Tale 8
Taylor Instability and Ship Waves

1. Why does water pour out of an overturned glass? Is the answer so obvious, as it seems? It is known that the atmospheric pressure can hold a water column as high as 10 meters. So, why cannot it hold a glass of water? To answer this question consider first the waves on the surface of a heavy liquid. Since the main force, acting on the liquid, is the force of gravity, the dispersion relation $\omega(q)$ must contain the free fall acceleration g. From the dimensional analysis, we get

$$\omega^2(q) = gq; \qquad \omega(q) = \sqrt{gq}. \tag{1}$$

If the surface tension α is also taken into account, then the only combination that has the dimensions of ω^2 is $\alpha q^3/\rho$, where ρ is the density of the liquid. Combining the gravity and the surface tension, we obtain the following dispersion relation for the gravity-capillary waves:

$$\omega^2(q) = gq + \frac{\alpha}{\rho}q^3 \quad \omega(q) = \sqrt{gq + \frac{\alpha}{\rho} \cdot q^3} \tag{2}$$

Now consider an inverted geometry, when the heavy liquid is above a light one, or the air. Then the free fall acceleration changes its direction with respect to the normal to the surface of the heavy liquid, and the dispersion relation acquires the form

$$\omega(q) = \sqrt{-gq + \frac{\alpha}{\rho} \cdot q^3}. \tag{3}$$

For small q the frequency $\omega(q)$ is imaginary, which means that the flat surface is unstable with respect to the ripple formation (the Rayleigh-Taylor instability). If the radius R of the glass is smaller than $\sqrt{\alpha/g\rho}$ $(q > \sqrt{g\rho/\alpha})$, then the surface of the liquid is stable and the glass be better called a capillary.

2. Let us now restrict ourself to small q. Then $\omega(q) = \sqrt{gq}$. Consider the ripples on the surface of water caused by a stone, dropping into the water. The surface profile $\xi(r,t)$ looks roughly as

$$\xi(r,t) \sim \int \frac{d\omega}{2\pi} \int \frac{d^2q}{(2\pi)^2} \cdot \frac{e^{iqr - i\omega t}}{\omega^2 - gq \pm i\delta}$$
$$= \frac{i}{(2\pi)^2} \int \frac{d^2q}{2\sqrt{gq}} \cdot \exp[iqr\cos\alpha - i\sqrt{gq}t] =$$
$$= \frac{i}{4\pi\sqrt{g}} \int_0^\infty dq\sqrt{q} J_0(qr) e^{-i\sqrt{gq}t}$$

where $J_0(x)$ is the Bessel function. Introducing a dimensionless variable $z^2 = gqt^2$, we can simplify the above expression:

$$\xi(r,t) = \frac{i}{2\pi g^2 t^3} \int_0^\infty dz z^2 J_0(\frac{rz^2}{gt^2}) e^{-iz} \tag{4}$$

Introducing also a new function

$$\Phi(y) = \int_0^\infty z^2 dz J_0(yz^2) e^{-iz}, \tag{5}$$

of the variable $y = r/gt^2$, we obtain $\xi(r,t)$ in the form

$$\xi(r,t) = \frac{i}{2\pi g^2 t^3} \Phi(\frac{r}{gt^2}) \tag{6}$$

So, the ripples look like concentric circles shown in Fig 1. The leading crest moves with acceleration, and its radius obeys the law

$$r_1 \sim gt^2$$

To find the surface profile $\xi(r,t)$ well ahead of the leading wave, i.e. at $r \gg gt^2$, we must find $\Phi(y)$ for $y \gg 1$. The Bessel function oscillates rapidly, leading to the decay of $\Phi(y)$. At $z \ll 1$ the exponential e^{-iz} in Eq (5) could be set to unity. The result is that $\Phi(y) \sim y^{-3/2}$ and

$$\xi(r,t) \sim \frac{1}{g^2t^3}\Phi(\frac{r}{gt^2}) \sim \frac{1}{g^2t^3}(\frac{gt^2}{r})^{3/2} \sim \frac{1}{g^{1/2}r^{3/2}} \tag{7}$$

This result could be foreseen, because the profile well ahead of the leading wave must be time-independent, and, therefore, the asymptote of $\Phi(y)$ should be a power of its argument that compensate t^{-3} in the pre-factor in Eq (6).

To find the form of the ripples at $r \ll r_1 \sim gt^2$, i.e. behind the leading wave, we go back to the wave vector representation:

$$\xi \sim \int dq q^{1/2} \int d\alpha \exp[iqr\cos\alpha - i\sqrt{gq}t]. \tag{8}$$

At $r \ll gt^2$, the exponential in the integrand oscillates, so, $qr \gg 1$. The integral over α is determined by the small values of α, when $\cos\alpha = 1 - \alpha^2/2$. So,

$$\xi \sim \int dq \exp[iqr - i\sqrt{gq}t]. \tag{9}$$

The phase $\phi(r,t) = qr - \sqrt{gq}t$ of the integrand changes rapidly. Therefore, the integral is determined by those values of the wave vector $q(r,t)$ which correspond to the stationary values of the phase. The stationary phase conditions for $q(r,t)$ and $\phi(r,t)$ look like

$$r - \frac{t}{2}\sqrt{\frac{g}{q(r,t)}} = 0 \qquad q(r,t) = \frac{gt^2}{4r^2} \tag{10}$$

$$\phi(r,t) = \sqrt{gq}t - qr = \frac{gt^2}{4r} \tag{11}$$

Thus,

$$\xi \sim \exp[-igt^2/4r] \sim \sin[gt^2/4r],$$

and the n-th crest corresponds to $\phi_n = 2\pi n + \pi/2$. Therefore, the radius r_n of the n-th ripple is

$$r_n = \frac{gt^2}{2\pi(4n+1)}$$

We can see that the phase of a monochromatic wave changes with the velocity

$$v_{ph} = \frac{\omega}{q(\omega)} = \frac{\omega g}{\omega^2} = \frac{g}{\omega}$$

On the other hand, a wave packet is a bunch of monochromatic waves, propagating with different velocities. As a result, the wave vector $q(r,t)$, associated with the wave packet, changes in space and time and the radius $r(t)$ of a circular ripple varies as $r(t) = gt^2/4\phi_n$ (Eq (11)). Therefore, the group velocity is

$$v_{gr} = \frac{dr}{dt} = \frac{gt}{2\phi_n} = \frac{g}{2\omega} = \frac{v_{ph}}{2}.$$

3. Consider now the waves generated in deep water by a ship moving at a constant velocity $\mathbf{V}$. This problem differs from that of a dropping stone in that the perturbation is not localized like $\delta(t)\delta(\mathbf{r})$, but remains stationary in the moving frame ($\delta(\mathbf{r}-\mathbf{V}t)$). Under these conditions, the propagator $G(\omega, q) = (\omega^2 - gq)^{-1}$ must be substituted by $G_{\mathbf{V}} = [(\omega - \mathbf{qV})^2 - gq]^{-1}$. The stationary character of the wave pattern means that $\omega = 0$. Thus,

$$\xi(r,t) = \int \frac{d^2q}{(2\pi)^2} \cdot \frac{\exp[i\mathbf{qr}]}{(\mathbf{qV})^2 - gq} \tag{12}$$

The integration over the angle θ ($\cos\theta = \mathbf{qV}/qV$) gives

$$\int \frac{d\theta}{(qV\cos\theta)^2 - gq \pm i\delta} \sim \frac{1}{2\sqrt{gq}qV|\sin\theta|}\int d\theta\delta(\cos\theta - \frac{\sqrt{gq}}{qV})$$

and

$$\cos\theta = v_{ph}/V.$$

This relation is known as Cherenkov's condition. So,

$$\xi \sim \int \exp[iqr\cos(\alpha - \theta)]$$

The moving ship generates a circular wave at every point of its course. The resulting wave wave front is the envelope of these waves. Given the relations

$$r = v_{gr}t = \frac{v_{ph}t}{2} = \frac{Vt\cos\theta}{2}$$

$$\cos\theta = \frac{v_{ph}}{V} \qquad v_{ph} = V\cos\theta,$$

the coordinates x and y of the wave front as the functions of time t are:

$$x(t) = Vt - r\cos\theta = \frac{Vt}{2}(2 - \cos^2\theta) \tag{13}$$

$$y(t) = r\sin\theta = \frac{Vt}{2}\cdot\cos\theta\sin\theta \tag{14}$$

Since the phase $\phi(x, y, t)$ is constant along the wave front, it is convenient to express all the arguments through it:

$$\phi(r, t) = \frac{gt^2}{4r} = \frac{gt}{2V\cos\theta}, \tag{15}$$

$$t = \frac{2\phi V}{g}\cdot\cos\theta. \tag{16}$$

Substituting Eq (16) into Eqs (13) and (14), we obtain the parametric equations of the wave front:

$$x = \frac{V^2}{g} \cdot \phi (2 - \cos^2 \theta) \cos \theta \tag{17}$$

$$y = \frac{V^2}{g} \cdot \phi \cos^2 \theta \sin \theta \tag{18}$$

The consecutive crests of the waves behind the ship (shown in Fig. 2) are given by the consecutive semi-integer values of n in the equality $\phi = 2\pi(n + 1/2)$. One can see that our wave front has a folding point. Indeed, both $x(\theta)$ and $y(\theta)$ have maxima at the same value of $\theta = \theta_0$ ($\cos \theta_0 = \sqrt{2/3}$), which means that the wave front folds as it is shown in Fig 2. The reason for this is that, due to dispersion, the phase velocity increases with time, and, therefore, the wave front folds and propagate in the same direction as the ship at the same velocity.

Example 1. *Show that the shape of the wave front near its folding point is semi-cubic. i.e. its shape is given by the equation* $\eta^2 = \zeta^3$*, where* η *and* ζ *are appropriate coordinates.*

Example 2. *Show that the wave fronts, corresponding to different phases, lie within a wedge (the Kelvin ship-wave wedge) of semi-angle* $\alpha = 19.5^o \quad (\tan \alpha = 2^{-3/2})$.