Tale 4
Drunkard and Policemen

This short tale is about the problem of a random walk in the presence of traps. Imagine a drunkard wandering at random in a typical American town. The latter is particularly suitable for our purposes, because its map is a rectangular grid of streets and avenues. Let our drunkard, who is initially at a cross-roads, go one block of length l in time τ in a random direction. At the next cross-roads he makes a new decision and goes another block. The poor fellow keeps wandering around the town until he meets one of the policemen, who are distributed randomly with the density c and remain in the same place. The policeman seizes the drunkard ending his random walk. The question is: what is the probability that the drunkard will survive for a time t, after he has started his walk, assuming that t is much larger, than the time he needs to go one block $(t \gg \tau)$.

The solution may be found without going into any details of American town building, since for large time intervals the relevant distances are large as well. The probability $w_t(\mathbf{r})$ of finding our drunkard at a point $\mathbf{r}$ in a time t, if he starts from the origin, is given by the diffusion equation:

$$[\frac{\partial}{\partial t} - D\nabla^2 + U_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i)]w_t(\mathbf{r}) = \delta(\mathbf{r})\delta(t). \tag{1}$$

where $D = 2l^2/\tau$ is the diffusion coefficient. The sum is taken over all policemen. U_0 is a constant which has the same dimensions as D. The term in U_0 in Eq (1) is responsible for decreasing the probability $w_t(\mathbf{r})$ due to drunkard's encounters with the policemen. (This is why it is important that U_0 is positive.) It is convenient to use the Laplace transform

$$w_s(\mathbf{r}) = \int_0^\infty w_t(\mathbf{r}) e^{-st} dt \tag{2}$$

Then Eq (1) can be rewritten in the form

$$[s - D\nabla^2 + U(\mathbf{r})]w_s(\mathbf{r}) = \delta(\mathbf{r}); \quad U(\mathbf{r}) = U_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i). \tag{3}$$

Equation (3) is similar to the time-independent Schroedinger equation, where -s plays the role of energy. If $\phi_n(\mathbf{r})$ is the eigenfunction, which corresponds to the eigenvalue E_n, then

$$w_s(\mathbf{r}) = \sum_n \frac{\phi_n(\mathbf{r})\phi_n^*(0)}{s + E_n}$$

Since $U > 0$, all $E_n \geq 0$. As a result, all the poles of $w(s, \mathbf{r})$ lie on the negative half of the real axis. Using the inverse transform, we get

$$w_t(\mathbf{r}) = \sum_n \phi_n(\mathbf{r})\phi_n^*(0)e^{-E_n t}$$

To make the problem simpler, let us consider an ensemble of drunkards, who have started their walks at all cross-roads of the town, and find the total number of survivals to time t. This allows us to ignore the $\mathbf{r}$-dependence of w. As a result, we obtain, with exponential accuracy:

$$w_t \propto \sum_n e^{-E_n t} = \int_0^\infty dE g(E) e^{-Et} \tag{4}$$

where

$$g(E) = \sum_n \delta(E - E_n)$$

is the density of states. In the 2D case and for $U(r) = 0$ the density of states is

$$g(E) = \int \delta(E - Dk^2)\frac{d^2k}{(2\pi)^2} = \frac{1}{2\pi D}\theta(E)$$

If $U(r) \neq 0$, then $g(E)$ looks like it is shown in Fig 1. In the mean field approximation the real density of states is replaced by a shifted step-function:

$$g(E) = \frac{1}{2\pi D}\theta(E - E_0), \quad E_0 \propto cU_0, \tag{5}$$

which leads to

$$w(t) \propto e^{-E_0 t}, \quad E_0 \propto cU_0.$$

But this approximation is not good, because, using it, we miss the tail of the density of states at very low energies. As we can see from

Eq (4), it is this tail that determines the long time asymptote of the probability w_t.

The low energy states are the bound states of our fictitious Schroedinger equation caused by rare fluctuations of the random potential, like those shown in Fig.3. If the energy E is very small, these fluctuations of the random potential correspond to $U(\mathbf{r}) = 0$ in the inner area of a circle of radius R and about cU_0 outside the circle. For $E \ll cU_0$ the potential walls could be considered infinitely high. The wave function of the ground state in such a well is the zeroth order Bessel function $J_0(\sqrt{E/D}r)$ that obeys the condition

$$J_0(\sqrt{\frac{E}{D}}R) = 0.$$

This condition determines the minimal radius R_E of the potential fluctuation, which has a bound state of energy E.

$$R_E = \mu_1 \sqrt{\frac{E}{D}},$$

where $\mu_1 = 2.4048$ is the first root of the Bessel function $J_0(r)$.

The probability $P(R_E)$ of such a fluctuation is given by the Poisson formula

$$P(R_E) = \exp[-\pi c R_E^2],$$

so, the density of states is

$$g(E) \propto \exp\left[-c\pi\mu_1^2 \cdot \frac{D}{E}\right]. \tag{6}$$

Using Eq (4) and Eq (6), we arrive at the following expression for the probability

$$w_t \propto \int dE e^{-Et-\frac{E_*}{E}}, \quad E_* = c\pi\mu_1^2 D. \tag{7}$$

This integral can be calculated by the method of steepest descent:

$$0 = t - \frac{E_*}{\tilde{E}^2}$$

$$\tilde{E} = \sqrt{\frac{E_*}{t}}, \tilde{E}t + \frac{E_*}{\tilde{E}} = 2\sqrt{E_* t}$$

and, finally,

$$w_t \propto \exp[-2\sqrt{c\pi\mu_1^2 Dt}] \tag{8}$$

So, our drunkard will survive to time t if:

1. he comes to an empty space of size

   $$\tilde{L} = \sqrt{c^{-1/2}L_t}, \qquad L_t \gg \tilde{L} \gg c^{-1/2}$$

   where $L_t = \sqrt{Dt}$ is the characteristic length of diffusion in time t, and $c^{-1/2}$ is the mean distance between the policemen. So the drunkard must be lucky in the first place;

2. he does not leave this area.

A better strategy would be, of course, not to drink at all.