The First Tale
Adiabatic Pendulum and Semi-classical Approximation.

1. The string of a pendulum is gently pulled up, so that its length changes adiabatically. How does the amplitude of oscillations depend on time? The answer could be found in analogy with the basic problem of quantum mechanics. Indeed, the equation of motion

$$\frac{d^2x}{dt^2} + \omega^2(t)x = 0 \tag{1}$$

has the form, which is very similar to the Schroedinger equation

$$\psi'' + 2[E - U(r)]\psi = 0 \tag{2}$$

The adiabatic condition means that the frequency $\omega(t)$ changes over the period of oscillations by a small fraction of ω. In other words, $\dot{\omega}/\omega \ll \omega$, or $\dot{\omega} \ll \omega^2$.

This condition applied to the potential $U(r)$ in Eq. (2) means that $U(r)$ obeys the semi-classical condition. Therefore, it is convenient to

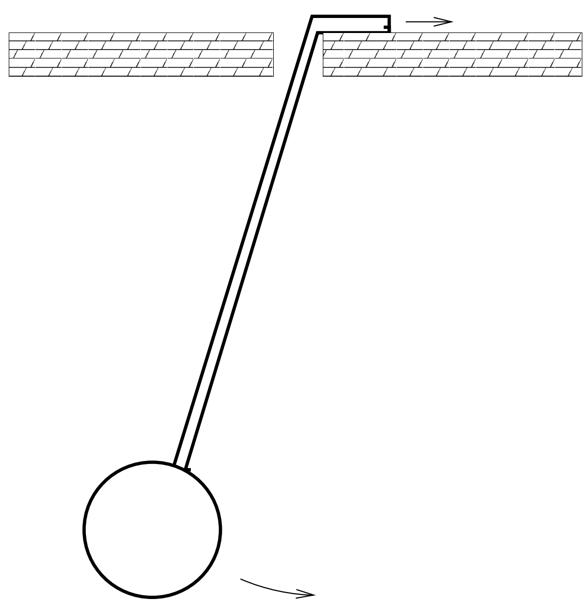

Figure 1: Adiabatic pendulum

look for the solution of Eq. (1) in the form

$$x(t) = \mathrm{Re}\exp[iS] = \mathrm{Re}\psi$$

where

$$\psi' = iS'\psi; \qquad \psi'' = -(S')^2\psi + iS''\psi$$

In the leading approximation we can neglect S'', which is smaller than $(S')^2$. Hence, representing $S(t)$ as

$$S(t) = S_1 + S_2, \quad S_1 \gg S_2, \tag{3}$$

we get from Eqs. (1) and (2)

$$(S_1')^2 = \omega^2(t), \qquad S_1 = \pm \int^t dt'\omega(t'). \tag{4}$$

In the next approximation

$$2S_1'S_2' + iS_1'' = 0 \qquad 2\omega(t)S_2' = \omega'(t). \tag{5}$$

Therefore,

$$\psi(t) = \frac{C_\pm}{\sqrt{\omega(t)}} \exp\left[\pm i \int^t dt'\omega(t')\right], \tag{6}$$

where $C_\pm$ are arbitrary constants. and

$$x(t) = x(-\infty)\sqrt{\frac{\omega(-\infty)}{\omega(t)}} \cos\left[\int_{-\infty}^t dt'\omega(t')\right]. \tag{7}$$

The last expression allows to calculate the mean value of the kinetic energy

$$< T >= \frac{1}{2} < \left(\frac{dx}{dt}\right)^2 > . \tag{8}$$

The mean value of the potential energy

$$< U >= \frac{\omega^2}{2} < x^2 > \tag{9}$$

is exactly equal to $< T >$ (the virial theorem). Thus, the total energy $E = T + U$ changes in time as

$$E(t) = \omega^2 < x^2 >= \frac{\omega(-\infty)\omega(t)}{2} x^2(-\infty) = \frac{E(-\infty)}{\omega(-\infty)}\omega(t) \tag{10}$$

This means that the quantity

$$I = \frac{E(t)}{\omega(t)}$$

is approximately constant and does not change if $d\omega/dt \ll \omega^2$. As for the amplitude of the oscillations, it changes as

$$x_{\max}(t) = \sqrt{\frac{2E}{\omega^2(t)}} = \sqrt{\frac{2I}{\omega(t)}} \propto \frac{1}{\sqrt{\omega(t)}}. \tag{11}$$

The result looks astonishing, because it means that our classical system "knows" that it can be considered a quantum oscillator, which has the quantum number $n = E/\hbar\omega$. Under adiabatic variation of the oscillation frequency, this quantum number does not change, which corresponds to the conservation of I in the classical motion. The conservation is, certainly, approximate and in the next section we will consider it more thoroughly.

2. The main defect of our reasoning is that the wave function $\psi(t)$ corresponds to a wave, running, say, only from left to right without any reflection. It is an artifact of the semi-classical approximation. The general solution contains a nonzero reflection amplitude and has the form

$$\psi(t) = \frac{1}{\sqrt{\omega(t)}} \begin{cases} Te^{i\int^t \omega(t')dt'}, & t = -\infty, \\ e^{i\int^t \omega(t')dt'} + Re^{-i\int^t \omega(t')dt'}, & t = +\infty, \end{cases} \tag{12}$$

where T and R are the transmission and reflection coefficients respectively. This form is slightly unusual, because we have swapped the left and right sides on the real axis around, but it suits our purposes better. In the semi-classical approximation $R = 0$, which means that $R \ll 1$. As a result $|T| = 1$ and $T = e^{ib}$. If the function (12) is used to determine the adiabatic invariant $I = E(t)/\omega(t)$, then

$$\begin{aligned} x(t) &= \mathrm{Re}\ \psi(t) \\ &= \frac{1}{\sqrt{\omega(t)}} \begin{cases} \cos\left[\int^t \omega(t')dt'\right], & t = -\infty \\ \cos\left[\int^t \omega(t')dt'\right] + \mathrm{Re}\ R\cos\left[\int^t \omega(t')dt'\right] & \\ +\mathrm{Im}\ R\sin\left[\int^t \omega(t')dt'\right] & t = +\infty \end{cases} \end{aligned} \tag{13}$$

and

$$E(t) = \omega^2(t) < x^2(t) >=$$
$$= E(-\infty)\omega(t) \begin{cases} 1, & t=-\infty \\ (1+\text{Re } R)^2 + (\text{Im } R)^2, & t=+\infty \end{cases} \tag{14}$$

Thus, in the leading approximation, $R \ll 1$, we have

$$\Delta I = 2I\text{Re}R. \tag{15}$$

Consider now a specific example of the frequency time dependence:

$$\omega^2(t) = \omega_0^2 + \omega_1^2 \frac{\alpha t}{\sqrt{1+\alpha^2 t^2}}, \tag{16}$$

where

$$\alpha \ll \omega_1 < \omega_0. \tag{17}$$

Then the frequency

$$\omega(t) = \sqrt{\omega^2(t)} = \sqrt{\omega_0^2 + \omega_1^2 \frac{\alpha t}{\sqrt{1+\alpha^2 t^2}}} \tag{18}$$

as a function of complex time t has a zero at $t = t_*$,

$$t_* = \pm \frac{i}{\alpha} \frac{\omega_0}{\sqrt{\omega_0^2 + \omega_1^2}}. \tag{19}$$

Near the complex turning point

$$\omega(t) = \left[\frac{i\alpha(\omega_0^2+\omega_1^2)^{3/2}(t-t_*)}{\omega_1} \right]^{1/2} = e^{i\pi/4} \left[\frac{\alpha^2(\omega_0^2+\omega_1^2)^3}{\omega_1^2} \right]^{1/4} (t-t_*)^{1/2} \tag{20}$$

and the action $S(t)$ is equal to

$$S(t) = \int^t dt' \omega(t') \cong \frac{2}{3} e^{i\pi/4} \left[\frac{\alpha^2}{\omega_1^2} (\omega_0^2+\omega_1^2)^3 \right]^{1/4} (t-t_*)^{3/2} \tag{21}$$

Therefore, Im $S(t) = 0$ along the dashed lines in Fig 2. These so called Stokes lines are very important, since the exponential $\exp[iS(t)]$ changes the character of its variation after crossing these lines, i.e. a

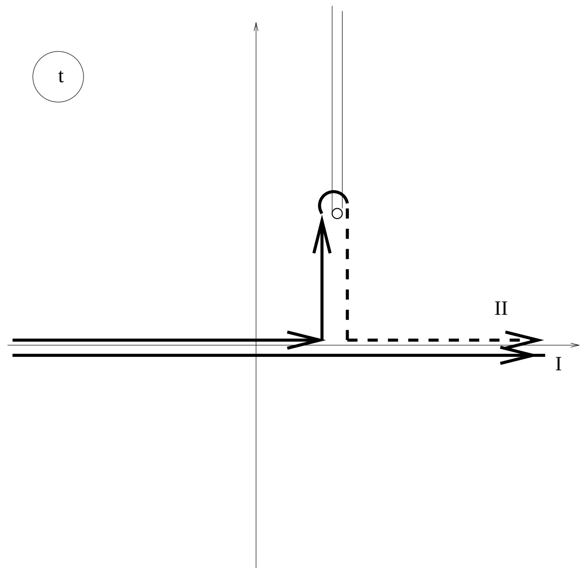

Figure 2: Plane of complex time t. Contours of integrations are denoted I and II as discribe in text. Branch cut originate from the turning point t_*.

growing exponential becomes a decaying one and vice versa. In the triangle, formed by two Stokes lines and the real axis, $\mathrm{Im}S(t) > 0$. Thus, the solution of the Schroedinger equation contains the integral $\int dt\omega(t)$ which is taken along the real axis and can be deformed without crossing the Stokes lines to prevent exponentially growing contributions. One can see from the Fig.2, that the contour I and all equivalent contours give a zero reflection amplitude, while the contours like the contour II give finite R, since $S(t)$ changes its sign, going around the turning point t_*. Thus,

$$R \propto \exp[2 \int_0^{t_*} dt\ \omega(t)] = \exp[-2 \int_0^{-it_*} d\tau\ \mathrm{Im}\ \omega(i\tau)]. \qquad (22)$$

The result is equal, with exponential precision, to $\exp[-\Omega/\alpha]$, where $\Omega \propto \omega_1, \omega_2$.

3. Consider the generalization of the problem of the adiabatic pendulum to the case of two coupled pendulums. The equations of their motion have the form

$$\frac{d^2x}{dt^2} = -\omega^2(t)x - \beta(x - y)$$

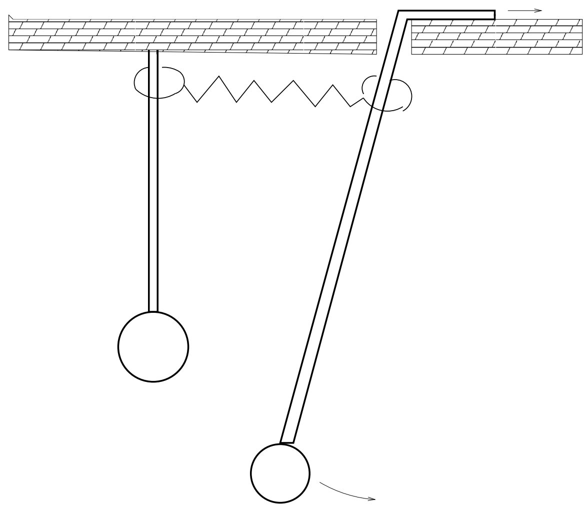

Figure 3: Coupled pendulae

$$\frac{d^2y}{dt^2} = -\omega_0^2 y + \beta(x - y)$$

At any current time t this coupled system has instantaneous frequency $\Omega(t)$, which obeys the equation

$$\det \left\| \begin{array}{cc} \Omega^2 - \omega^2(t) - \beta & \beta \\ \beta & \Omega^2 - \omega_0^2(t) - \beta \end{array} \right\| = 0, \tag{23}$$

so

$$\Omega_{1,2}^2(t) = \frac{\omega_0^2 + \omega^2(t) + 2\beta}{2} \pm \sqrt{(\frac{\omega^2(t) - \omega_0^2}{2})^2 + \beta^2}. \tag{24}$$

Two branches of the square root correspond to two different normal modes of the system. The time dependences of the frequencies is shown in Fig 3. If $\omega_0^2 \gg \beta$, the two modes are strongly separated everywhere, except near the intersection. This means that if at $t \mapsto -\infty$ only the pendulum with the variable string oscillates, then at $t \mapsto +\infty$, when it stops, all the energy of its oscillation goes to the other pendulum. The mixing of the modes due to a weak violation of the adiabatic condition is called, in this case, the Landau-Zener breakdown.

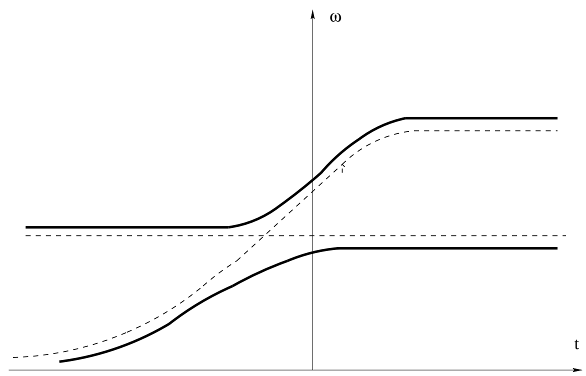

Figure 4: Eigen-frequencies of coupled pendulae