Field Theory of Euclidean Matrix Ensembles

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Abstract

Recently, a class of Hermitian matrix ensembles was proposed by Mézard, Parisi and Zee [1] in which the matrix elements depend on the Euclidean separation of coordinates drawn from a random distribution. Matrix ensembles of this kind appear in a variety of physical contexts. In this paper, we formulate and investigate the spectral properties of these Euclidean random matrix ensembles within the framework of a supersymmetric statistical field theory. We discuss applications of these results to various model systems.

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I. INTRODUCTION

Theories and applications of Hermitian random matrix ensembles are ubiquitous in the scientific literature [2,3]. Leaving aside their intrinsic interest as a field of mathematical research, the considerable attention paid to random matrix theories can, in part, be attributed to the universality of the low-energy (or long-time) response of irregular or stochastic quantum structures. The spectral statistics of random matrix ensembles provide a reference point for the consideration of phase coherence phenomena in irregular quantum structures.

Recently, a new class of random matrix ensembles has been identified and investigated [1], where the elements of (in this case) a real symmetric $N \times N$ matrix are specified by a given function of the Euclidean distances between a set of N points,

$$f_{ij} = f(\mathbf{r}_i - \mathbf{r}_j). \tag{1}$$

Here the coordinates are chosen at random from a d-dimensional box of size L^d, and $f(\mathbf{r})$ represents, for example, an interaction potential between the N identical particles. For simplicity, the interaction is assumed here to depend only on the pairwise distance between the points, although the formalism developed below can be extended to the consideration of a more general two-body potential. From this Hamiltonian, we can generate a random matrix ensemble by specifying the space of all possible configurations of the coordinates, in general weighted by some probability distribution function $P(\mathbf{r})$. Such an ensemble, which depends on the spatial separation of some coordinate distribution, has been christened a "Euclidean Matrix Ensemble" [1]. Distributions of this kind have been considered in a variety of physical contexts, from the vibration spectra of glasses [4] and the statistical mechanics of the interacting classical gas [5], to theories of combinatorial optimization [6], and the classical Coulomb gap [7].

To understand the nature of spectral correlations in this class of matrix ensembles, it is useful to first place in context the generality or 'universality' of these results, and to comment on the implications for potential applications. The current insight into theories of random disordered systems allows us immediately to draw some intuition about the nature of spectral correlations of the Euclidean matrix ensembles. To facilitate our discussion, it is useful to begin by briefly reviewing the properties of a quantum particle subject to a random impurity potential. This system is specified by the random Schrödinger operator or Anderson Hamiltonian,

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}), \tag{2}$$

where $\hat{\mathbf{p}} = -i\hbar\partial$, and $V(\mathbf{r})$ represents a weak, say, δ-correlated random impurity potential.

In the quasi-classical limit (i.e. where the transport mean-free path ℓ is greatly in excess of the wavelength λ), the properties of the Anderson Hamiltonian can be separated according to three characteristic energy or time scales. On time scales smaller than the typical transport or scattering time τ associated with the random potential V, the particle dynamics is regular or ballistic. However, on time scales in excess of τ a density fluctuation relaxes diffusively. In this regime, mechanisms of quantum interference lead to a renormalization of the effective diffusion constant D, and ultimately induce the complete localization of all states in dimensions $d \leq 2$. In systems of finite extent, the (renormalized) diffusive dynamics is truncated at the typical transport time $t_D = L^2/D$ across the system, beyond which the particle relaxation is "ergodic". Here one enters a universal regime in which all spectral and dynamical properties become independent of the material properties of the system, i.e. at energy scales $\Omega \ll \hbar/t_D$ ensemble average properties of the disordered system become indistinguishable from those of, say, Gaussian distributed random matrix ensembles exhibiting the same fundamental symmetry.

With this in mind, let us turn to the consideration of the class of Euclidean random matrix ensembles. As with any other random Hamiltonian (i.e. one which exhibits no continuous or discrete symmetries), a Euclidean random matrix will belong to one of three fundamental symmetry classes specified by the invariance of the Hamiltonian under time-reversal or spin rotation — the 'Dyson classification scheme'; e.g. the particular case of real symmetric matrices belongs to the orthogonal ensemble. Moreover, as with usual random matrix ensembles (i.e. ensembles defined on an invariant measure), for a given Euclidean matrix ensemble the average density of states (DoS), and indeed any other single particle property, is a non-universal quantity, specific to the given distribution (i.e. it depends sensitively on the function $f(\mathbf{r})$). By contrast, one can expect that, below some characteristic (system dependent) energy scale E_c, the statistical fluctuations of the DoS and wavefunctions are universal and depend only on the symmetry class of the Hamiltonian, independent of the detailed properties of the system, reflecting the ergodicity of the Hamiltonian dynamics.

However, keeping in mind the phenomenology of the disordered conductor, what can be said about the properties at short and intermediate time scales? At short time scales one expects the nature of density relaxation to depend sensitively on the range of the coupling $f(\mathbf{r})$. In short-ranged models, one expects to identify some characteristic scattering time scale τ beyond which density relaxation is fundamentally diffusive. Here, the spectral and localization properties of the system should parallel those of the Anderson Hamiltonian. By contrast, in systems which exhibit long-ranged coupling one expects the ergodic regime to extend to high energy scales, as it does in, say, Gaussian distributed random matrix ensembles. Finally, at the shortest time scales one expects properties to be highly non-universal, and depend sensitively on the precise short-distance nature of the coupling, density, etc.

With this background, let us review the existing theory of the Euclidean random matrix ensembles. In Ref. [1] an ingenious procedure was devised to present the ensemble average of the single-particle Green function in terms of a field integral involving a single replicated scalar field (c.f. Ref. [5]). Their formalism gave access to a controlled expansion of the average DoS of the bulk distribution, and provided a vehicle to study the asymptotic regions of the spectrum.

The aim of the present paper is two-fold. Firstly, we aim to generalize the procedure of Ref. [1] to the investigation of the two-particle properties (and higher) of the Euclidean matrix ensembles. The latter provide access to transport and localization properties, as well as fluctuation phenomena of the matrix ensemble. Secondly, within the same formalism, we will expose a class of soft modes of density relaxation which affect the long-time fluctuation properties of the system. These soft modes mimic the massless diffusion modes of the quantum disordered Anderson Hamiltonian, and provide a bridge between the Euclidean matrix ensembles and the conventional theory of disordered conductors. For such purposes experience has shown that the supersymmetric scheme is more convenient than the replica formalism. We will therefore generalize the replica approach of Ref. [1] to the supersymmetric formalism of Efetov [8].

As mentioned above, for short-ranged potentials we will find the phenomenology of the Anderson model to be applicable: All states are localized in dimensions $d \leq 2$, while for higher dimensions there is a transition or mobility edge separating localized and delocalized states. However, previous studies of Euclidean random matrix ensembles with a long-range coupling (i.e. where $\lim_{|\mathbf{r}|\to\infty} f(\mathbf{r}) \sim |\mathbf{r}|^{-\alpha}$) indicate that the situation is more subtle [9,10], and lead to the following conclusions. If $\alpha > d$, the localization properties are similar to the short-ranged case, with complete localization for low dimensionalities and a localization-delocalization transition in higher dimensions. However, if $\alpha \leq d$ all states outside the band tails are found to be *delocalized*, however weak the hopping matrix elements. In the particular case where $\alpha = d$ the localization properties are predicted to be "critical", with only a power-law decay of correlations. Although the formulation of the statistical field theory is general, in this work we will limit our considerations to short-ranged potentials, in which the low-energy effective field theory for bulk states will assume a standard diffusive form.

This completes our introduction. The remainder of the paper is organized as follows. In section II we formulate a supersymmetric field theory describing average properties of general n-point correlation functions of the Euclidean random matrix ensemble. Here we will make use of the approach of Ref. [1] to facilitate the ensemble averaging procedure. Subjecting the action to a mean-field analysis in section III, we obtain an expression for the non-universal average DoS over a wide range of energy scales in section IV. These results are compared with numerics for a variety of couplings. By developing an expansion of the action about the mean-field, in section V we cast the low-energy properties of the field theory as a diffusive, supersymmetric non-linear σ-model. Finally, in section VI, we discuss the implications of these results for the nature of the fluctuation phenomena and localization in this class of systems.

II. FORMULATION OF FIELD THEORY

In this section we formulate the generating function of spectral correlations of the Euclidean random matrix ensembles as a statistical field theory. Following the now standard procedure of Efetov [8], we employ a supersymmetric scheme to express Green functions (and their correlators) of the Hermitian matrix f_{ij} as a functional field integral. Since we are interested in the limit of high density, where the physics is expected to be insensitive to the precise value of N, we will ultimately find it convenient to perform the ensemble average over the random coordinate distribution using a grand canonical ensemble [1].

Statistical n-point correlators of the matrix $[\hat{f}]_{ij} \equiv f_{ij}$ can be determined from the generating function,

$$\mathcal{Z}_N[J, \{\mathbf{r}_i\}] = \int D[\bar{\Psi}, \Psi] \exp[i\bar{\Psi} \cdot (\hat{f} - \hat{\epsilon}) \cdot \Psi + (\bar{\Psi} \cdot J + \bar{J} \cdot \Psi)], \tag{3}$$

where $\hat{\epsilon} = \{\epsilon_1, \ldots, \epsilon_n\}$, and Ψ represent $2 \times 2 \times n \times N$-component supervectors. As usual, the super-index (boson B, fermion F) structure of the elements Ψ is used to enforce the normalization $\mathcal{Z}_N = 1$. The remaining elements index the n-points of the correlation function, the N-components of the matrix Hamiltonian and, finally, two conjugate components indexing a 'time-reversal' sector. At the expense of introducing some redundancy in the formalism, the introduction of conjugate elements provides a convenient way of taking into account the time-reversal symmetry of the Hamiltonian. Altogether, the supervector fields assume the structure

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}_{\mathrm{TR}}, \qquad \bar{\Psi} = \Psi^T C, \qquad C = \begin{pmatrix} \sigma_1^{\mathrm{tr}} & 0 \\ 0 & -i\sigma_2^{\mathrm{tr}} \end{pmatrix}_{\mathrm{BF}}. \tag{4}$$

The source vectors J are defined to have the same structure, $J^T = (j^T, \bar{j})/\sqrt{2}$. Defining $\Lambda = -\mathrm{sgn}(\mathrm{Im}[\hat{\epsilon}])$, convergence of the field integral demands $\bar{\psi} = \psi^\dagger \iota$, where $\iota = \Lambda \otimes E_{11}^{\mathrm{BF}} + \mathbb{1} \otimes E_{22}^{\mathrm{BF}}$. (Here the projection matrix E_{ij}^{BF} takes the value of unity on the element ij in the BF-space, and zero on all other elements.)

Formally, subjecting the generating function to the ensemble average entails integrating over the particle positions weighted by some distribution function,

$$\langle \mathcal{Z}_N[J] \rangle = \prod_{i=1}^{N} \int_0^L \frac{d\mathbf{r}_i}{L^d} P(\mathbf{r}_i) \mathcal{Z}_N[J, \{\mathbf{r}_i\}].$$

In the simplest case the distribution function is taken to be uniform. Expressed in discrete coordinate form, applying the ensemble average directly to the generating functional seems problematic. To circumvent this difficulty, we follow the approach of Ref. [1] and present the generating function using fields which do not depend explicitly on the number of particles.

Accordingly we define the fields,

$$\Theta(\mathbf{r}) = \sum_i \Psi_i \delta^d(\mathbf{r} - \mathbf{r}_i), \qquad \bar{\Theta}(\mathbf{r}) = \sum_i \bar{\Psi}_i \delta^d(\mathbf{r} - \mathbf{r}_i),$$

where Θ and $\bar{\Theta}$ are $2 \times 2 \times n$-component supervector fields, with the same supervector structure as Ψ and $\bar{\Psi}$. The change of variables can be enforced using a δ-functional,

$$\delta\left[\Theta(\mathbf{r}) - \sum_i \Psi_i \delta^d(\mathbf{r} - \mathbf{r}_i)\right] = \int D\bar{\Phi}(\mathbf{r}) e^{i\int d\mathbf{r}\bar{\Phi}(\mathbf{r})\left[\Theta(\mathbf{r}) - \sum_i \Psi_i\delta^d(\mathbf{r}-\mathbf{r}_i)\right]},$$

where, once again, Φ and $\bar{\Phi}$ inherit the same supervector structure. When substituted into the generating functional we obtain,

$$\langle \mathcal{Z}_N[J]\rangle = \int D[\bar{\Phi},\Phi]\int D[\bar{\Theta},\Theta]\int D[\bar{\Psi},\Psi]\prod_{i=1}^{N}\frac{d\mathbf{r}_i}{L^d}P(\mathbf{r}_i)e^{iS},$$

with

$$\begin{aligned} S = \int d\mathbf{r}\int d\mathbf{r}'\bar{\Theta}(\mathbf{r})f(\mathbf{r},\mathbf{r}')\Theta(\mathbf{r}') + \int d\mathbf{r}(\bar{\Phi}(\mathbf{r})\Theta(\mathbf{r}) + \bar{\Theta}(\mathbf{r})\Phi(\mathbf{r})) \\ -\sum_i\left[\bar{\Psi}_i\hat{\epsilon}\Psi_i + \bar{\Psi}_i(\Phi(\mathbf{r}_i) + iJ_i) + (\bar{\Phi}(\mathbf{r}_i) + i\bar{J}_i)\Psi_i\right] \end{aligned}$$

(With this expression, it is convenient to leave the term $\bar{\Psi}_i\hat{\epsilon}\Psi_i$ in its present form.)

Gaussian in the discrete variables Ψ_i and fields $\Theta(\mathbf{r})$, the functional integral can be evaluated, whereupon one obtains,

$$\langle \mathcal{Z}_N[J]\rangle = \int D[\bar{\Phi},\Phi]\left[\int\prod_{i=1}^{N}\frac{d\mathbf{r}_i}{L^d}P(\mathbf{r}_i)e^{i(\bar{\Phi}(\mathbf{r}_i)+i\bar{J}_i)\hat{\epsilon}^{-1}(\Phi(\mathbf{r}_i)+iJ_i)}\right]e^{-i\int d\mathbf{r}\int d\mathbf{r}'\bar{\Phi}(\mathbf{r})f^{-1}(\mathbf{r},\mathbf{r}')\Phi(\mathbf{r}')},$$

where $f^{-1}(\mathbf{r},\mathbf{r}')$ represents the inverse or Green function of the operator $f(\mathbf{r},\mathbf{r}')$, i.e. $\int d\mathbf{r} f^{-1}(\mathbf{r}_1,\mathbf{r})f(\mathbf{r},\mathbf{r}_2) \equiv \delta^d(\mathbf{r}_1 - \mathbf{r}_2)$. Note that, strictly speaking, at this stage we have not assumed $f(\mathbf{r},\mathbf{r}')$ to depend only on the Euclidean separation $\mathbf{r} - \mathbf{r}'$.

So that this may be completely expressed in a way which does not depend on the number of particles, it is also necessary to replace the source terms J with fields independent of N. However, in doing so we will wish to be able to relate what were originally averages defined for particular pairs of particles, to the corresponding expressions involving these source fields, where the identity of individual particles is lost. To achieve this we define within the integrals,

$$J(\mathbf{r}) = \sum_i \frac{J_i\delta^d(\mathbf{r}-\mathbf{r}_i)}{\delta^d(\mathbf{r}=0)}, \quad \text{i.e. } J_i = J(\mathbf{r}_i), \quad \frac{\partial}{\partial J_i} = \int d\mathbf{r}\frac{\delta}{\delta J(\mathbf{r})}\frac{\delta^d(\mathbf{r}-\mathbf{r}_i)}{\delta^d(\mathbf{r}=0)},$$

where the symbol $\delta^d(\mathbf{r}=0)$ is defined explicitly below. This definition is unambiguous provided $\mathbf{r}_i \neq \mathbf{r}_j$, when what were the i'th and j'th particles become associated with J evaluated at the same point in space. However, since this coincidence occurs with zero measure, such a circumstance can be safely neglected.

As an example of this procedure, let us consider the general quantity,

$$\sum_{j_1,\ldots,j_m}\mathrm{tr}\left[\left(\frac{\partial}{\partial\bar{J}_{j_1}}\otimes\frac{\partial}{\partial J_{j_2}}\right)\left(\frac{\partial}{\partial\bar{J}_{j_2}}\otimes\frac{\partial}{\partial J_{j_3}}\right)\cdots\left(\frac{\partial}{\partial\bar{J}_{j_m}}\otimes\frac{\partial}{\partial J_{j_1}}\right)\langle\mathcal{Z}_N\rangle\right].$$

Upon changing variables from J_i to $J(\mathbf{r})$, this becomes,

$$\frac{1}{(\delta^d(\mathbf{r}=0))^m}\int d\mathbf{r}_1\ldots d\mathbf{r}_m \mathrm{tr}\Bigg[\left(\frac{\delta}{\delta\bar{J}(\mathbf{r}_1)}\otimes\frac{\delta}{\delta J(\mathbf{r}_2)}\right)\left(\frac{\delta}{\delta\bar{J}(\mathbf{r}_2)}\otimes\frac{\delta}{\delta J(\mathbf{r}_3)}\right) \\ \cdots\left(\frac{\delta}{\delta\bar{J}(\mathbf{r}_m)}\otimes\frac{\delta}{\delta J(\mathbf{r}_1)}\right)\langle\mathcal{Z}_N\rangle\Bigg], \tag{5}$$

where we can now write,

$$\langle\mathcal{Z}_N[J]\rangle = \int D[\bar{\Phi},\Phi]\left[\int\frac{d\mathbf{r}}{L^d}P(\mathbf{r})e^{i(\bar{\Phi}(\mathbf{r})+i\bar{J}(\mathbf{r}))\hat{\epsilon}^{-1}(\Phi(\mathbf{r})+iJ(\mathbf{r}))}\right]^N e^{-i\int d\mathbf{r}\int d\mathbf{r}'\bar{\Phi}(\mathbf{r})f^{-1}(\mathbf{r},\mathbf{r}')\Phi(\mathbf{r}')}.$$

The factors $\delta^d(\mathbf{r}=0)\equiv\int d\mathbf{q}/(2\pi)^d$ are expected to cancel with other factors which are introduced when the functional derivatives are evaluated.

Although formally exact, in the following it will prove to be convenient to work with a "grand canonical ensemble" defined by [1],

$$\langle\mathcal{Z}[J]\rangle = \left(\sum_N\frac{\alpha^N}{N!}\right)^{-1}\sum_N\frac{\alpha^N}{N!}\langle\mathcal{Z}_N[J]\rangle,$$

where $\alpha=\langle N\rangle$ represents an effective 'fugacity' of the distribution. Changing variables to $(\Phi+iJ)$ and $(\bar{\Phi}+i\bar{J})$, one obtains $\langle\mathcal{Z}[J]\rangle=\int D[\bar{\Phi},\Phi]e^{-S}$, where

$$S=-\alpha\left[\int\frac{d\mathbf{r}}{L^d}P(\mathbf{r})e^{i\bar{\Phi}(\mathbf{r})\hat{\epsilon}^{-1}\Phi(\mathbf{r})}-1\right]+i\int d\mathbf{r}\int d\mathbf{r}'(\bar{\Phi}(\mathbf{r})-i\bar{J}(\mathbf{r}))f^{-1}(\mathbf{r},\mathbf{r}')(\Phi(\mathbf{r}')-iJ(\mathbf{r}')). \tag{6}$$

This expression, which has relied on no approximation, formally coincides with $\langle\mathcal{Z}_N\rangle$ in the limit $N\to\infty$.

This completes our formulation of the field theory of the Euclidean random matrix ensembles. We have succeeded in presenting the generating functional of the Green functions as a supersymmetric field integral involving a single supervector field. This result differs from that obtained by Ref. [1] in only two simple respects. The first is that this expression is cast within the framework of the supersymmetric formalism as opposed to that of replicas. Secondly, and more importantly, it is generalized to account for two-point and higher correlations. Next we will explore a range of parameters in which the low-energy properties of the system are reduced to a simple effective action. In this limit, the connection between the Euclidean matrix ensemble and the problem of the disordered metal will be made explicit.

For arbitrary values of the parameters ϵ_i, an evaluation of the field integral does not seem feasible. The interaction of the fields generated by the ensemble average is highly non-linear. To proceed we are forced into looking for some appropriate approximation schemes. Such a choice depends sensitively on the particular application, and on the form of coupling $f(\mathbf{r})$. Drawing intuition from the strongly localized states of the disordered Anderson Hamiltonian, at the extrema of the spectrum one expects the action to be dominated by supersymmetry broken instanton configurations of the fields $\Phi(\mathbf{r})$, corresponding to rare or 'optimal conformations' of the effective "disorder potential" f_{ij}. (For a detailed investigation of tail states in the framework of the replica approach, see Ref. [11].) However, for the bulk of the spectrum one might expect the action to submit to a simpler, controlled perturbative expansion of the fields around some stable mean-field. Since our aim in this paper is to uncover the phenomenology of the Euclidean ensembles, we will focus here on the latter, keeping in mind the fact that many physically motivated problems will be concerned with the extrema of the spectra.

To proceed, we will seek an expansion in powers of $\bar{\Phi}\Phi/\bar{\epsilon}$, assuming the expectation of this quantity to be small. Here $\bar{\epsilon}$ represents some reference energy scale ($\bar{\epsilon} > 0$). The validity of this expansion will be checked self-consistently. In addition, we will restrict attention to short-range correlations in energy, when we can set

$$\epsilon_i = \bar{\epsilon} + \omega_i,$$

where $|\omega_i| \ll \bar{\epsilon}$. In this approximation, an expansion of the action to quadratic order obtains,

$$\begin{aligned} S \simeq i \int d\mathbf{r} \int d\mathbf{r}' \bar{\Phi}(\mathbf{r}) G^{-1}(\mathbf{r}, \mathbf{r}') \Phi(\mathbf{r}') + \frac{\alpha}{2} \int \frac{d\mathbf{r}}{L^d} P(\mathbf{r}) \left(\frac{\bar{\Phi}(\mathbf{r})\Phi(\mathbf{r})}{\bar{\epsilon}} \right)^2 \\ + \int d\mathbf{r} \int d\mathbf{r}' \left(\bar{J}(\mathbf{r}) f^{-1}(\mathbf{r}, \mathbf{r}') \Phi(\mathbf{r}') + \bar{\Phi}(\mathbf{r}) f^{-1}(\mathbf{r}, \mathbf{r}') J(\mathbf{r}') - i \bar{J}(\mathbf{r}) f^{-1}(\mathbf{r}, \mathbf{r}') J(\mathbf{r}') \right), \end{aligned} \tag{7}$$

where

$$G^{-1}(\mathbf{r}, \mathbf{r}') = f^{-1}(\mathbf{r}, \mathbf{r}') - \rho P(\mathbf{r}) \hat{\epsilon}^{-1} \delta(\mathbf{r} - \mathbf{r}')$$

represents the supermatrix Green function, and $\rho = \alpha / L^d$ represents the particle density.

At this level, the form of the effective action bares comparison with the corresponding field theory of the disordered metal. As with the Euclidean random matrix ensemble above, one can construct the corresponding (source-free) generating function of n-point correlations,

$$\mathcal{Z}[J = 0] = \int D[\bar{\Phi}, \Phi] e^{i \int d\mathbf{r} \bar{\Phi}(\hat{H} - \hat{\epsilon})\Phi},$$

where the superfields $\Phi(\mathbf{r})$ have the same index structure as that employed previously, and $\hat{H}$ represents the Anderson Hamiltonian defined by Eq. (2). Taking the random impurity potential to be δ-correlated, with zero mean and correlation $\langle V(\mathbf{r}) V(\mathbf{r}') \rangle = (\hbar / 2\pi\nu\tau) \delta^d(\mathbf{r} - \mathbf{r}')$, where τ represents the scattering time and ν the average DoS, the ensemble average generates $\langle \mathcal{Z}[J = 0] \rangle = \int D[\bar{\Phi}, \Phi] e^{-S}$, where

$$S = -i \int d\mathbf{r} \bar{\Phi}(\mathbf{r}) \left(\frac{\hat{\mathbf{p}}^2}{2m} - \hat{\epsilon} \right) \Phi(\mathbf{r}) + \int d\mathbf{r} \frac{\hbar}{4\pi\nu\tau} (\bar{\Phi}(\mathbf{r})\Phi(\mathbf{r}))^2.$$

A comparison of this action with that of Eq. (7) reveals an immediate connection between the field theory of the disordered metal, and the approximate theory of the Euclidean ensemble. Within this mapping, the effective bare or clean Hamiltonian is identified with the operator $\hat{f}^{-1} - \rho/\bar{\epsilon}$, and the strength of the scattering potential with $\rho/\bar{\epsilon}^2$. This mapping is listed in Table I, and using it some important energy and length scales are collected in Table II.

In principle, we could proceed with the analysis of the action (7) by forming an expansion in the quartic interaction, and collecting the resulting series into a diagrammatic perturbation theory. However, experience with the study of the theory of quantum transport in disordered metals suggests that an effective low-energy theory can be developed in a non-perturbative scheme. In particular, it is well established that the low-energy properties of the disordered metal are governed by the interaction of two soft modes of density relaxation: ‘diffusons’ and ‘Cooperons’. With this background, we therefore proceed to analyse the action of the Euclidean field theory (7) in a standard manner. However, since the precise connection between the parameters of the two theories is sufficiently oblique, we believe that it is instructive to develop the low-energy theory in some detail.

III. MEAN-FIELD ANALYSIS

As a first step in this program, we begin by subjecting the quartic interaction of the fields to a Hubbard-Stratonovich decoupling, which is chosen so that the effective theory is present in slow fluctuations of the decoupling field about a saddle point. Taking into account the two soft modes of the effective theory, this entails introducing a decoupling in two channels. Both are contained within different elements of a $4n \times 4n$ supermatrix field Q,

$$e^{-\frac{\alpha}{2}\int \frac{d\mathbf{r}}{L^d}P(\mathbf{r})\left(\frac{\bar{\Phi}(\mathbf{r})\Phi(\mathbf{r})}{\bar{\epsilon}}\right)^2} = \int DQ e^{-\frac{\alpha}{4}\int \frac{d\mathbf{r}}{L^d}P(\mathbf{r})\mathrm{str}Q^2 - \frac{\alpha}{\bar{\epsilon}}\int \frac{d\mathbf{r}}{L^d}P(\mathbf{r})\bar{\Phi}(\mathbf{r})Q\Phi(\mathbf{r})}.$$

Note that this expression is strictly not exact. In particular, as written, the decoupling makes sense only if the fields Q are interpreted as slowly varying in space. The two soft modes of the theory are both present in slow fluctuations of Q, and are duplicated as fast fluctuations; roughly speaking, if we restrict Q to being slowly varying we must count these degrees of freedom twice. The legitimacy of this approximation will be established presently on the level of the saddle-point. Finally, the symmetry of the fields Φ induces a corresponding symmetry in the fields Q which reflects that of the dyadic product $\Phi \otimes \bar{\Phi}$. Accordingly, one finds,

$$Q^T = CQC^T.$$

Gaussian in fields Φ, an evaluation of the functional integral obtains $\mathcal{Z} = \int DQ e^{-S-S_\mathrm{J}}$, where

$$S[Q] = \frac{\alpha}{4}\int \frac{d\mathbf{r}}{L^d}P(\mathbf{r})\mathrm{str}\, Q^2 - \frac{1}{2}\mathrm{str}\ln G^{-1}, \tag{8a}$$

$$S_\mathrm{J}[Q] = \frac{i\alpha}{\bar{\epsilon}}\int \frac{d\mathbf{r}}{L^d}P(\mathbf{r})\bar{J}(\mathbf{r})\left(1 + iQ(\mathbf{r})\right)J(\mathbf{r}) \tag{8b}$$
$$+\frac{i\alpha^2}{\bar{\epsilon}^2}\int \frac{d\mathbf{r}}{L^d}\int \frac{d\mathbf{r}'}{L^d}\bar{J}(\mathbf{r})P(\mathbf{r})\left(1 + iQ(\mathbf{r})\right)G(\mathbf{r},\mathbf{r}')\left(1 + iQ(\mathbf{r}')\right)P(\mathbf{r}')J(\mathbf{r}').$$

In the leading order of expansion in $\hat{\omega}/\bar{\epsilon}$, the supermatrix Green function takes the form

$$G^{-1}(\mathbf{r},\mathbf{r}') = f^{-1}(\mathbf{r},\mathbf{r}') - \frac{\rho P(\mathbf{r})}{\bar{\epsilon}}\left(1 - \frac{\hat{\omega}}{\bar{\epsilon}} + iQ(\mathbf{r})\right)\delta(\mathbf{r}-\mathbf{r}').$$

To establish the validity of the Hubbard-Stratonovich decoupling, we proceed by subjecting the effective action to a saddle-point expansion. Varying the action with respect to Q, and neglecting the sources J and ω, one obtains the saddle-point equation,

$$Q_{\rm sp}(\mathbf{r}) = -\frac{i}{\bar{\epsilon}} G_{\rm sp}(\mathbf{r}, \mathbf{r}). \tag{9}$$

Taking the interaction potential f to be translationally invariant, and the distribution to be uniform ($P(\mathbf{r}) = 1$), the supermatrix Green function assumes a simple form. Applying the *Ansatz* that $Q_{\rm sp}$ is diagonal in the internal indices and homogeneous in space, one obtains,

$$G_{\rm sp}(\mathbf{r}, \mathbf{r}') = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{e^{-i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})}}{f^{-1}(\mathbf{q}) - (1 + iQ_{\rm sp})\rho/\bar{\epsilon}}, \tag{10}$$

where $f^{-1}(\mathbf{q}) \equiv f^{-1}(|\mathbf{q}|) = \int d\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} f^{-1}(0, \mathbf{r})$. In this form, one can interpret the saddle-point equation as representing the self-consistent Born approximation for the self-energy. Note that $\langle \Phi \otimes \bar{\Phi}/\bar{\epsilon} \rangle = -(i/\bar{\epsilon})\langle G(\mathbf{r}, \mathbf{r}) \rangle$, and hence the limit in which the expansion of the action (7) is justified corresponds here to the condition $Q_{\rm sp} \ll 1$.

Generally, the diagonal elements of $Q_{\rm sp}$ exhibit both real and imaginary parts: $Q_{\rm sp} = X + iY$, where X and Y are real diagonal matrices. Typically, one can expect these components to be similar in magnitude. However, while the self-consistent solution of the saddle-point equation suggests that the shift in the real part of the self-energy is of the form $Y \propto \mathbb{1}$, the analytical properties of the advanced and retarded components of the Green function demand that $X = a\Lambda$. Hence, while $Y \ll 1$ (as demanded by the applicability of the expansion of the Euclidean matrix action) the elements of Y simply bring about a small shift in the effective 'Fermi energy' $\rho/\bar{\epsilon}$ and can be safely neglected. A second issue concerns the existence of the momentum integration itself. In particular, $G(\mathbf{r}, \mathbf{r}')$ will typically diverge if $f(\mathbf{r} \to 0)$ is also divergent. Clearly, in such situations, one must introduce an appropriate regularization to exclude self-interactions (see, for example, the case of the Coulomb interaction below). More generally, a divergence of $G(\mathbf{r}, \mathbf{r}')$ signals the breakdown of the self-consistent approximation scheme, and the applicability of the present approach. In this case one must seek an alternative route to explore the nature of the effective action.

Applying the trial solution $X = a\Lambda$ to the saddle-point equation, one obtains the following equation for a,

$$1 = \frac{S_d}{(2\pi)^d} \frac{\rho}{\bar{\epsilon}^2} \int_0^\infty dq \frac{q^{d-1}}{(f^{-1}(q) - \rho/\bar{\epsilon})^2 + (\rho a/\bar{\epsilon})^2},$$

where S_d is the d-dimensional solid angle. Since $a \ll 1$, one can expect the integrand to exhibit a pole (or set of poles) close to $f^{-1}(q_0) = \rho/\bar{\epsilon}$. (For simplicity, let us assume that the integral is dominated by a single pole.) If the resonance here is 'sharp', i.e. if the ratio of q_0 to the bandwidth of the resonance is much greater than one, we can make the approximation $f^{-1}(q) \approx f^{-1}(q_0) + (q - q_0)\partial f^{-1}/\partial q|_{q_0}$, and also extend the lower limit of this integral to $-\infty$. In doing so, one obtains the pole approximation

$$a = \frac{\pi S_d}{(2\pi)^d} \frac{q_0^{d-1}}{\bar{\epsilon}} \left| \left. \frac{\partial f^{-1}}{\partial q} \right|_{q_0} \right|^{-1} \ll 1. \tag{11}$$

The sharp resonance condition then corresponds to the limit,

$$\frac{S_d}{(2\pi)^{d-1}} \frac{\rho q_0^{d-2}}{\bar{\epsilon}^2} \left(\left. \frac{\partial f^{-1}}{\partial q} \right|_{q_0} \right)^{-2} \ll 1. \tag{12}$$

The position of the pole in Eq (10) implies that $G_{\rm sp}(\mathbf{r}) \sim e^{-r/2\ell}$, where the effective "mean free path" $\ell = (\bar{\epsilon} L^d/2a\alpha)(\partial f^{-1}/\partial q)_{q_0}$, so that the sharp resonance limit is equivalent to the semi-classical limit $q_0 \ell \gg 1$.

As with the disordered metal, the matrix structure of the saddle-point equation results in a manifold of degeneracy for $Q_{\rm sp}$ for $\hat{\omega} = 0$. In particular, it is simple to check that, by subjecting the solution $Q_{\rm sp} \equiv X = a\Lambda$ to a rotation $Q \mapsto TQ_{\rm sp}T^{-1}$, where T represents an arbitrary superunitary rotation, homogeneous in space, the saddle-point equation is left invariant. Later, in section V, by taking into account fluctuations around the saddle-point manifold we will obtain a low-energy effective field theory for the Euclidean matrix system. However, before doing so, we will explore the validity and ramifications of the self-consistent Born approximation established above.

IV. DENSITY OF STATES AND EIGENVECTOR CORRELATIONS

To construct the low-energy action, it is convenient to subject the matrices Q to the rescaling $Q \mapsto aQ$, such that the saddle-point manifold $Q_{\rm sp}^2 = \mathbb{1}$ is parameterized by $Q_{\rm sp} = T\Lambda T^{-1}$. When investigating the ensemble average of the one-point function this reduces to a single point $Q_{\rm sp} = \Lambda$ (since $\Lambda \propto \mathbb{1}$). This makes the study of the average DoS an ideal test of the validity of the self-consistent Born approximation. To this end, let us begin by considering the Green function of the Euclidean matrix,

$$\hat{g}_N = \frac{1}{\bar{\epsilon} - \hat{f}} = \mathrm{str} \left[iE_{11}^{\rm BF} \left(\frac{\partial}{\partial \bar{j}} \otimes \frac{\partial}{\partial j} \right) \langle \mathcal{Z}_N \rangle \Big|_{j=0} \right].$$

With this definition, the DoS of the matrix f_{ij} can be obtained as,

$$\nu(\bar{\epsilon}) = \frac{1}{\pi} \mathrm{tr}\, \mathrm{Im}\, \hat{g}_N(\bar{\epsilon} - i0)$$

Expressed in terms of $j(\mathbf{r})$ using Eq. (5), this becomes,

$$\nu(\bar{\epsilon}) = \frac{1}{\pi \delta^d(\mathbf{r}=0)} \mathrm{Im} \int d\mathbf{r}\, \mathrm{str} \left[iE_{11}^{\rm BF} \Lambda \left(\frac{\delta}{\delta \bar{j}(\mathbf{r})} \otimes \frac{\delta}{\delta j(\mathbf{r})} \right) \langle \mathcal{Z} \rangle \Big|_{j=0} \right].$$

If the functional derivatives are applied to the first of the two terms in (8b), the leading contribution will generate a factor of $\delta^d(\mathbf{r}=0)$ which cancels with that in the denominator, so that the expression as a whole is finite. On the other hand, if the contraction is applied to the second of the source terms there will be an overall factor of $1/\delta^d(\mathbf{r}=0)$ which can be neglected in the thermodynamic limit. Altogether, one obtains

$$\nu(\bar{\epsilon}) = \frac{a\rho}{\pi\bar{\epsilon}} \left\langle \mathrm{Re} \int d\mathbf{r}\, \mathrm{str}\, [E_{11}^{\mathrm{BF}} \otimes E_{11}^{\mathrm{TR}} \Lambda Q] \right\rangle_Q, \tag{13}$$

where the projection matrix E_{ij}^{TR} takes the value of unity on the element ij in the TR-space, and zero on all other elements. Substituting the saddle-point solution $Q_{\mathrm{sp}} = \Lambda$ into this equation one obtains,

$$\nu(\bar{\epsilon}) = \frac{a\alpha}{\pi\bar{\epsilon}}, \tag{14}$$

and from which we obtain the average level spacing $\Delta(\bar{\epsilon}) = 1/\nu(\bar{\epsilon})$. (As usual, in the semi-classical limit where the pole approximation is valid, the average DoS remains unaffected by field fluctuations around the saddle-point.) To test the validity of this result, let us consider its application to several examples.

A. Gaussian Potential

As a first example, let us consider the Gaussian potential,

$$f(\mathbf{r}) = \epsilon_0 \exp\left(-\frac{\mathbf{r}^2}{2\sigma^2}\right), \qquad f^{-1}(\mathbf{q}) = \frac{1}{\epsilon_0 (2\pi\sigma^2)^{d/2}} \exp\left(\frac{\sigma^2 \mathbf{q}^2}{2}\right),$$

which is the particular case studied in Ref. [1]. Poles q_0 are solutions to the equation $f^{-1}(q_0) = \rho/\bar{\epsilon}$. Since f^{-1} has a minimum at $q = 0$, this equation can only be satisfied if $\bar{\epsilon}$ is in the interval $0 < \bar{\epsilon} < \epsilon_{\max} \equiv \epsilon_0 \rho (2\pi\sigma^2)^{d/2}$; outside of this region there is no support for the DoS, i.e. the 'Fermi energy' $\rho/\bar{\epsilon}$ is within an 'energy gap'. Within this interval we find,

$$q_0 = \left(\frac{2}{\sigma^2} \ln\left[\frac{\epsilon_{\max}}{\bar{\epsilon}}\right]\right)^{1/2},$$

and hence, applying Eq. (11),

$$a = \frac{\pi S_d \epsilon_0}{2\pi^{d/2} \epsilon_{\max}} \left(\ln\left[\frac{\epsilon_{\max}}{\bar{\epsilon}}\right]\right)^{(d-2)/2} \ll 1.$$

Thus we find that, in d-dimensions, the average DoS takes the form,

$$\nu(\bar{\epsilon}) = \frac{1}{\bar{\epsilon}} \frac{S_d \alpha \epsilon_0}{2\pi^{d/2} \epsilon_{\max}} \left(\ln\left[\frac{\epsilon_{\max}}{\bar{\epsilon}}\right]\right)^{(d-2)/2}. \tag{15}$$

The requirement that resonance is sharp (12) translates to the condition, $a \ll \ln[E_{\max}/\bar{\epsilon}]$.

In two-dimensions, these results, which reduce to the form,

$$\nu(\bar{\epsilon}) = \frac{(L/\sigma)^2}{2\pi\bar{\epsilon}}, \qquad \frac{1}{2\rho\sigma^2} \ll 1, \qquad \bar{\epsilon} \lesssim \epsilon_{\max}$$

are compared in Fig. 1 with measurements taken from numerical simulation. Over a wide range of the spectrum a good agreement between theory and numerics is observed. In three-dimensions the DoS reduces to the form

$$\nu(\bar{\epsilon}) = \frac{(L/\sigma)^3}{\sqrt{2}\pi^2\bar{\epsilon}} \left(\ln\left[\frac{\epsilon_{\max}}{\bar{\epsilon}}\right]\right)^{1/2},$$

and coincides with the result obtained in Ref. [1] for a high density expansion (with $\sigma = 1$, $\epsilon_0 = (2\pi)^{-3/2}$, and setting '$P(\epsilon) = \nu(\epsilon)/\alpha$'). Again, comparison with numerics shows good agreement where the theory is valid (Fig. 2).

B. Exponential Potential

As a second example, we consider a potential in which the interaction decays exponentially with separation,

$$f(\mathbf{r}) = \epsilon_0 e^{-\kappa|\mathbf{r}|}.$$

Solving the self-consistent equation, and making use of Eq. (11), in one dimension one obtains

$$\nu(\bar{\epsilon}) = \frac{\epsilon_0 \alpha}{\pi \bar{\epsilon}^2} \left(\frac{\epsilon_{\max}^{(1)}}{\bar{\epsilon}} - 1 \right)^{-1/2},$$

valid in the interval $\epsilon_0 \kappa / 2\rho \ll \bar{\epsilon} \ll \epsilon_{\max}^{(1)} \equiv 2\epsilon_0 \rho/\kappa$. Meanwhile, in two-dimensions, within the energy interval $\pi \epsilon_0 \kappa/(54\rho)^{1/2} \ll \bar{\epsilon} \ll \epsilon_{\max}^{(2)} \equiv 2\pi\epsilon_0 \rho/\kappa^2$, the DoS is given by,

$$\nu(\bar{\epsilon}) = \frac{\epsilon_0 \alpha}{3 \bar{\epsilon}^2} \left(\frac{\epsilon_{\max}^{(2)}}{\bar{\epsilon}} \right)^{-1/3}.$$

In both cases, over the range of validity of the expansion, agreement with numerical simulation is good (see Figs. 3 and 4).

C. Coulomb Potential

As a final example, let us consider the Coulomb potential in three-dimensions,

$$f(\mathbf{r}) = \frac{\epsilon_0 r_0}{|\mathbf{r}|}.$$

In this case, an application of the self-consistent theory above obtains,

$$\nu(\bar{\epsilon}) = \frac{4}{\sqrt{\pi}} \left(\frac{\epsilon_0 r_0 \alpha}{L} \right)^{3/2} \frac{1}{\bar{\epsilon}^{5/2}}.$$

Note, however, that the real part of expression (10) diverges at the upper limit of the momentum integration for this choice of potential, and moreover that the particles have an infinite self-energy — some kind of regularization[1] is required if the result above is to be valid. A simple solution is to redefine the potential to be,

$$f(\mathbf{r}) = 4\pi\epsilon_0 r_0 \int^{q_{\max}} \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2}.$$

where $q_{\max}$ is set by the condition $\int^{q_{\max}} d^3\mathbf{q}/(2\pi)^3 = \rho$. The only effect of this is to limit the potential at short distances, so that $f(0)$ remains finite. This will not affect our result so long as $q_0 < q_{\max}$, or $\bar{\epsilon} > \epsilon_0 r_0 (16\rho/9\pi)^{1/3}$. Furthermore, the conditions for the validity of our approximations, including that Y, the imaginary part of Q, remains small, are all satisfied if $\bar{\epsilon} \gg \epsilon_0 r_0 (16\pi\rho)^{1/3}$. In this limit, the numerics are in agreement (Fig. 5).

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[1]Despite the fact that the model is strictly not renormalizable [11], the regularization scheme employed here allows us to stay within a domain in which the pole approximation is legitimate.

D. Eigenvector correlations

The calculation of the DoS requires only the first of the two terms in the source action (8b), but there are also quantities which depend on the second term. An example of these is the eigenvector correlation function $C(\mathbf{r}) = \langle v_\alpha(\mathbf{r}_0) v_\alpha(\mathbf{r}_0 + \mathbf{r})\rangle$, where $v_\alpha(\mathbf{r})$ is the component of the α'th normalized eigenvector at position $\mathbf{r}$, assuming this to be one of the coordinates $\{\mathbf{r}_i\}$. This can be obtained from the one-point function using, for $\mathbf{r} \neq 0$,

$$\begin{aligned}
\nu\rho^2 C(\mathbf{r}) &= \sum_{ij}\sum_{\alpha} \langle i|\alpha\rangle\langle\alpha|j\rangle \delta^d(\mathbf{r}_0 - \mathbf{r}_i)\delta^d(\mathbf{r}_0 + \mathbf{r} - \mathbf{r}_j)\delta(E - E_\alpha) \\
&= \frac{1}{\pi}\mathrm{Im}\sum_{ij}\langle i|\hat{g}_N(\bar{\epsilon} - i0)|j\rangle \delta^d(\mathbf{r}_0 - \mathbf{r}_i)\delta^d(\mathbf{r}_0 + \mathbf{r} - \mathbf{r}_j) \\
&= \frac{1}{\pi}\mathrm{Im}\,\mathrm{str}\left[iE_{11}^{\mathrm{BF}}\Lambda \frac{\delta}{\delta\bar{j}(\mathbf{r}_0)} \otimes \frac{\delta}{\delta j(\mathbf{r}_0 + \mathbf{r})}\langle Z\rangle\right].
\end{aligned}$$

This depends only on the second of the source terms. Meanwhile $C(0) = \langle v_\alpha(\mathbf{r}_0) v_\alpha(\mathbf{r}_0)\rangle = 1/\alpha$ is fixed by the normalization. Hence,

$$C(\mathbf{r}) = \begin{cases} \frac{1}{\alpha}, & \mathbf{r} = 0 \\ \frac{\Lambda}{\alpha a \bar{\epsilon}}[(1 - a^2)\mathrm{Im}\, G(\mathbf{r}) + 2aQ_{\mathrm{sp}}\mathrm{Re}\, G(\mathbf{r})], & \mathbf{r} > 0 \end{cases} \tag{16}$$

where $G(\mathbf{r}) = G_{\mathrm{sp}}(0, \mathbf{r})$ defined by Eq. (10).

In the limit $a \to 0$ the terms higher order in a can be neglected, and then $C(\mathbf{r}) = (\Lambda/\alpha a \bar{\epsilon})\mathrm{Im}\, G(\mathbf{r})$ is continuous at $\mathbf{r} = 0$. However, it is infeasible in practice to numerically diagonalize matrices which are very large, and this can constrain how small a can be made when comparing the theory with numerical results. In that case, the other terms in (16) can become significant, and result in a discontinuity at $\mathbf{r} = 0$. This is apparent in Fig (6), where this comparison is made for the Gaussian potential of section IV A.

E. Discussion

In general, the domain of applicability of the self-consistent Born approximation defines the same interval over which the quartic expansion of the action (7) is legitimate. The breakdown of the saddle-point approximation requires a reconsideration of the full Euclidean field theory action. In principle, one might still find that the extrema of the spectra are dominated by saddle-point or mean-field configurations in which the Q_{sp} assume instanton-like configurations reflecting optimal configurations of the coordinates [11]. As with usual stochastic Hamiltonians, the analysis of the DoS for different interaction potentials above confirms that one-point properties of the Euclidean matrix ensembles are non-universal. Staying within the domain of applicability of the quartic expansion, in the following section we will investigate the role of massless fluctuations of the fields Q on spectral correlations involving many-point functions.

V. FLUCTUATIONS AND THE σ-MODEL

The degenerate manifold of saddle-point solutions implies the existence of massless or Goldstone mode fluctuations of the fields Q. The effective low-energy field theory of these fluctuations can be found by expanding the action (8a) about the saddle point solution, by setting $Q(\mathbf{r}) = Q_{\rm sp} + \delta Q(\mathbf{r})$, where δQ can be divided into transverse and longitudinal fluctuations. Transverse fluctuations δQ_t are those which anticommute with the saddle point $Q_{\rm sp}$, and hence remain on the saddle point manifold — they represent the Goldstone modes. Longitudinal fluctuations δQ_l, on the other hand, commute with $Q_{\rm sp}$, and cause Q to move off the manifold; consequently these modes are massive. Applying this expansion, one obtains,

$$G^{-1}(\mathbf{r},\mathbf{r}') = G^{-1}_{\rm sp}(\mathbf{r},\mathbf{r}') + \frac{\rho}{\bar\epsilon}\left(\frac{\hat\omega}{\bar\epsilon} - ia\delta Q(\mathbf{r})\right)\delta(\mathbf{r}-\mathbf{r}'),$$

and the total effective action takes the form,

$$\begin{aligned} S = \frac{a^2\alpha}{4}\int\frac{d\mathbf{r}}{L^d}{\rm str}(\delta Q^2) - \frac{i\alpha a}{2\bar\epsilon}\int\frac{d\mathbf{r}}{L^d}{\rm str}(Q\hat\omega) \\ - \left(\frac{\alpha a}{2\bar\epsilon}\right)^2\int\frac{d\mathbf{r}}{L^d}\int\frac{d\mathbf{r}'}{L^d}{\rm str}\left[G_{\rm sp}(\mathbf{r}',\mathbf{r})\delta Q(\mathbf{r})G_{\rm sp}(\mathbf{r},\mathbf{r}')\delta Q(\mathbf{r}')\right] + \ldots \end{aligned}$$

valid in the diffusive limit $\hat\omega \ll a\bar\epsilon$. In this limit fluctuations of Q vary slowly over the length scale ℓ on which G decays, so that we can employ the gradient expansion,

$$\delta Q(\mathbf{r}') = \delta Q(\mathbf{r}) + \partial_\alpha\delta Q(\mathbf{r})\cdot(\mathbf{r}'-\mathbf{r})_\alpha + \frac{1}{2}\partial_\alpha\partial_\beta\delta Q(\mathbf{r})\cdot(\mathbf{r}'-\mathbf{r})_\alpha(\mathbf{r}'-\mathbf{r})_\beta + \ldots$$

Note that if we have a long-ranged potential we must take more care, as an action local in the fields Q may not be adequate (see Ref. [9]). Assuming that this is not the case, we proceed by applying again the sharp resonance condition to evaluate the integral,

$$\int\frac{d\mathbf{r}}{L^d}G_{\rm sp}(\mathbf{r})AG_{\rm sp}(\mathbf{r})B = \frac{\bar\epsilon^2}{2\alpha}(AB - Q_{\rm sp}AQ_{\rm sp}B).$$

Hence to zeroth order in the gradient expansion the transverse fluctuations are confirmed to be massless, while the longitudinal fluctuations have a mass and can be neglected in the limits $a^2\alpha \gg 1$ and $\hat\omega \ll a\bar\epsilon$. The first non-vanishing contribution from δQ_t occurs at second order in the gradient expansion; again this integral can be evaluated in the sharp resonance limit, where one obtains the identity,

$$\int\frac{d\mathbf{r}}{L^d}G_{\rm sp}(\mathbf{r})\mathbf{C}.\mathbf{r}G_{\rm sp}(\mathbf{r})\mathbf{C}.\mathbf{r} = \frac{\pi S_d}{2(2\pi)^d d}\frac{q_0^{d-1}}{L^d}\left(\frac{\bar\epsilon L^d}{a\alpha}\right)^3\left|\left.\frac{\partial f^{-1}}{\partial q}\right|_{q_0}\right|\mathbf{C}.\mathbf{C},$$

for matrices $\mathbf{C}$ which anticommute with $Q_{\rm sp}$.

Altogether, making use of the mean-field expression for the ensemble averaged DoS (14), we obtain the non-linear σ-model action:

$$S[Q] = \frac{\pi\nu}{8}\int\frac{d\mathbf{r}}{L^d}\mathrm{str}\left[\hbar D\left(\partial Q\right)^2 - 4i\hat{\omega}Q\right], \tag{17}$$

where supermatrices Q span the coset manifold $Q = T\Lambda T^{-1}$, and

$$D = \frac{\bar{\epsilon}^2 L^{2d}}{2\pi\hbar\nu\alpha d}\left(\left.\frac{\partial f^{-1}}{\partial q}\right|_{q_0}\right)^2,$$

denotes the effective diffusion constant. The length scale on which Q varies is given by $\sqrt{\hbar D/\omega} \gg \ell$, justifying self-consistently the gradient expansion. Meanwhile the action for the sources, S_{J}, takes the form,

$$\begin{aligned}S_{\mathrm{J}} = &\frac{i\alpha}{\bar{\epsilon}}\int\frac{d\mathbf{r}}{L^d}\bar{J}(\mathbf{r})\left(1 + iaQ(\mathbf{r})\right)J(\mathbf{r})\\ &+\frac{i\alpha^2}{\bar{\epsilon}^2}\int\frac{d\mathbf{r}}{L^d}\frac{d\mathbf{r}'}{L^d}\bar{J}(\mathbf{r})\left(1 + iaQ(\mathbf{r})\right)G(\mathbf{r},\mathbf{r}')\left(1 + iaQ(\mathbf{r}')\right)J(\mathbf{r}').\end{aligned}$$

As expected, within its domain of applicability, spectral fluctuations of the Euclidean random matrix ensembles are described by a diffusive supersymmetric non-linear σ-model. From this result we can immediately draw conclusions about the localization properties of the states. In particular, in dimensions $d \leq 2$ all states are localized, while in dimensions above two the system admits the existence of an Anderson transition from localized to delocalized behaviour. Similarly, at energy scales $\Delta, \omega \ll E_c \equiv \hbar D/L^2$, the action becomes dominated by the zero spatial mode contribution,

$$S[Q_0] = -\frac{i\pi}{2\Delta(\bar{\epsilon})}\mathrm{str}\left[\hat{\omega}Q_0\right].$$

In this limit, spectral correlations become universal, independent of the nature of potential $f(\mathbf{r})$, and coincident with those of, say, Gaussian distributed random matrix ensembles.

To be concrete, as an example let us consider the two-point correlator of DoS fluctuations,

$$R_2 = \langle\nu\rangle^{-2}\langle\nu(\bar{\epsilon}+\Omega/2)\nu(\bar{\epsilon}-\Omega/2)\rangle - 1.$$

By making use of the approximation $\langle\hat{g}_N(\epsilon_1 - i0)\hat{g}_N(\epsilon_2 - i0)\rangle \simeq \langle\hat{g}_N(\epsilon_1 - i0)\rangle\langle\hat{g}_N(\epsilon_2 - i0)\rangle$, implied by the analytical properties of the Green function, we have the identity

$$\langle\nu(\bar{\epsilon}+\Omega/2)\nu(\bar{\epsilon}-\Omega/2)\rangle \simeq \left(\frac{1}{2\pi i}\right)^2\mathrm{Re}\left\langle\left(\mathrm{tr}\left[\hat{g}_N(\bar{\epsilon}-\Omega/2-i0) - \hat{g}_N(\bar{\epsilon}+\Omega/2+i0)\right]\right)^2\right\rangle.$$

Making use of the generating function derived above, one obtains

$$R_2(\Omega) = \frac{1}{64}\mathrm{Re}\left\langle\left(\int\frac{d\mathbf{r}}{L^d}\mathrm{str}(\Lambda\otimes\sigma_3^{\mathrm{BF}}Q)\right)^2\right\rangle_Q - 1.$$

With the corresponding two-point matrix source,

$$\hat{\omega} = -(\Omega/2 + i0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

this result can be identified with the standard integral expression for the DoS correlator [8].

VI. DISCUSSION

This completes our survey of the properties of the Euclidean matrix ensembles. While the analysis above shows that the DoS is non-universal, depending sensitively on the nature of the interaction potential $f(\mathbf{r})$, at intermediate energy scales the bulk many-particle correlations of the system coincide with those of weakly disordered conductors. In particular, as well as the diffusive character of density relaxation, the Euclidean system inherits the localization properties of the disordered metal. Moreover, the spectral properties exhibit characteristic fluctuation phenomena and universal low-energy spectral correlations characteristic of random matrix ensembles. In summary, the irregular lattice structure of the Hamiltonian becomes visible only in the thermodynamic single-particle properties of the system.

Although the low-energy effective field theory (17) becomes invalid in the vicinity of the extrema of the spectra, the fundamental action (6) on which it is based is exact. Further investigation of the low-energy properties of the Euclidean matrix ensembles in the vicinity of the band edge requires a fundamentally different non-perturbative approach to the effective action whose basis depends on the particular nature of the interaction $f(\mathbf{r})$ (see Ref. [11]). The development of such theories remain the subject of future research.

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TABLES

Disordered metal		Euclidean RMT
Hamiltonian	$E_F - \frac{\mathbf{p}^2}{2m}$	$f^{-1}(q) - \frac{\rho}{\bar{\epsilon}}$
Frequency source	$\hat{\omega} = \hat{\epsilon} - E_F$	$\frac{\rho\hat{\omega}}{\bar{\epsilon}^2}$
Variance of random potential	$\frac{\hbar}{2\pi\nu\tau}$	$\frac{\rho}{\bar{\epsilon}^2}$
DoS at Fermi energy	$\nu \equiv \frac{S_d p_F^{d-1}}{(2\pi\hbar)^d}\left[\frac{d}{dp}\left(\frac{p^2}{2m}\right)\right]_{p_F}^{-1}$	$\frac{a\bar{\epsilon}}{\pi} = \frac{S_d q_0^{d-1}}{(2\pi)^d}\left[\frac{d}{dq}f^{-1}(q)\right]_{q_0}^{-1}$

TABLE I. Mapping from the disordered metal to Euclidean RMT for parameters entering the action. Note that corresponding quantities have quite different physical interpretations.

	Disordered metal	Euclidean RMT
Level spacing Δ	$\frac{1}{\nu L^d}$	$\frac{\pi\bar{\epsilon}}{a\alpha}$
Thouless energy E_c	$\frac{\hbar D}{L^2} = \frac{\hbar\tau}{L^2 d}\left[\frac{d}{dp}\left(\frac{p^2}{2m}\right)\right]_{p_F}^{2}$	$\frac{\bar{\epsilon}^3}{2a\rho^2 L^2 d}\left[\frac{d}{dq}f^{-1}(q)\right]_{q_0}^{2}$
Scattering energy	$\frac{\hbar}{\tau}$	$2a\bar{\epsilon}$
Mean free path ℓ	$\left[\frac{d}{dp}\left(\frac{p^2}{2m}\right)\right]_{p_F}\tau$	$\frac{\bar{\epsilon}}{2\rho a}\left[\frac{d}{dq}f^{-1}(q)\right]_{q_0}$

TABLE II. Comparison of various energy and length scales between the disordered metal and Euclidean RMT.

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FIGURES

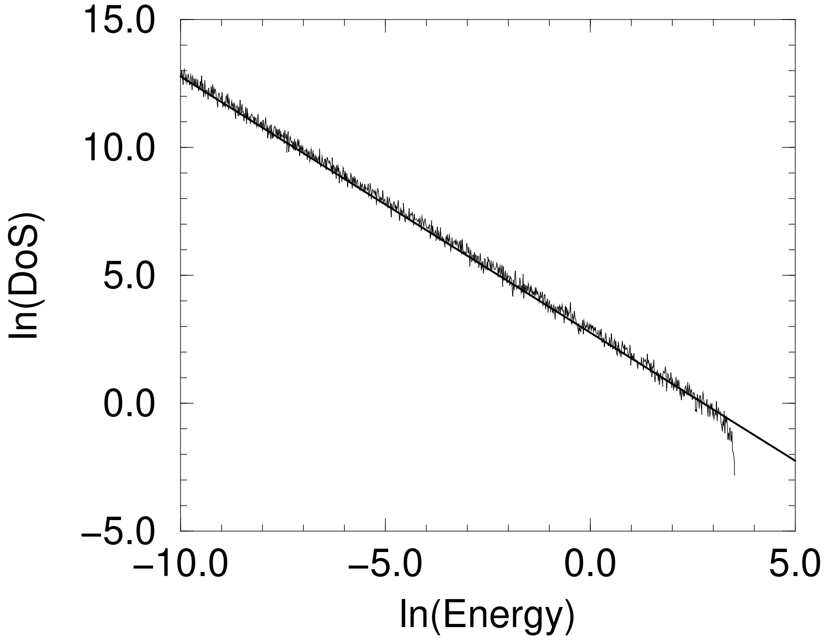

FIG. 1. Log-log plot of the DoS for the Gaussian potential in two dimensions, with $\alpha = 500$, $L = 1$, $\sigma = 0.1$, $\epsilon_0 = 1$. Data from numerical simulation is compared to the theoretical prediction taken from the text.

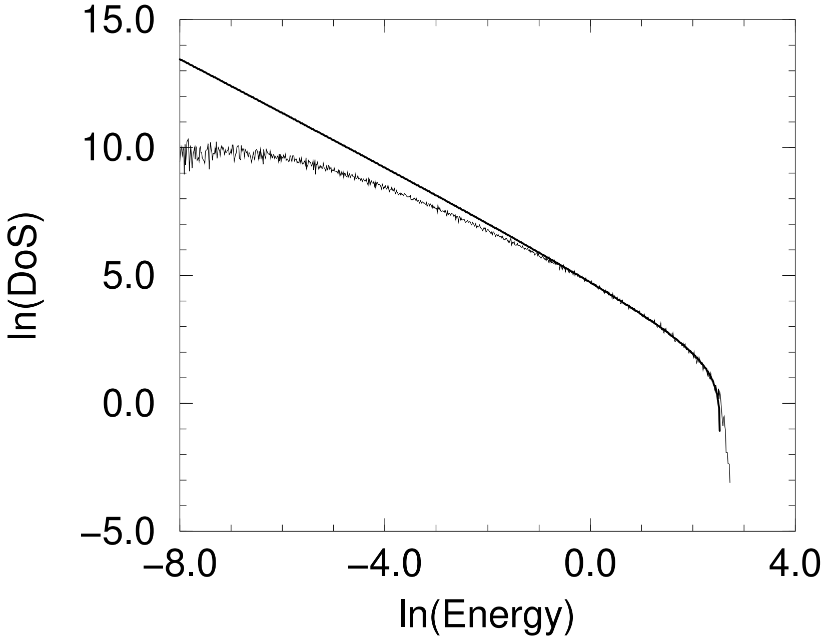

FIG. 2. Log-log plot of the DoS for the Gaussian potential in three dimensions, with $\alpha = 800$, $L = 1$, $\sigma = 0.1$, $\epsilon_0 = 1$. Data from numerical simulation is compared to the theoretical prediction taken from the text.

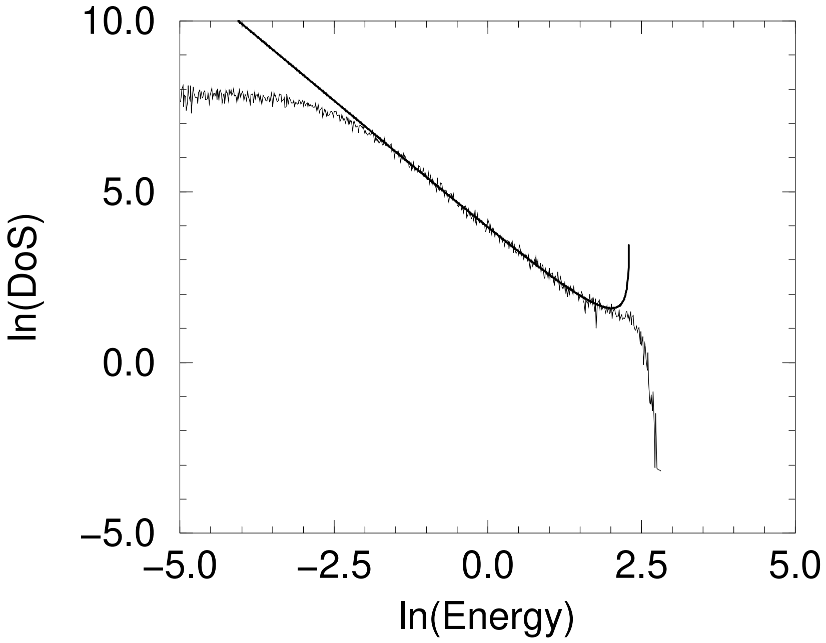

FIG. 3. Log-log plot of the DoS for the exponential potential in one dimension, with $\alpha = 500$, $L = 1$, $\kappa = 100$, $\epsilon_0 = 1$. Data from numerical simulation is compared to the theoretical prediction taken from the text.

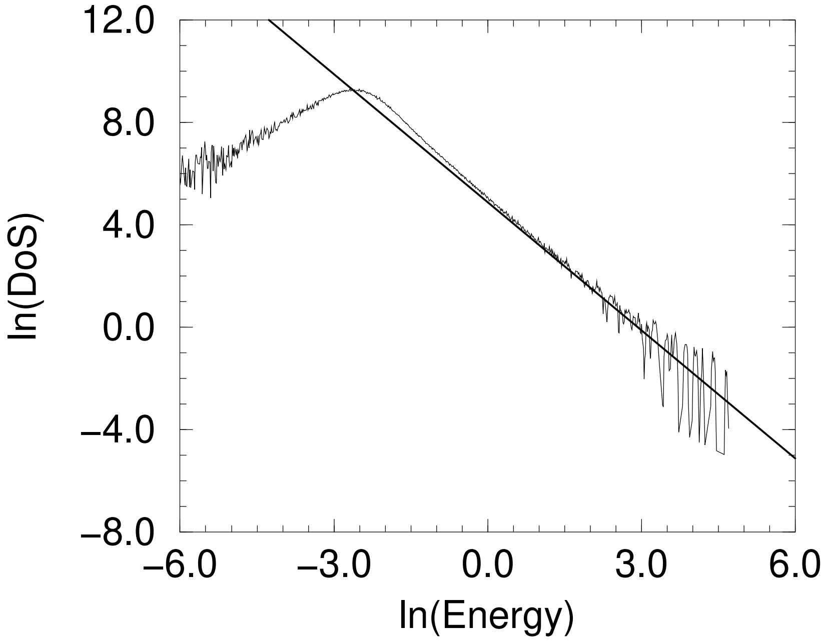

FIG. 4. Log-log plot of the DoS for the exponential potential in two dimensions, with $\alpha = 2000$, $L = 1$, $\kappa = 10$, $\epsilon_0 = 1$. Data from numerical simulation is compared to the theoretical prediction taken from the text.

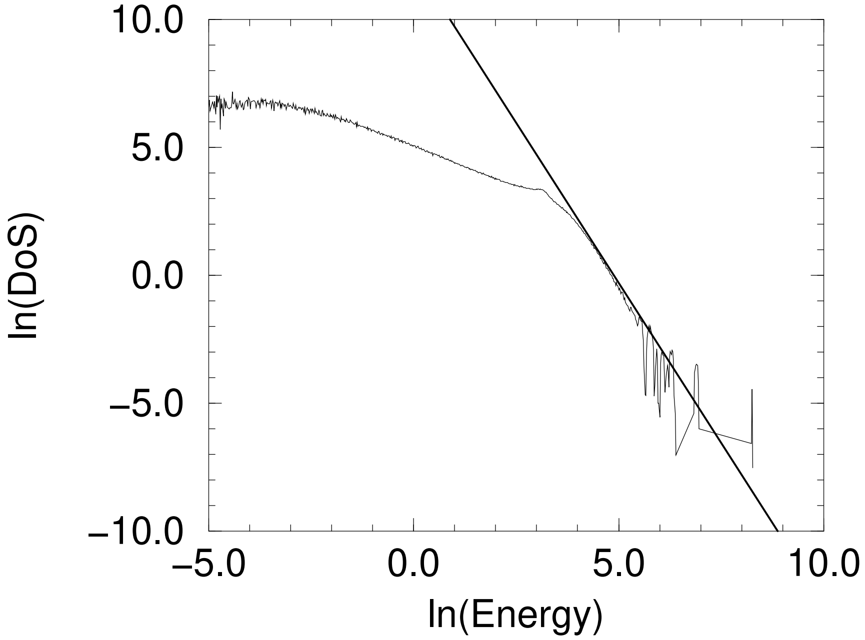

FIG. 5. Log-log plot of the DoS for the regularized coulomb potential in three dimensions, with $\alpha = 2000$, $L = 1$, $r_0 = 1$, $\epsilon_0 = 1$. Data from numerical simulation is compared to the theoretical prediction taken from the text.

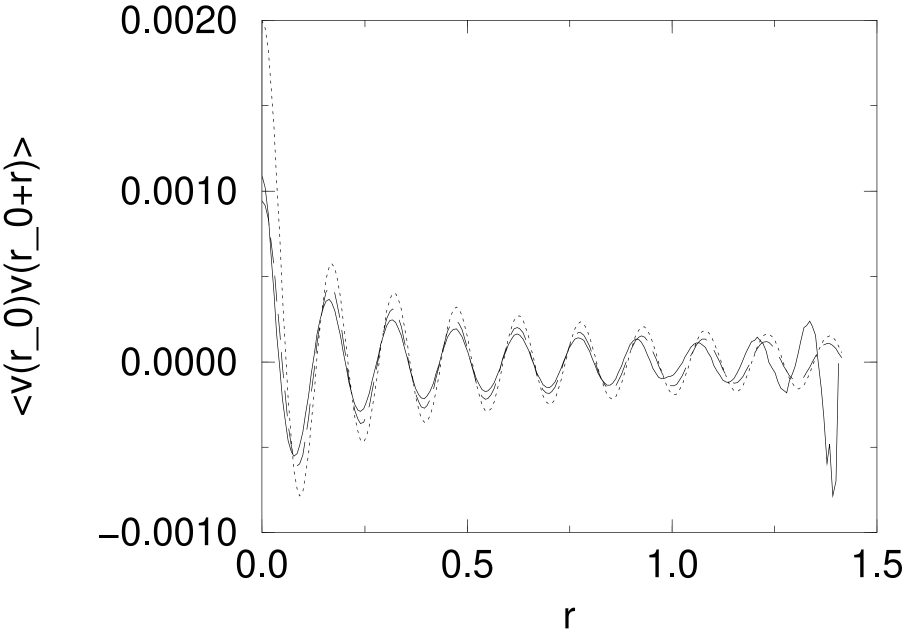

FIG. 6. Plot of the eigenvector correlation function $C(\mathbf{r})$ for the Gaussian potential in two dimensions, with $\alpha = 500$, $L = 1$, $\sigma = 0.1$ and $\epsilon_0 = 1$, and at energy $\bar{\epsilon} = 0.006$. A numerical simulation (solid line) is compared to both the theory assuming $a \to 0$ (dotted line; theory predicts $a = 0.1$), and also Eq. (16) evaluated at finite a (dashed line; numerical solution of the saddle point equation gives $a = 0.129$). At large r effects due to the finite size of the system are apparent.