2} - m^2R - k^2(x^2-1)\ R = 0. \tag{39}$$

Solutions of Eq (39) give the eigenfunctions on the stereographic hyperboloid, which may be transformed the inverted stereo-graphy into the functions on the momentum plane.

# Zero energy

At zero energy

$$r = \frac{2m\alpha}{\mathbf{p}^2} \tag{40}$$

$$dS^2 = \left(\frac{2m\alpha}{\mathbf{p}^2}\right)^2 (dp_x^2 + dp_y^2) \tag{41}$$

This provokes an inversion $\xi = \mathbf{p}/p^2$, which immediately gives

$$dS^2 = 2m\alpha(d\xi_x^2 + d\xi_y^2) \tag{42}$$

This implies that every trajectory in the $(\xi_x, \xi_y)$ -plane is a straight line and the trajectories in the $\mathbf{p}$-plane are the circles $(\mathbf{p} - \mathbf{p}_p/2)^2 = p_p^2/4$ which pass through the origin. Here $\mathbf{p}_p$ is the momentum in the pere-helium.