Second Tale
Waves on Shallow Water

KdV Equation

This story started in Scotland near Edinburgh and relates to tsunami in the ocean. Consider waves in shallow water, whose depth h is much smaller than the lengths of the waves on its surface. The equation of motion and the continuity condition for this waves have the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}, \tag{1}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv) = 0, \tag{2}$$

where v is the velocity in the water and g is the gravitational constant.

Introducing the height of the wave $u(x,t) = h - h_0$, excluding the velocity $v(x,t)$ and taking into account the lowest nonlinearity,

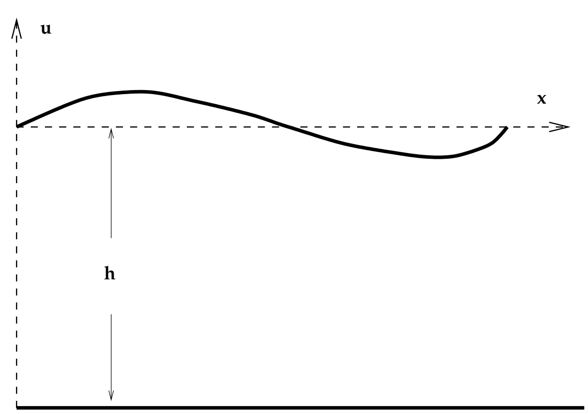

Figure 1: Adiabatic pendulum

we have

$$\frac{\partial^2 u}{\partial t^2} = gh_0\frac{\partial^2 u}{\partial x^2} + 3gu\frac{\partial^2 u}{\partial x^2} \tag{3}$$

If we seek the solution of this equation in the form of waves running either to the right or to the left, i.e.

$$u(x,t) = u_{2,1}(x \pm t, t), \tag{4}$$

we reduce the order of the time derivative:

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} = 0, \qquad \alpha = \frac{3}{2}\sqrt{\frac{g}{h_0}}. \tag{5}$$

This equation describes nonlinear evolution in the reference frame running with the wave velocity. If we take into also account the dispersion of this velocity due to the finite depth

$$\omega(k) = \sqrt{gh_0}\cdot\left[1 - \frac{(kh_0)^2}{2}\right] \tag{6}$$

our equation will take on the following form

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0 \tag{7}$$

which has a splendid name of the Korteveg - de Vries (KdV) equation. This equation describes the time evolution of a perturbation which runs to the right in the frame, which itself runs to the right with the velocity of the waves of infinitesimal amplitude. The weak dispersion and weak nonlinearity in the wave propagation are taken into account . All the constants (the velocity of the wave, its dispersion and nonlinearity) have been eliminated by an appropriate choice of the units for t, x and u. If we consider waves of finite amplitude, then in our reference frame their crests run with a positive and their wells with a negative velocity. What shape should a localized bump on the surface of the water have to move steady, and to what its velocity is equal to? The answer was, perhaps, already known to Rayleigh:

$$u(x,t) = A\cosh^{-2}\left[\frac{x-vt}{a}\right], \qquad v = 6A, \quad a^{-2} = \frac{3}{2}A \tag{8}$$

This wave is called a solitary wave or a soliton. The higher and the narrower a soliton is, the faster it moves. Therefore, if a high soliton moves behind a small one, they collide. The numerical solution of the KdV equation shows that, as a result of such a collision, the heights of both solitons remain exactly the same and they only swap their places. This result obtained by Kruscal and Zabussky has given rise to the suspicion that the KdV equation is an unusual nonlinear equation, which does not mix the modes. Perhaps, there are special conservation laws, which guarantee the conservation of the heights of colliding solitons.

The Inverse Scattering Problem for Schroedinger Equation and the Solutions of KdV Equation

Let us consider the Schroedinger equation

$$\hat{L}\phi = k^2\phi, \qquad \hat{L} = -\frac{d^2}{dx^2} + u(x,t), \tag{9}$$

which contains $u(x,t)$ as a potential. One can prove that the comutator of the Schroedinger operator $\hat{L}$ with an operator $\hat{A}$:

$$\hat{A} = 4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right) \tag{10}$$

is equal to

$$[L, A] = -u_{xxx} + 6u_x u \tag{11}$$

Thus, if $u(x,t)$ obeys the KdV equation, the time derivative of $\hat{L}$ obeys the so called Lax equation

$$\frac{\partial L}{\partial t} = [L, A]. \tag{12}$$

The Lax equation means, in particular, that the eigenvalue k^2 of the Schroedinger operators (9) does not depend on time t, while the potential does. Indeed,

$$\frac{\partial k_n^2}{\partial t} = \langle n| \frac{\partial L}{\partial t} |n\rangle = \langle n| LA - AL \Big| n\rangle = k^2 \cdot 0 = 0 \tag{13}$$

Taking the time derivative of the both parts of the Schroedinger equation (9) and using the Lax equation (12) for u, we have

$$L\left(\frac{\partial \phi}{\partial t} + A\phi\right) - k^2\left(\frac{\partial \phi}{\partial t} + A\phi\right) = 0. \tag{14}$$

Therefore,

$$\frac{\partial \phi}{\partial t} + \hat{A}\phi \tag{15}$$

is the eigenfunction of $\hat{L}$, corresponding to the eigenvalue k^2, and proportional to ϕ itself. The proportionality coefficient can be found if we use the asymptote at $x = -\infty$, where $\phi = e^{-ikx}$. One has

$$\frac{\partial \phi}{\partial t} + A\phi = 4ik^3\phi \tag{16}$$

Therefore, if at $t = 0$ the asymptote, as $x = +\infty$, is

$$\phi = a(k,0)e^{-ikx} + b(k,0)e^{+ikx}$$

then

$$\frac{\partial a(k,t)}{\partial t} = 0; \qquad \frac{\partial b(k,t)}{\partial t} = 8ik^3 b(k,t), \tag{17}$$

and we have the Gardner-Green-Kruscal-Miura (GGKM) solutions

$$a(k,t) = a(k,0); \quad b(k,t) = b(k,0)e^{+i8k^3 t}. \tag{18}$$

Similarly, for the bound states

$$\hat{L}\phi = -\kappa^2\phi$$

the relation

$$(\hat{L} + \kappa^2)(\frac{\partial\phi}{\partial t} + A\phi) = 0$$

is valid, and, since

$$\varphi|_{|x|\to\infty} = be^{-\kappa x} \to A\phi = -4\kappa^3\phi$$

we have the GGKM solution

$$b(t) = b(0)\exp[8\kappa^3 t]$$

Thus, we have found the following recipe to solve the KdV equation with a given initial condition, vanishing at the infinity:

1. Use the initial condition $u(x,0)$ to find the scattering data: $a(k,0), b(k,0)$, as well as the parameters of the bound states $\kappa(n)$ and $b(n,0)$.

2. Using the GGKM solution, find the scattering data at time t.

3. Use the solution of the inverse scattering problem to restore $u(x,t)$. To solve the inverse scattering problem, one does not need to know $a(k)$ and $b(k)$ separately but only their ratio, i.e. the reflection coefficient $r(k) = b(k)/a(k)$, $k(n)$ and $b(n)$.

Solitons

It is clear that there are special potentials, playing an exclusive role in the solution of the KdV equation. These are the potentials which have zero reflection at all positive energies. A soliton, taken with a negative sign,

$$u(x,t) = A\cosh^{-2}\left[\frac{x-vt}{a}\right], \qquad v = 6A, \quad a^{-2} = \frac{3}{2}A \tag{19}$$

is an example of such a potential, which has one bound state. It is clear now why solitons in the Kruscal-Zabussky experiments did not change their individuality and did not mix with each other. Indeed, two initially separated solitons form a potential, which has zero reflection coefficient and bound states in both soliton wells. In the course of time these energies will not change and the reflection-less character of the potential will not change either. Therefore, in the process of collision of two solitons, the water profile $u(x,t)$ corresponds to a reflection-less potential, which has the same energies of the bound states. After collision we have again two separated reflection-less potentials with the same energies of the bound states. Thus, it is not surprising that the solitons keep their individualities throughout the collision.

There is another important observation concerning the solutions of the KdV. If the initial potential has non-zero reflection coefficient and several bound states, it cannot be reduced to any set of solitons, but the ripples, associated with non-zero reflection, interfere with each other and, therefore, the long time asymptote looks like a set of solitons running at distances, determined by the initial condition.

Appendix. Solitons

The concept of the Lax LA-pair of operators allows to reduce the solution of the non-linear KdV equation to a solution of the inverse scattering problem, i.e. to find the potential from the results of scattering by this potential. A general solution of the inverse scattering problem is not an easy task. Fortunately, the problem of finding solitons is not that difficult.
We have to find the eigenfunctions of the Schroedinger operator

$$L = -\frac{d^2}{dx^2} + u(x) \tag{20}$$

In the continuous spectrum these functions obey the equation

$$-\frac{d^2\psi}{dx^2} + u(x)\psi = k^2\psi. \tag{21}$$

there are two sets of linearly independent solutions of Eq. (21)

$$\psi_{1,2}(x) = \exp[\mp ikx], \qquad x \to +\infty; \tag{22}$$

$$\phi_{1,2}(x) = \exp[\mp ikx], \qquad x \to -\infty. \tag{23}$$

functions $\psi_{1,2}(x)$ and $\phi_{1,2}(x)$ are coupled by complex conjugation:

$$\bar{\psi}_1 = \psi_2 = \psi_1(-k), \quad \bar{\phi}_1 = \phi_2 = \phi_1(-k).$$

These two sets are not independent and are coupled by the scattering matrix

$$\phi_i = t_{ij}\psi_j, \quad t = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}. \tag{24}$$

where a(k) and b(k) are connected with the reflection, $r(k)$, and transmition, $t(k)$, coefficients

$$t(k) = a^{-1}(k); \quad r(k) = b(k)a^{-1}(k). \tag{25}$$

Thus,

$$\phi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k). \tag{26}$$

The Wronsky determinant W for both pairs of functions does not depend on the coordinates x and is equal to $2ik$:

$$W(\phi,\bar{\phi}) = W(\psi,\bar{\psi}) = \psi\frac{d\bar{\psi}}{dx} - \bar{\psi}\frac{d\psi}{dx}. \tag{27}$$

This independence on x exhibits the current conservation. This conservation also means that

$$|a(k)|^2 - |b(k)|^2 = 1, \quad |t(k)|^2 + |r(k)|^2 = 1. \tag{28}$$

A reflection-less potential has $r(k) = 0$ for all k. This also means that $b(k) = 0$, and $|a(k)|^2 = |t(k)|^2 = 1$. Since $a(k)$ has zeros at the values $k = i\kappa_n$, connected with the bound states ($\kappa_n^2 = E_n$) and $a(k) = 1$ at $k \to \infty$, the only way to satisfy the condition (28) is to take

$$a(k) = \prod_{n=1}^{N} \frac{k + i\kappa_n}{k - i\kappa_n}, \tag{29}$$

where N is the total number of bound states in this potential (the N-soliton solution). For $N = 1$ (single soliton)

$$b(k) = 0, \quad a(k) = \frac{k + i\kappa}{k - i\kappa}, \quad a'(k = i\kappa) = \frac{1}{2i\kappa}. \tag{30}$$

And, finally, the wave function, that corresponds to the ground state, has an asymptote

$$\phi(x) = be^{-\kappa x}, \qquad x \to +\infty \tag{31}$$

What potential $u(x)$ gives these scattering data? First of all, Eq (26) reads now as

$$\phi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k). \tag{32}$$

It is convenient to use the Schroedinger equation in its integral form

$$\phi(x,k) = e^{-ikx} - \frac{1}{k}\int_x^{+\infty} dx' \sin k(x - x')u(x')\phi(k,x') \tag{33}$$

Multiplying Eq (33) by e^{ikx}, we obtain for $\chi_+(k,x) = \phi(k,x)e^{ikx}$

$$\chi_+(x,k) = 1 - \frac{1}{2ik}\int_x^{+\infty} dx'[e^{2ik(x-x')} - 1]u(x')\chi_+(k,x') \tag{34}$$

Similarly, for $\chi_-(k,x) = \psi(k,x)e^{-ikx}$ we have

$$\chi_-(x,k) = 1 + \frac{1}{2ik}\int_{-\infty}^{x} dx'[e^{2ik(x-x')} - 1]u(x')\chi_-(k,x') \tag{35}$$

The functions $\chi_{\pm}(x,k)$ are analytic in the upper (lower) half-planes of the plane of complex k respectively. Multiplying Eq (32) by $a^{-1}e^{ikx}$, we obtain the equation

$$\frac{\chi_+(x,k)}{a(k)} = \bar{\chi}_-(x,k). \tag{36}$$

Therefore, the function

$$\Phi(x,k) = \begin{cases} a^{-1}(k)\chi_+(x,k), & \mathrm{Im}k > 0, \\ \chi_-(x,k), & \mathrm{Im}k < 0. \end{cases} \tag{37}$$

is meromorphic with a single pole at $k = i\kappa$ and a value at infinity $\Phi(k = \infty) = 1$. There is only one function of this sort, namely

$$\Phi(x,k) = 1 + \frac{\Gamma(x)}{k - i\kappa}. \tag{38}$$

where

$$\Gamma(x) = 2i\kappa\phi(x,i\kappa)e^{-\kappa x} = 2i\kappa\psi(x,-i\kappa)e^{-\kappa x} = 2i\kappa b\chi_-(x,-i\kappa)e^{-2\kappa x}$$

Using Eq (37) and setting $k = -i\kappa$ in Eq (38), we obtain closed equation for $\Gamma(x)$:

$$\Gamma(x) = 2i\kappa b e^{-2\kappa x}\left(1 + \frac{\Gamma(x)}{k - i\kappa}\right), \tag{39}$$

where

$$\Gamma(x) = 2i\kappa\phi(x,i\kappa)e^{-\kappa x} = 2i\kappa\psi(x,-i\kappa)e^{-\kappa x} = 2i\kappa b\chi_-(x,-i\kappa)e^{-2\kappa x}$$

Using Eq (37) and setting $k = -i\kappa$ in Eq (38), we obtain a closed equation for $\Gamma(x)$:

$$\Gamma(x) = 2i\kappa b e^{-2\kappa x}\left(1 + \frac{i\Gamma(x)}{2\kappa}\right). \tag{40}$$

Eqs (35, 37) and (39) allow to express the potential $u(x)$ through the residue of Γ:

$$u(x) = -2i\frac{d\Gamma}{dx}. \tag{41}$$

Using

$$b(t) = be^{8\kappa^3 t}$$

and introducing a phase

$$\phi = \frac{1}{2\kappa} \ln b,$$

we arrive at the single soliton solution:

$$u(x,t) = \frac{2\kappa^2}{\cosh^2 \kappa(x - 4\kappa^2 t - \phi)}. \tag{42}$$