# Mesoscopic Physics

Smaller is different

1. Theories of Anderson localization
2. Weak localization: theory and experiment
3. Universality and Random Matrix Theory
4. Metal-Insulator Transitions
5. Mesoscopic physics beyond condensed matter

**References:**

**1. Thouless, Phys. Rep. 13, 93 (1974).**
**2. Lee, Ramkrishnan, Rev. Mod. Phys. 57,387 (1985) .**
**3. Kramer, MacKinnon, Rep. Prog. Phys. 56, 1469 (1993).**
**4. Boris Altshuler, Boulder Colorado Lectures**
**http://boulder.research.yale.edu/Boulder-2005/Lectures/index.html**

What is mesoscopic physics?

1. Interference\tunneling effects in a solid.
2. These effects usually occur at intermediate scales and at relatively low temperatures.
3. Disorder plays a role in most materials.

Why is mesoscopic physics interesting?

1. Reveals universal features of quantum physics.
2. Continuation of quantum mechanics.
3. Technological applications.

Are 4+1 lectures enough?

Problems are easy to understand but difficult to solve rigorously.

# Lecture I: From Anderson to Anderson: perturbative formalism and scaling theory of localization

1. Intuition about quantum dynamics in a disordered potential. Anderson localization

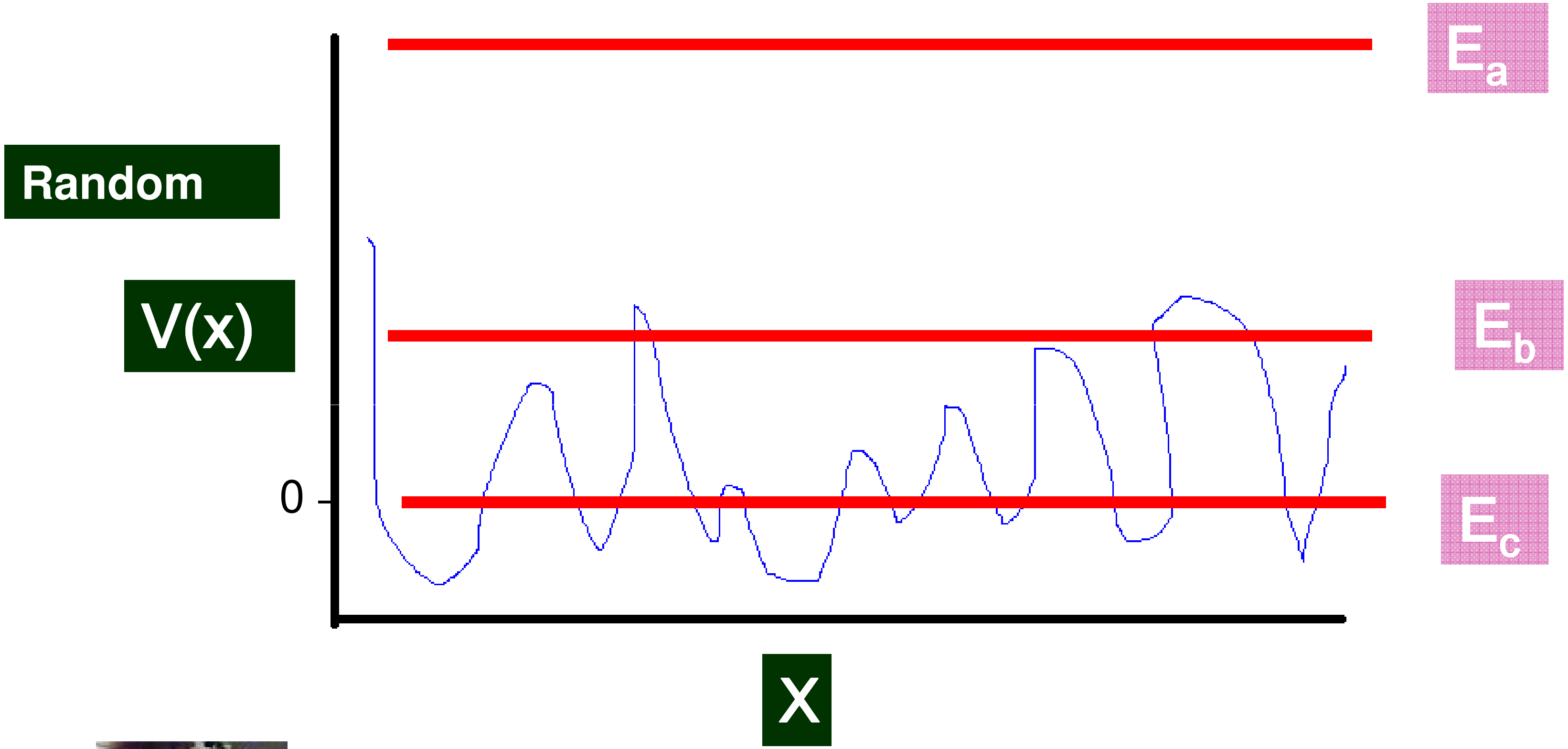
2. Theories of localization: Locator expansions

a. Anderson 1957: “Absence of diffusion in certain random lattices”

b. Anderson, Abou-Chacra, Thouless, 1973: “A self-consistent theory of localization”

3. Abrahms, Anderson, et al., 1979: “Scaling theory of localization”

# Your intuition about localization



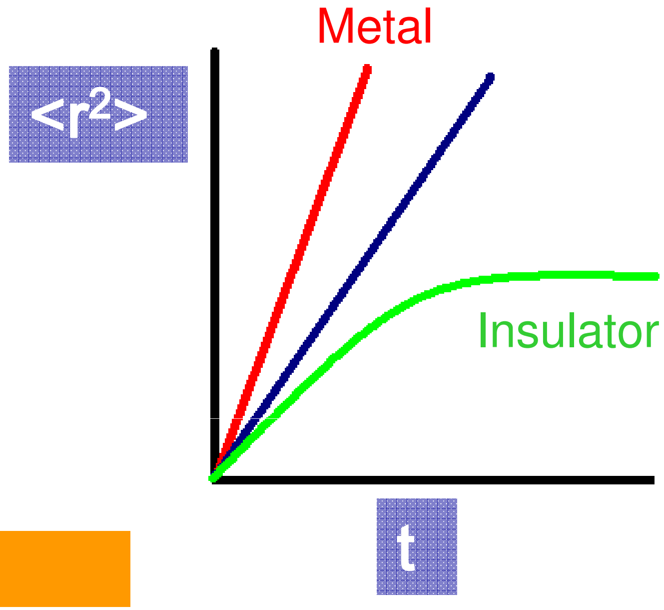

P. W. Anderson

Will the classical motion be strongly affected by quantum effects?

$$\hat{H}\Psi_\alpha = E_\alpha \psi_\alpha$$

$$\hat{H} = -\frac{\nabla^2}{2m} + V(\vec{r})$$

$$\langle V(\vec{r})V(\vec{r}\,')\rangle = V_0^{\,2}\delta(\vec{r}-\vec{r}\,')$$




**Metal**

$$\langle r^2(t)\rangle \underset{t\to\infty}{\propto} t$$

Abs. Continuous

$$|\Psi_\alpha(r)| \propto 1/\sqrt{V}$$

$k_F l >> 1$

$$\langle P(t)\rangle \underset{t\to\infty}{\to} 0$$

**Insulator**

$$\langle r^2(t)\rangle \underset{t\to\infty}{\propto} const$$

Pure point spectrum

$$|\Psi_\alpha(r)| \propto e^{-r/\xi_{loc}}$$

$k_F l > 1$

$$\langle P(t)\rangle \underset{t\to\infty}{\to} cons$$

# Theories of localization

Locator expansions

One parameter scaling theory

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

# Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
*Bell Telephone Laboratories, Murray Hill, New Jersey*

**6203 citations!**

## What if I place a particle in a random potential and wait?

**Tight binding model**

$\varepsilon_i \in [-W, W]$

$$\varepsilon_i a_i^\alpha + \sum_j V_{ij} a_j^\alpha = E^\alpha a_i^\alpha$$

1. (Locator) expansion around V=0

2. Probability distribution needed

3. At $V=V_c$ perturbation breaks down $\rightarrow$ Metal

$$\left(\mathrm{i}\frac{\partial}{\partial t}-\epsilon_i\right)\mathcal{G}_{ij}(t)-\sum_k V_{ik}\,\mathcal{G}_{kj}(t)=\mathrm{i}\delta_{ij}\,\delta(t)$$

$$\mathcal{G}_{ij}(t)=\sum_\alpha a_i^{(\alpha)}\,a_j^{(\alpha)*}\exp(-\mathrm{i}E_\alpha t)\,,$$

$$(E-\epsilon_i)\,G_{ij}(E)-\sum_k V_{ik}\,G_{kj}(E)=\delta_{ij}\,.$$

$$G_{ij}(E)=\sum_\alpha a_i^{(\alpha)}\,a_j^{(\alpha)*}\,(E-E_\alpha)^{-1}\,.$$

$$n_i(E)=-\pi^{-1}\lim_{\eta\to 0+}\operatorname{Im}G_{ii}(E+\mathrm{i}\eta)$$

$$\lim_{t\to\infty}|\mathcal{G}_{00}(t)|^2=\lim_{u\to 0}u\int_0^\infty|\mathcal{G}_{00}(t)|^2\,\mathrm{e}^{-ut}\,\mathrm{d}t=\lim_{u\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}u\,G_{00}(E+\mathrm{i}u)\,G_{00}(E-\mathrm{i}u)\,\mathrm{d}E$$ =0 Metal

$$S_0(E)=E-\epsilon_0-[G_{00}(E)]^{-1}$$

$$S_0(E+\mathrm{i}u)=H_0(E+\mathrm{i}u)-\mathrm{i}\Delta_0(E+\mathrm{i}u)$$

$$S_i(E)=\sum_{j\neq i}V_{ij}\{E-\epsilon_j-S_j^{(i)}(E)\}^{-1}V_{ji}-\sum_{j\neq i}\sum_{k\neq i,j}V_{ik}\{E-\epsilon_k-S_k^{(ij)}\}^{-1}$$
$$\times V_{kj}\{E-\epsilon_j-S_j^{(i)}\}^{-1}V_{ji}+\sum_{j\neq i}\sum_{k\neq i,j}\sum_{l\neq i,j,k}V_{il}\{E-\epsilon_l-S_l^{(ijk)}\}^{-1}V_{lk}\ldots$$

Increase V until perturbation theory breaks down

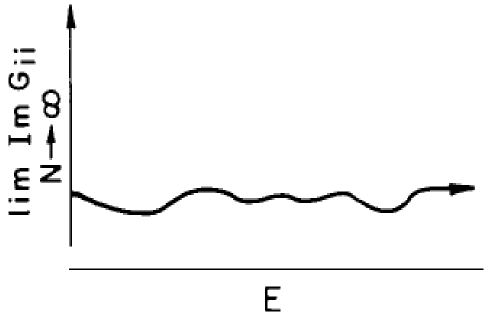
$$\lim_{u\to 0}\frac{1}{u}\Delta_0(E+\mathrm{i}u)=-\frac{\mathrm{d}S_0}{\mathrm{d}E}=-1+\frac{\mathrm{d}G_{00}}{\mathrm{d}E}(G_{00})^{-2}=-1+\frac{1}{|a_0|^2}=\sum_{i\neq 0}\frac{|a_i|^2}{|a_0|^2}$$

= ∞ Metal

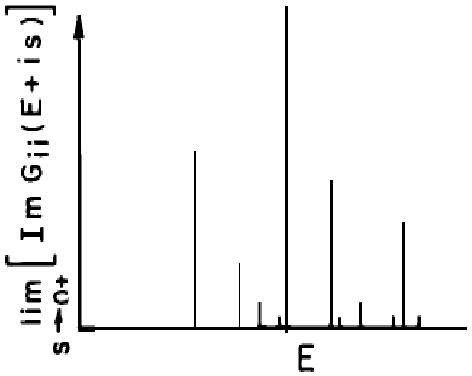
However:

1. Problem with small denominators

2. Uncorrelated paths?




$V > V_c$




$V < V_c$

**Correctly predicts a metal-insulator transition in 3d and localization in 1d**

**Interactions?**

**Disbelief?, against band theory**

**But my recollection is that, on the whole, the attitude was one of humoring me.**

# A selfconsistent theory of localization

R Abou-Chacra†, P W Anderson‡§ and D J Thouless†

J. Phys. C Vol. 6, 1973 1734

250 citations

Perturbation theory around the insulator limit (locator expansion).

$$\epsilon_i a_i^\alpha - \sum_j V_{ij} a_j^\alpha = E^\alpha a_i^\alpha .$$

$$E - \epsilon_i - S_i(E) = \{G_{ii}(E)\}^{-1}$$

No control on the approximation.

It should be a good approx for d>>2.

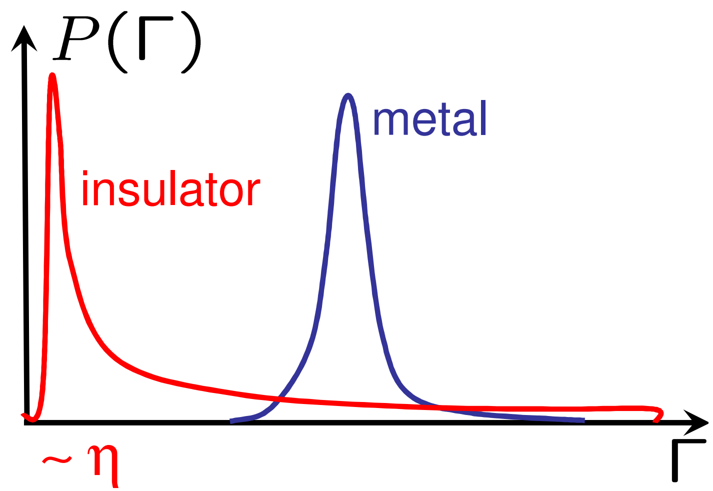
It predicts correctly localization in 1d and a transition in 3d

$$S_i(E) = \sum_{j \neq i} \frac{|V_{ij}|^2}{E - \epsilon_j - S_j(E)}$$

The distribution of the self energy $S_i$ (E) is sensitive to localization.

$$\Gamma = \mathrm{Im} S_i(E + i\eta)$$

$$\lim_{\eta \to 0} \lim_{V \to \infty} P(\Gamma) \begin{cases} > 0 & \text{metal} \\ = 0 & \text{insulator} \end{cases}$$



$$S_i = \sum_j V_{ij}^2/(E - \epsilon_j - S_j)$$

$$E = R + i\eta$$

$$S_i(R + i\eta) = E_i - i\Delta_i$$

$$E_i = \sum_j \frac{|V_{ij}|^2}{(R - \epsilon_j - E_j)},$$

$$\Delta_i = \sum_j \frac{|V_{ij}|^2(\eta + \Delta_j)}{(R - \epsilon_j - E_j)^2}.$$

Solution only provided that the homogenous equation has solutions with $\lambda \geq 1$

$$\lambda^2 \Delta_i = \sum_j \frac{|V_{ij}|^2 \Delta_j}{(R - \epsilon_j - E_j)^2}$$

We need probabilities distributions!!

$$F(k_1, k_2) = \left\{ \frac{1}{2\pi} \int \mathrm{d}x \int \mathrm{d}k'_1 \, P(k'_1) F\left(k'_1, \frac{k_2 V^2}{x^2}\right) \times \exp\left(ik'_1 R - \frac{ik_2 V^2}{x^2}\eta - ik'_1 x - \frac{ik_1 V^2}{x}\right) \right\}^K$$

$k_1$ $k_2$ Fourier transform of $E_i$, $\Delta_i$

Neglect $E_i$, $s \leftrightarrow k_1$

$$f(s) = \left\{ \int \mathrm{d}x \, p(R - x) f\left(\frac{sV^2}{x^2}\right) \exp\left(\frac{-sV^2\eta}{x^2}\right) \right\}^K$$

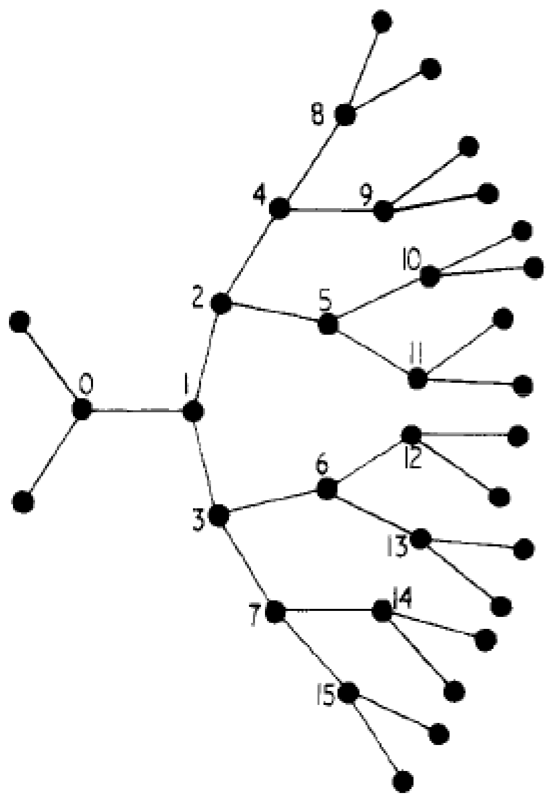
$$f(s) \approx 1 - As^\beta$$

Guess

$$f(s) = \left\{ 1 - A \int p(R - x) \frac{V^{2\beta}}{x^{2\beta}} s^\beta \, \mathrm{d}x + \mathrm{O}(s^{1/2}) \right\}^K$$

$$= 1 - As^\beta K V^{2\beta} \int \frac{p(R - x)}{x^{2\beta}} \mathrm{d}x + \mathrm{O}(s^{1/2}, s^{2\beta}).$$

$$\frac{2KeV_c}{W} \ln\left(\frac{W}{2V_c}\right) = 1$$




Accurate
for
d >> 1

# Localization in mathematics literature

Rigorous proof of localization for strong disorder

1. “Absence of diffusion in the Anderson tight binding model for large disorder or low energy”
Jürg Fröhlich and Thomas Spencer, Comm. Math. Phys. 88, 151 (1983).

2. “Localization at Large Disorder and at Extreme Energies: an Elementary Derivation.”
M. Aizenman, S. Molchanov, Comm. Math. Phys. **157**, 245 (1993).

# Experiments

Vol 453 | 12 June 2008 | doi:10.1038/nature07000 nature

LETTERS

## Direct observation of Anderson localization of matter waves in a controlled disorder

Juliette Billy[1], Vincent Josse[1], Zhanchun Zuo[1], Alain Bernard[1], Ben Hambrecht[1], Pierre Lugan[1], David Clément[1], Laurent Sanchez-Palencia[1], Philippe Bouyer[1] & Alain Aspect[1]

Vol 453 | 12 June 2008 | doi:10.1038/nature07071 nature

LETTERS

## Anderson localization of a non-interacting Bose–Einstein condensate

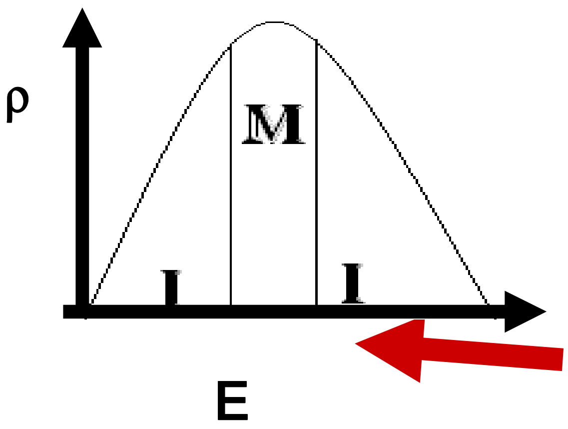
Giacomo Roati[1,2], Chiara D'Errico[1,2], Leonardo Fallani[1,2], Marco Fattori[1,2,3], Chiara Fort[1,2], Matteo Zaccanti[1,2], Giovanni Modugno[1,2], Michele Modugno[1,4,5] & Massimo Inguscio[1,2]

# State of the art:

d = 1 An insulator for any disorder

d =2 An insulator for any disorder

d > 2 Localization only for disorder strong enough




Metal Insulator Transition

Still not well understood

Why?

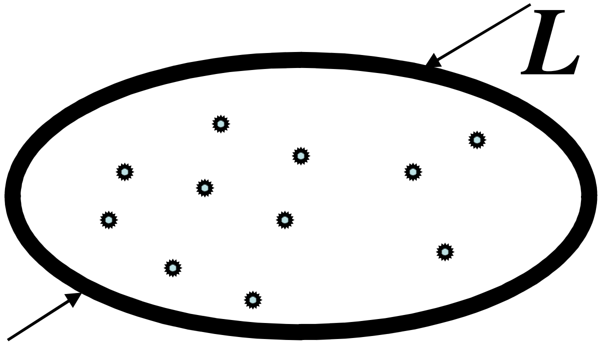
# Anderson localization

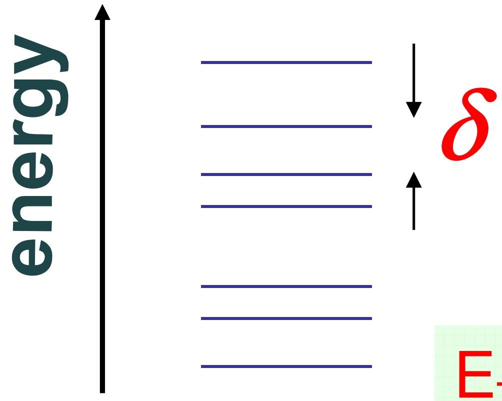
| | | |
|---|---|---|
| 50' | Perturbative locator expansion | Anderson |
| 60' | Self consistent condition | Abou Chakra, Anderson, Thouless |
| | 1d | Kotani, Pastur, Molchanov, Sinai, Jitomirskaya |
| 70' | Weak localization | Larkin, Khmelnitskii, Altshuler, Lee… |
| 80' | Scaling theory | Thouless, Wegner, Gang of four, Frolich, Spencer, Molchanov, Aizenman |
| | Field theory, RMT | Efetov, Wegner |
| 90' | Metal-Insulator | Efetov, Fyodorov, Mirlin, Vollhardt |
| | Computers! | Aoki, Schreiber, Kramer |
| 00' | Experiments! | Aspect, Fallani, Segev |

# Scaling ideas (*Thouless, 1972*)

## 1. Mean level spacing

$$\delta \propto L^{-d}$$




*L* = *system size;*

*D* = *diffusion constant*

δ = *mean- level spacing*

## 2. Thouless energy

$$E_T = hD/L^2$$

$E_T$ = inverse diffusion time ~ sensitivity of the spectrum to change of boundary conditions

$E_T >> \delta \quad g >> 1$ → **Metal**

$E_T << \delta \quad g << 1$ → **Insulator**

## Electric conductivity

$$\sigma = \frac{e^2 n}{\hbar} \frac{l}{k_F} = e^2 \varrho(E_F) D$$

$$D = \frac{v_F^2 \tau}{3}$$

## Ohm's Law

$$\frac{1}{R} = G = \sigma L^{d-2}$$

**for a cubic sample of the size *L***

## Conductance

$$G = \frac{e^2}{h} g(L)$$

## Dimensionless conductance

$$g(L) = \frac{E_T}{\delta}$$

Experiments OK

Volume 42, Number 10 PHYSICAL REVIEW LETTERS 5 March 1979

## Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions

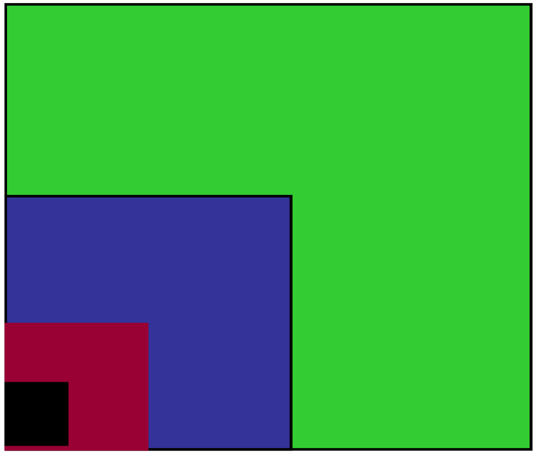
E. Abrahams
*Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854*

and

P. W. Anderson,[a] D. C. Licciardello, and T. V. Ramakrishnan[b]
*Joseph Henry Laboratories of Physics, Princeton University, Princeton, New Jersey 08540*

3724 citations

$L = 2L = 4L = 8L$ ....



$E_T \gg \delta \quad g \gg 1$ → Metal

$E_T \ll \delta \quad g \ll 1$ → Insulator

$$\frac{E_T}{\delta} \to \frac{E_T}{\delta} \to \frac{E_T}{\delta} \to \frac{E_T}{\delta}$$

Figure from Boris Altshuler Boulder Lectures

$$g \to g \to g \to g$$

$$\frac{d(\log g)}{d(\log L)} = \beta(g)$$

# Scaling theory of localization

*The change in the conductance with the system size only depends on the conductance itself*

$\beta(g)$

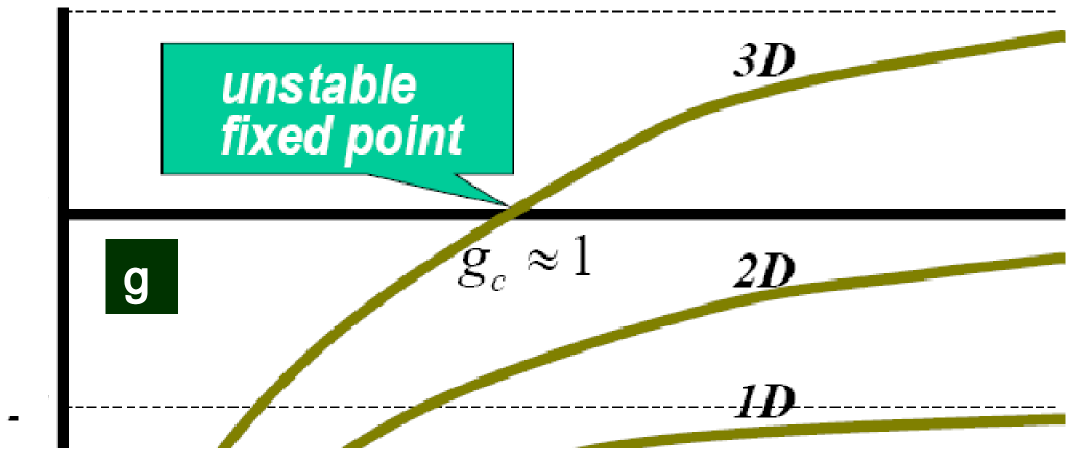



Figure from Boris Altshuler Boulder Lectures

$$\frac{d \log g}{d \log L} = \beta(g)$$

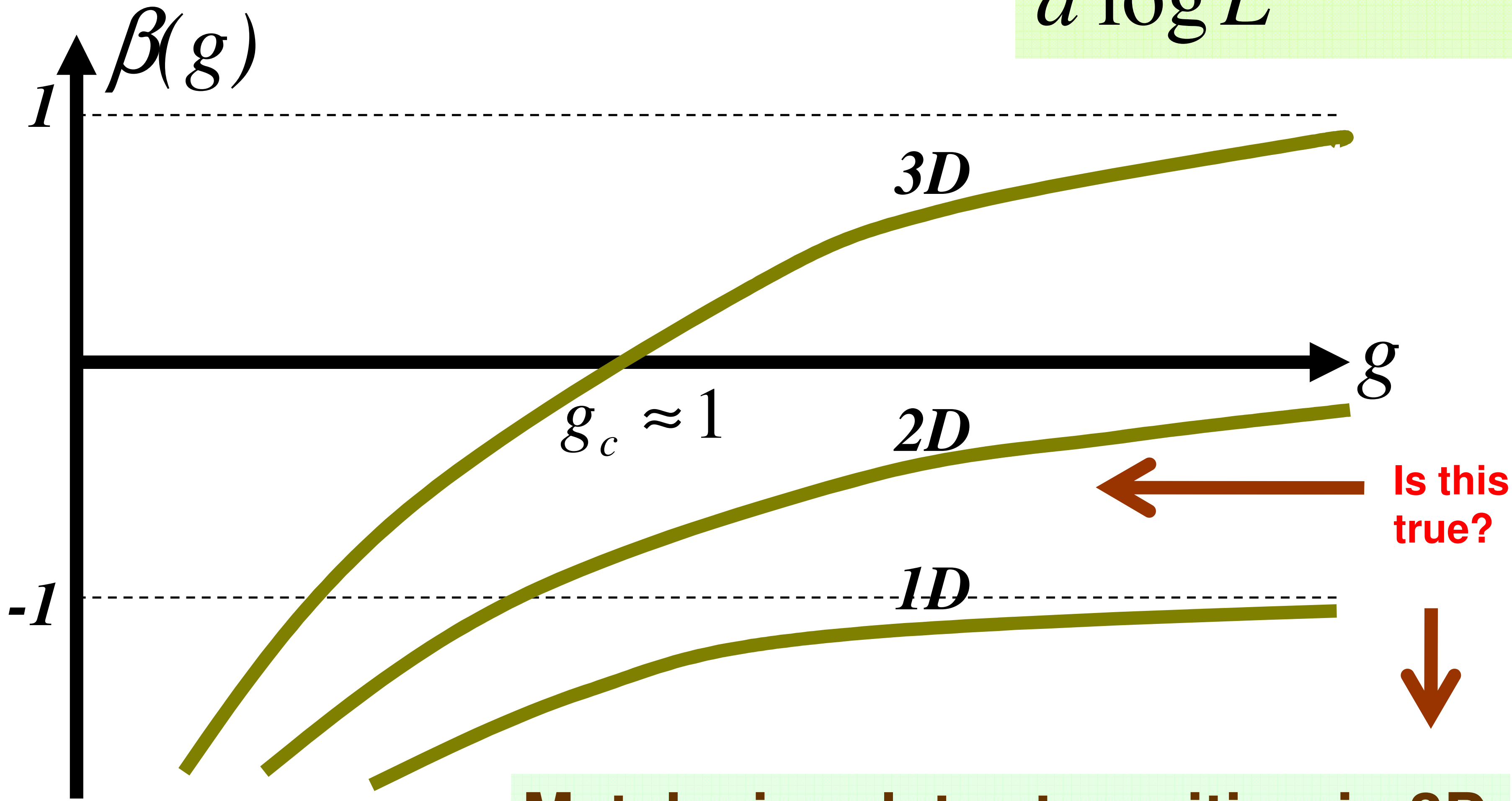
c > 0 ?

$$g >> 1 \quad g \propto L^{d-2} \quad \beta(g) = (d-2) - c/g$$

$$g << 1 \quad g \propto e^{-L/\zeta} \quad \beta(g) \approx \log g < 0$$

Lecture II: “Weak localization: the Russians, the cold war and experiments”

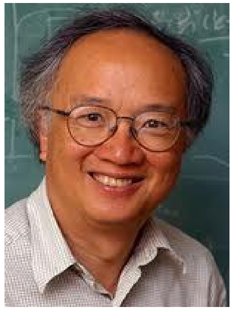
$$\frac{d \log g}{d \log L} = \beta(g)$$




**Metal – insulator transition in 3D**

**All states are localized for d=1,2**

**NO**, Patrick Lee, PRL 42,1492 (1979)
Scaling theory is wrong
Metal-Insulator transition in d=2!



**Particle conductivity in a two-dimensional random potential**

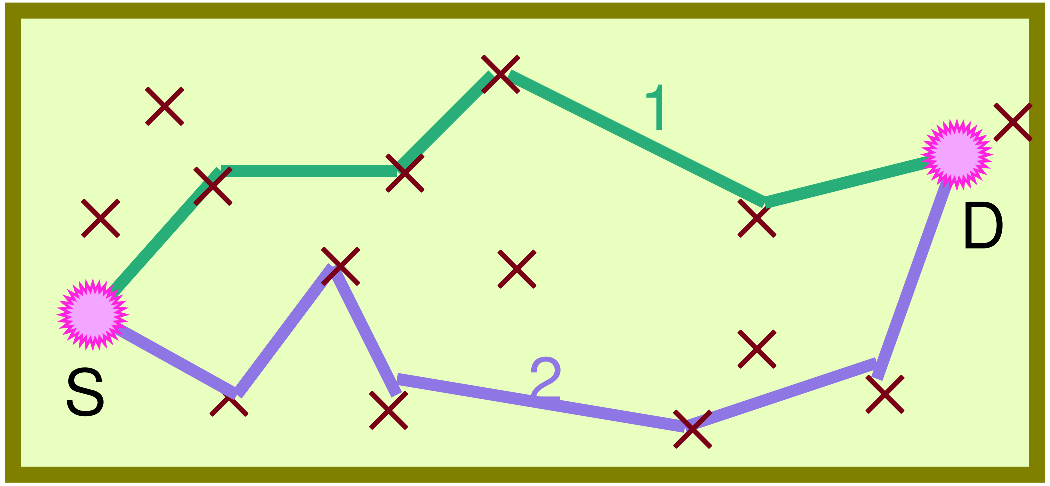
L. P. Gor'kov, A. I. Larkin, and D. E. Khmel'nitskiĭ

*L.D. Landau Institute of Theoretical Physics, USSR Academy of Sciences*

(Submitted 16 July 1979)

No metallic states in 2d. Scaling theory of localization is right!

# Interference and weak localization




$W_1, W_2$ probabilities  $A_1, A_2$ probability amplitudes

$$W_{1,2} = \left|A_{1,2}\right|^2$$

$$A_{1,2} = \left|A_{1,2}\right| e^{i\varphi_{1,2}}$$

**Total probability**

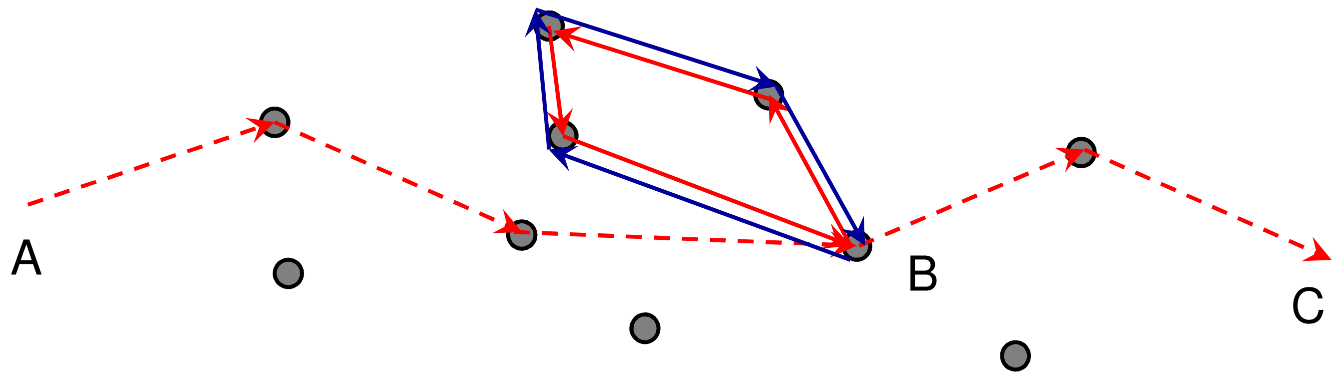
$$W = \left|A_1 + A_2\right|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right)$$

**Interference**

$$2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right) = 2\sqrt{W_1 W_2}\,\cos\left(\varphi_1 - \varphi_2\right)$$

Usually negligible but there are exceptions...

# Weak localization ?




**Constructive interference between clockwise and counter clockwise enhances return probability to B but suppresss the AC probability.**
**Langer and Neel PRL 16, 984 (1966).**

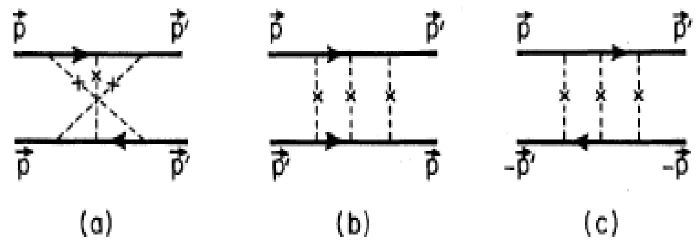



FIG. 5. (a) Example of maximally crossed diagram. (b) Redrawing of (a). (c) A particle-hole propagator derived from (b) using time-reversal symmetry.

?

**Negative correction to G of a metal at low T**

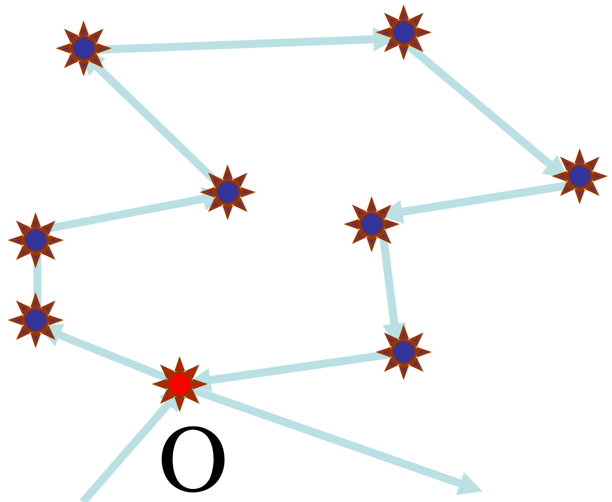



Figure from B. Altshuler Boulder Lectures

**What is the probability *P(t)* that such a loop is formed at time *t* ?**

**Probability to return to *dV* around O**

$$P\left(r\left(t\right)=0\right)dV=\frac{dV}{\left(Dt\right)^{d/2}}$$

$$dV=\lambda\!\!\!^{-}{}^{d-1}v_F\,dt$$

$$P\left(t\right)=-\lambda\!\!\!^{-}{}^{d-1}\int_{\tau}^{t}\frac{v_F\,dt'}{\left(Dt'\right)^{d/2}}$$

$$\frac{\delta g}{g}\approx P\left(t_{\max}\right)$$

$$t_{max} = min\left\{\frac{L^2}{D}, \tau_\varphi \ldots .\right\}$$

Decoherence/ dephasing time

$$\tau_\varphi \sim T^{-p}$$

Thouless time

$$\frac{L^2}{D}$$

$$P(t) = \lambda^{d-1} \int_{\tau}^{t} \frac{v_F dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

$$\frac{\delta g}{g} \approx -\frac{\lambda v_F}{D} \log \frac{L^2}{D\tau} = -\frac{2\lambda v_F}{D} \log \frac{L}{l}$$

$$\delta g = -\frac{2}{\pi} \log \frac{L}{l}$$

**d = 2**

**Universal**

$$\beta(g) = -\frac{2}{\pi g}$$

# Conductivity: The rigorous way

$$\sigma_{dc} = \frac{2e^2}{h} \lim_{\eta \to 0} 4\eta^2 \lim_{q \to 0} \nabla_q^2 \sum_{kk'} \langle G^+(k, k'+q) G^-(k', k-q) \rangle$$

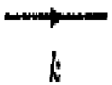
$$\langle \mathbf{G}^\pm \rangle = \mathbf{G}_0^\pm + \mathbf{G}_0^\pm \langle \mathbf{V}\mathbf{G}^\pm \rangle \equiv \mathbf{G}_0^\pm \sum_{n=0}^{\infty} \langle (\mathbf{V}\mathbf{G}_0^\pm)^n \rangle.$$

## Expansion parameter

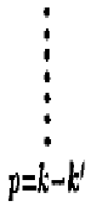
$$\frac{1}{k_F l}$$

$$\langle \mathbf{G}^+\mathbf{G}^- \rangle = \langle \mathbf{G}^+ \rangle \langle \mathbf{G}^- \rangle + \langle \mathbf{G}^+ \rangle \langle \mathbf{G}^- \rangle \mathbf{U} \langle \mathbf{G}^+\mathbf{G}^- \rangle$$

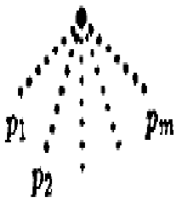
U = Irreducible vertex



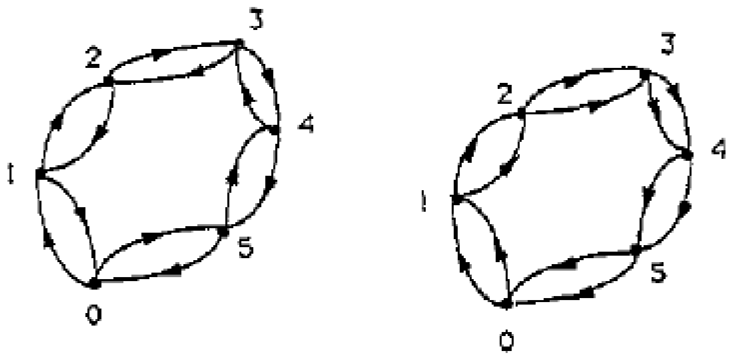

$\widehat{=}\ \mathbf{G}_0^+(k)$ free-particle Green's function

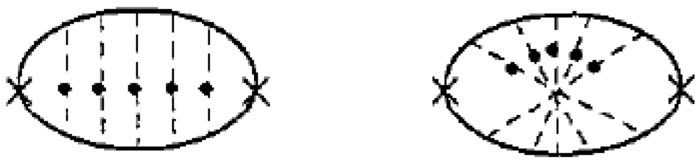



$\widehat{=}\ v(p)$ potential scattering




$\widehat{=}\ \frac{N}{\Omega}\delta(p_1 + p_2 + \ldots + p_m)$ $m$-fold scattering.

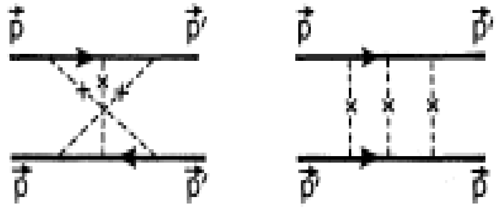


# Conductivity:
## The rigorous way

We must sum geometrical series of maximally crossed diagrams

$$\sigma_{\gamma\gamma'}(0)=\frac{e^2\hbar}{2\pi L^d}\sum_{p,p'}\left(\frac{p_\gamma}{m}\right)G^{\Pi+-}_{pp'}\left(\frac{p'_{\gamma'}}{m}\right)$$



(a) (b)

$$\boxed{\sigma=\frac{ne^2\tau}{m}-\frac{2e^2}{\hbar\pi}\frac{1}{L^d}\sum_Q\left(\frac{1}{Q^2}\right)}$$

$$\sigma_{3D}(L)=\sigma_0-\frac{e^2}{\hbar\pi^3}\left(\frac{1}{l}-\frac{1}{L}\right),$$

$$\sigma_{2D}(L)=\sigma_0-\frac{e^2}{\hbar\pi^2}\ln\left(\frac{L}{l}\right),$$

$$\sigma_{1D}(L)=\sigma_0-\frac{e^2}{\hbar\pi}(L-l).$$

$$L_{\varphi,Th}=\sqrt{D\tau_{\varphi,Th}}$$

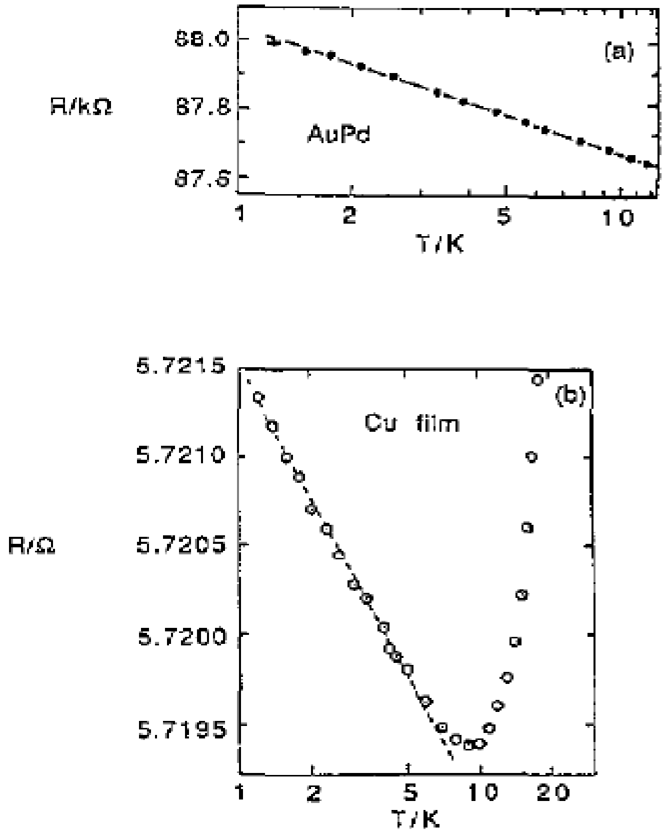
$$\sigma_{3D}(T)=\sigma_0+\frac{e^2}{\hbar\pi^3}\frac{1}{a}T^{p/2},$$

$$\sigma_{2D}(T)=\sigma_0+\frac{p}{2}\frac{e^2}{\hbar\pi^2}\ln\left(\frac{T}{T_0}\right),$$

$$\sigma_{1D}(T)=\sigma_0-\frac{ae^2}{\hbar\pi}T^{-p/2}.$$

**Dolan, Osherhoff, PRL 43, 721 (1979)**

# Resistance of thin metallic stripes increases as T decreases



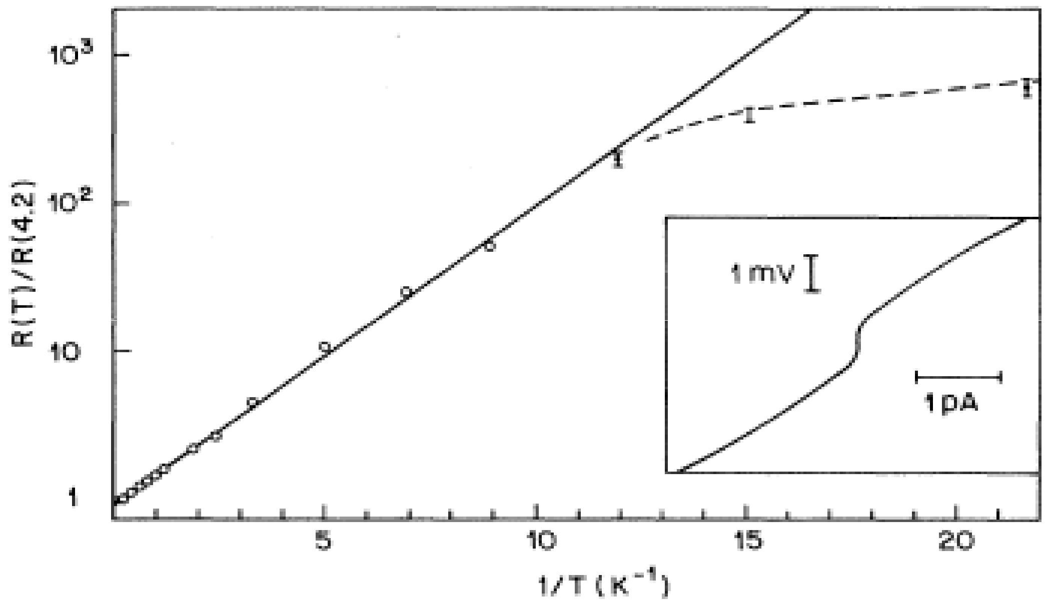
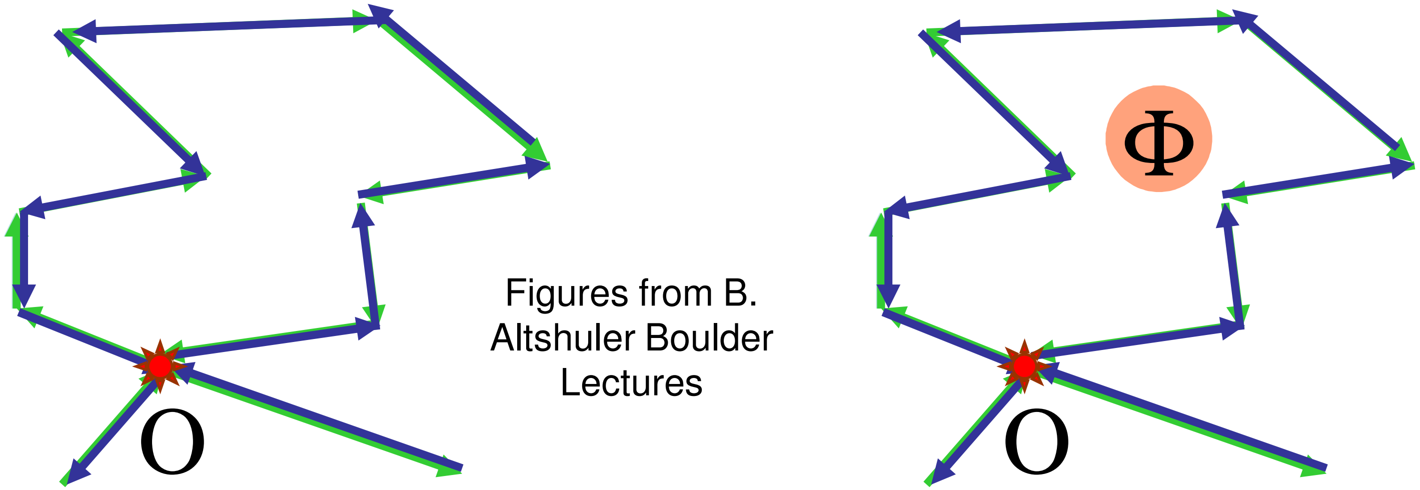




FIG. 3. The exponential dependence of (limit $V \to 0$) $R \equiv V/I$ vs $T$ for sample E in Table I. Inset: $I$–$V$ curve for this sample at $T \cong 20$ mK.

Experimental results also support scaling theory of localization

# Magnetic field/flux




Figures from B. Altshuler Boulder Lectures

***No magnetic field***

$$\varphi_1 = \varphi_2$$

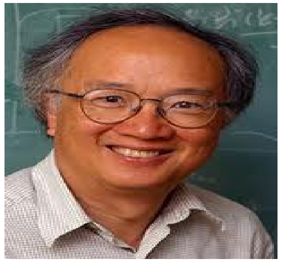
***Magnetic field*** H

$$\varphi_1 - \varphi_2 = 2 * 2\pi\, \Phi/\Phi_0$$

$$I = 2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right) = 2\sqrt{W_1 W_2}\,\cos\left(\varphi_1 - \varphi_2\right)$$

1. H suppresses weak localization

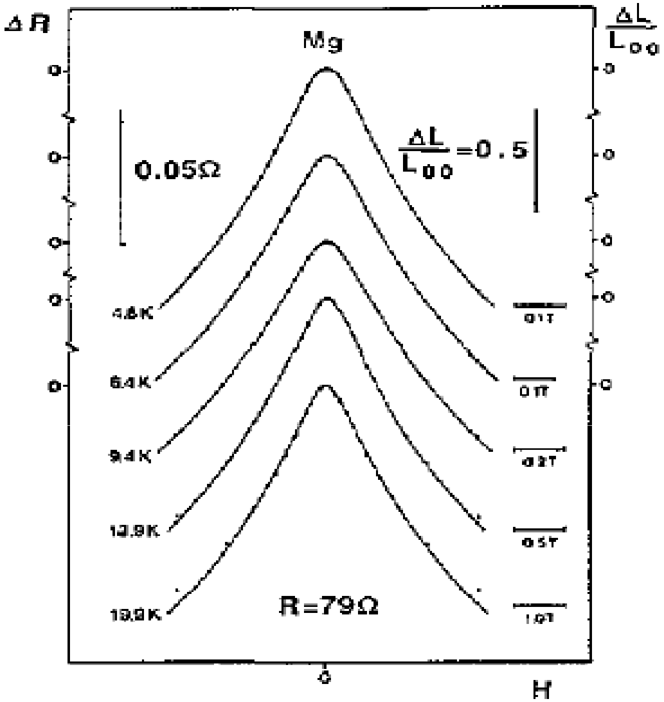
2. Oscillations in the conductance



PRB 21, 5142 (1980)

# Negative Magnetoresistance

**Chentsov (1949)**




Ref. 3

$$L_H = (eH/\hbar c)^{-1/2}$$

$$x = \frac{L_{Th}^2}{L_H^2}$$

$$\sigma(H,T) - \sigma(0,T) = \frac{e^2}{2\pi^2\hbar}\left[\psi\left(\frac{1}{2}+\frac{1}{x}\right)+\ln x\right]$$

$$x \gg 1 \qquad\qquad \sigma(H,T) - \sigma(0,T) \sim log\ H$$

# Bohm-Aharonov- effect

## Theory

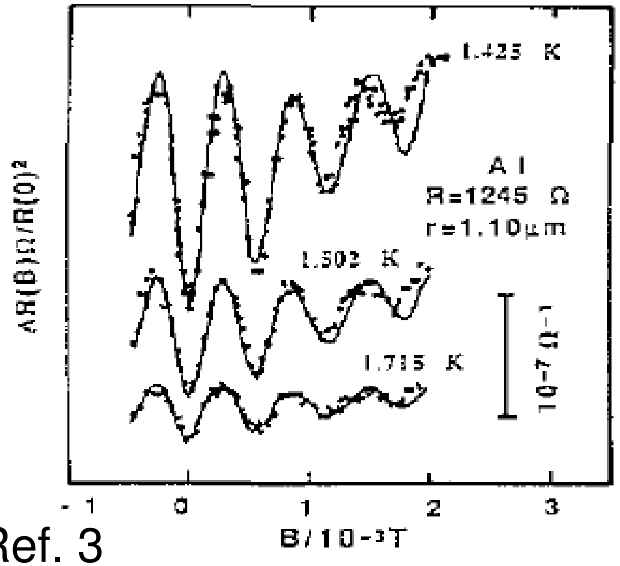
**Altshuler, Aronov, Spivak (1981)**

## Experiment

**Sharvin & Sharvin (1981)**

Select exp only those paths whose associated flux is the same

Thin metallic cylinders




Ref. 3

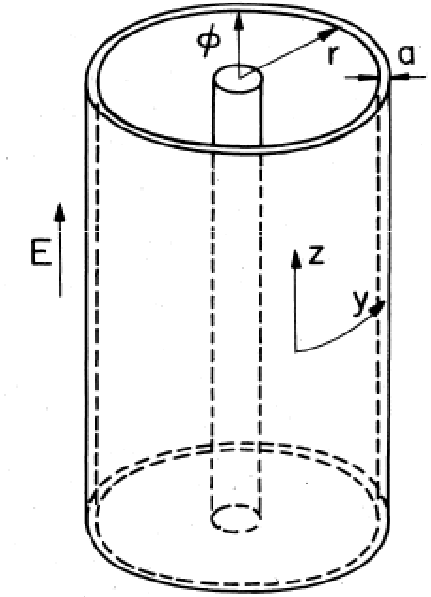
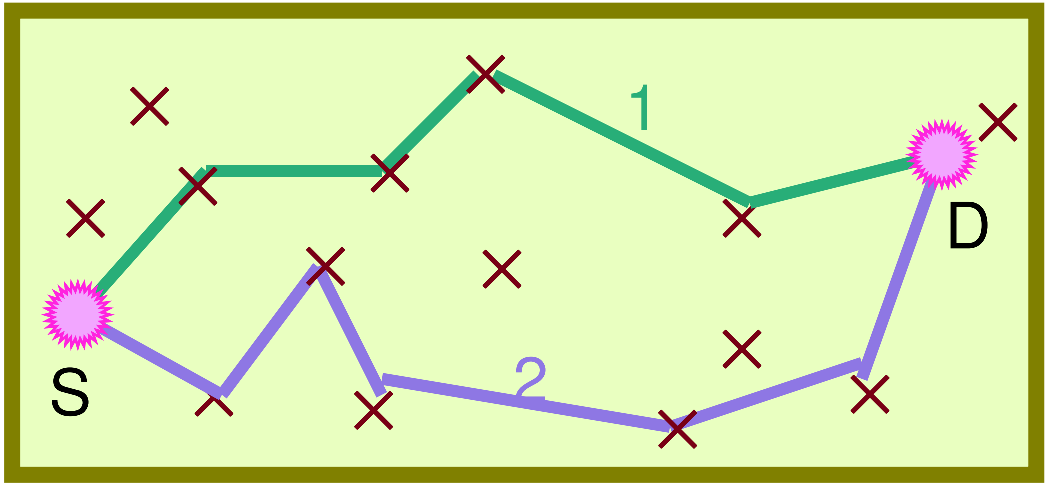



**Figure 28.** The oscillations of the magnetoresistance $\Delta R$ of an Al cylinder at different temperatures. Dots denote experimental data, curves are the fits to the theory of Altshuler *et al* (1981) (after Gijs *et al* (1984a)).

Figure from B. Alshuler Boulder lectures




$W_1, W_2$ probabilities

$A_1, A_2$ probability amplitudes

$$W_{1,2} = \left|A_{1,2}\right|^2$$

$$A_{1,2} = \left|A_{1,2}\right| e^{i\varphi_{1,2}}$$

**Total probability**

$$W = \left|A_1 + A_2\right|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right)$$

**Negative?**

$$2\ \mathrm{Re}\ \left(A_1 A_2^{\,*}\right) = 2\sqrt{W_1 W_2}\ \cos\ \left(\varphi_1 - \varphi_2\right)$$

Weak anti localization?

# Spin-Orbit interactions and anti-localization correction

Larkin

Hikami

$$V_{\boldsymbol{k}-\boldsymbol{k}'}[1+ic(\boldsymbol{k}\times\boldsymbol{k}')\boldsymbol{\sigma}]$$

$$\sigma=\sigma_0-\frac{e^2}{\hbar}\int G_R{}^2 G_A{}^2\left(\frac{v^2}{d}\nu\right)\Gamma_{\alpha\beta\beta\alpha}\,d\xi\, d^d q$$

$$\Gamma^0_{\alpha\beta,\gamma\delta}=\frac{\hbar}{2\pi N(\epsilon_F)}\left[\frac{1}{\tau_0}\delta_{\alpha\beta}\delta_{\gamma\delta}-\sum_i\frac{1}{\tau^i_{so}}\sigma^i_{\alpha\beta}\sigma^i_{\gamma\delta}\right]$$

$$\sigma=\sigma_0-\frac{\alpha e^2}{\pi^2\hbar}\ln L$$

$\alpha = 0,1,-1$

$$\beta(g)=+\frac{1}{2g}$$

Ref. 3

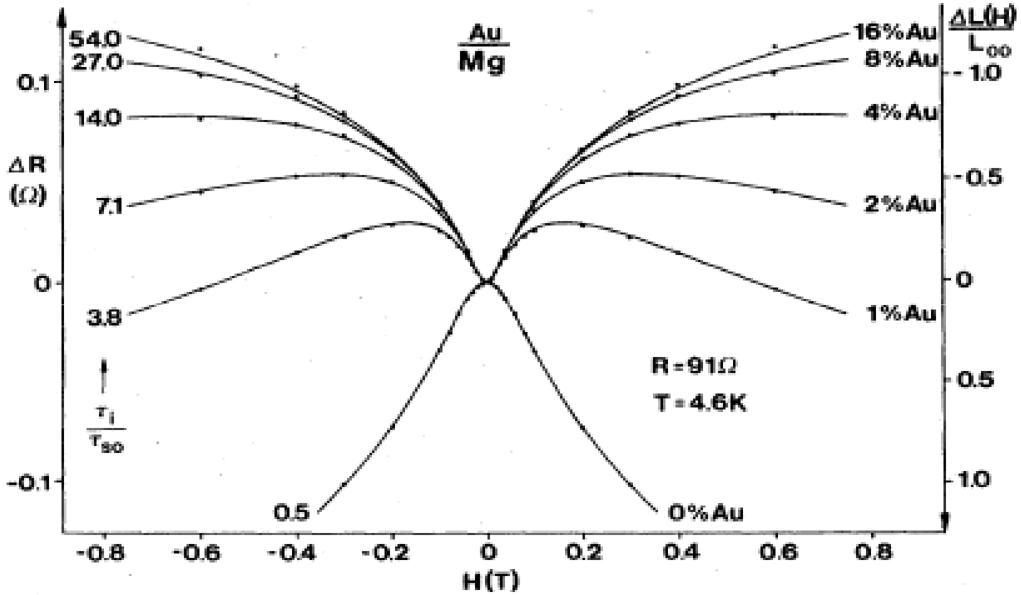
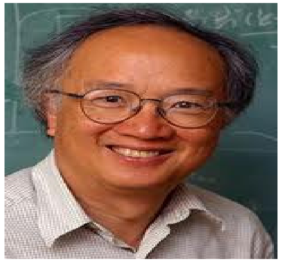


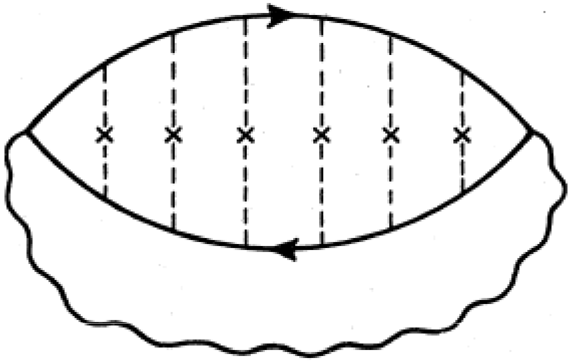

FIG. 17. The magnetoconductance curve of a Mg film with different coverages of Au. [$\Delta L(H)$ is the magnetoconductance, and $L_\infty = e^2/2\pi^2\hbar$.] The coverages shown are in percent of an atomic layer. Increasing Au coverage converts the positive magnetoconductance to negative. Full curves through the data points are fits using the theory of Hikami, Larkin, and Nagaoka (1980). Figure is taken from Bergmann (1982b).

Disorder +
Interactions


Altshuler


Aronov


Lee



$$\lambda \sim \lambda_0 \left[ 1 + \int_{\tau}^{h/E} \frac{v_F dt}{k_F^{d-1} (Dt)^{d/2}} \right] \sim \lambda_0 [1 + \Gamma_d(E)]$$

$$\frac{\delta\sigma(E)}{\sigma} \propto \Gamma_d(E) \propto \begin{cases} 1/\sqrt{E} \\ \log E\tau \\ \sqrt{E} \end{cases}$$

## Summary:

### Weakly localization

$$\sigma_{3D}(T)=\sigma_0+\frac{e^2}{\hbar\pi^3}\frac{1}{a}T^{p/2},$$

$$\sigma_{2D}(T)=\sigma_0+\frac{p}{2}\frac{e^2}{\hbar\pi^2}\ln\left[\frac{T}{T_0}\right],$$

$$\sigma_{1D}(T)=\sigma_0-\frac{ae^2}{\hbar\pi}T^{-p/2}.$$

Both n and A can have any sign

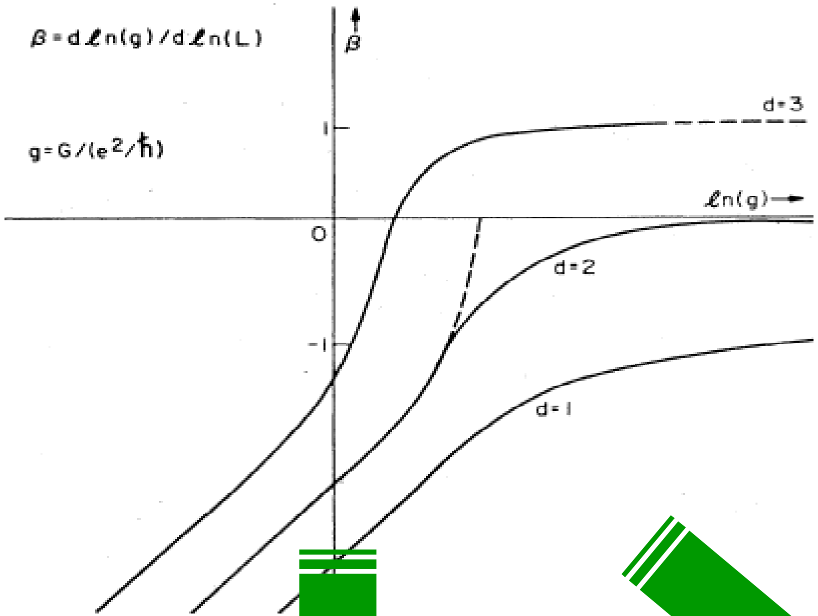
σ(H) can be oscillatory

### Boltzmann Picture

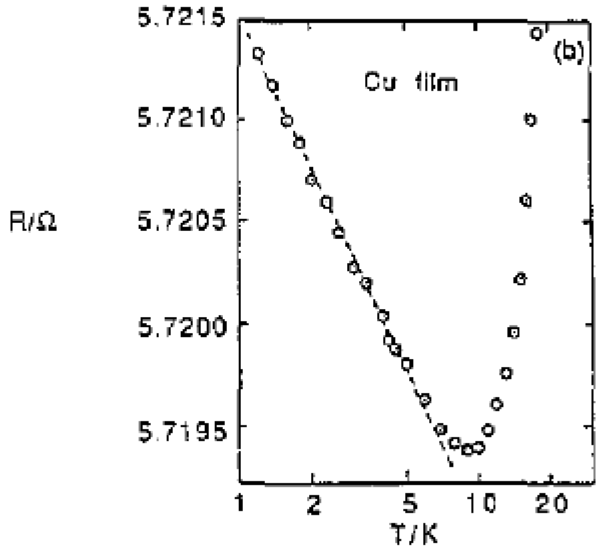
$$\sigma(T)=\sigma_0-AT^n$$

n integer > 2 (=2 for e-e collisions)
A>0

σ(H) monotonous



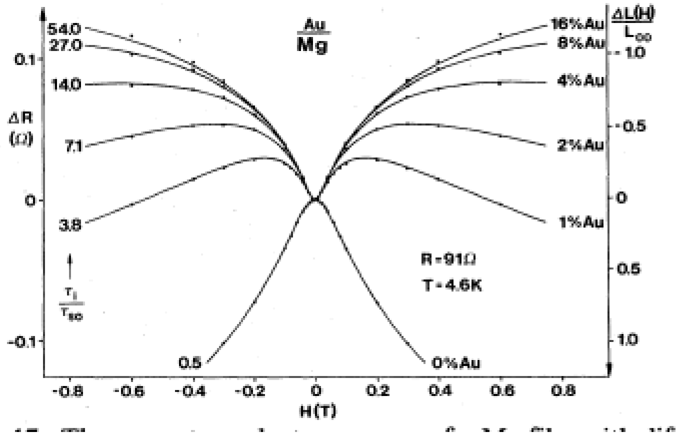
β = d ℓn(g) / d ℓn(L)
g = G / (e²/ħ)
β
ℓn(g)
d=3
d=2
d=1
1
0
-1

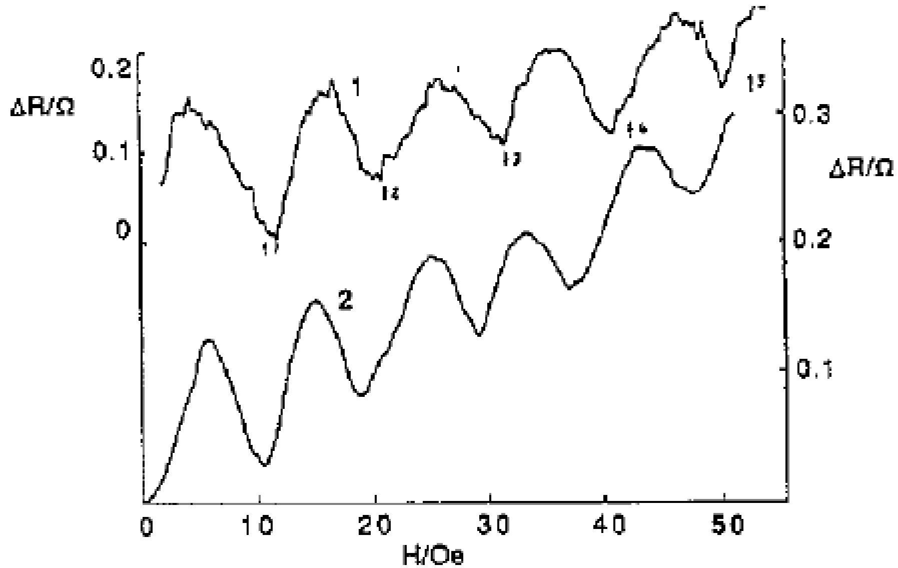
R/Ω
5.7215
5.7210
5.7205
5.7200
5.7195
(b)
Cu film
1
2
5
10
20
T/K


Ref. 3

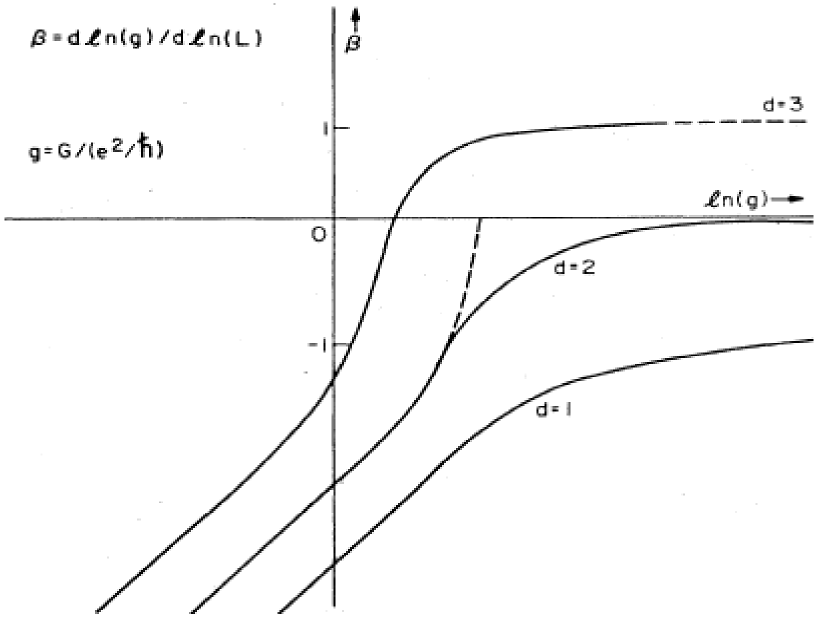
Ref. 2



Au
Mg
ΔR (Ω)
0.1
0
-0.1
54.0
27.0
14.0
7.1
3.8
0.5
16%Au
8%Au
4%Au
2%Au
1%Au
0%Au
ΔL(H)
L∞
1.0
0.5
0
0.5
1.0
R=91Ω
T=4.6K
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
H(T)

ΔR/Ω
0.2
0.1
0
1
2
0.3
0.2
0.1
ΔR/Ω
0
10
20
30
40
50
H/Oe




g >> 1... Universal properties?

Wigner-Dyson statistics?

Disordered Systems

$g >> 1$

$g = g_c$

Quantum Chaos

Classically chaotic motion

Lecture III

# Universality

Random Matrix theory

Nuclear Physics

High energy excitations

QCD

Infrared spectrum Dirac operator

Universality I

Field theory approach to disorder systems

Denominator problem

$$\hat{G}_{\mathrm{R,A}} = (E - \hat{H} \pm i\eta)^{-1}$$

$$(E - \hat{H} + i\eta)^{-1}_{kl} = -i\frac{\int [\mathrm{d}\phi^* \mathrm{d}\phi]\phi_k \phi_l^* \exp\{i\sum_{ij}\phi_i^*[(E+i\eta)\delta_{ij} - H_{ij}]\phi_j\}}{\int [\mathrm{d}\phi^* \mathrm{d}\phi] \exp\{i\sum_{ij}\phi_i^*[(E+i\eta)\delta_{ij} - H_{ij}]\phi_j\}}$$

1982-84: Grassmannian variables can help

$$\chi_k \chi_l = -\chi_l \chi_k$$

$$I = \int \exp\left(-\chi^+ A \chi\right) \prod_{i=1}^{n} d\chi_i^* d\chi_i = Det A$$

$$(E - \hat{H})^{-1}_{kl} = -i \int [\mathrm{d}\Phi^* \mathrm{d}\Phi] S_k S_l^* \exp\{i \sum_{ij} \Phi_i^\dagger [E\delta_{ij} - H_{ij}]\Phi_j\}$$

$$\Phi^\dagger = (S_1^*, \ldots, S_n^*, \chi_1^*, \ldots, \chi_n^*)$$

$$\left\langle \exp(i \sum_{ij} \Phi_i^\dagger H_{ij} \Phi_j) \right\rangle = \exp\left\{ -\frac{1}{2N} \sum_{ij} (\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) \right\}$$

Disorder Is integrated!!

Disordered system ⟷ Effective Field theory

$$R_2(\omega) = \frac{\langle \nu(E-\omega/2)\nu(E+\omega/2)\rangle}{\langle \nu(E)\rangle^2}$$

Density of probability that 2 eigenvalues are separated by ω

Efetov
Larkin
Wegner
Khmelnitskii

$$S[Q] = \frac{\pi\nu}{4}\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}[-D(\nabla Q)^2 - 2i\omega\Lambda Q]$$

$Q \sim \Phi\Phi^t$

$$R_2(\omega) = \left(\frac{1}{4V}\right)^2 \mathrm{Re}\int \mathrm{D}Q(\mathbf{r})\left[\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}Q\Lambda k\right]^2 e^{-S[Q]}$$

**$\nabla Q \approx 0$ Universal regime**

$$R_2(s) = \delta(s) + 1 - \frac{\sin^2(\pi s)}{(\pi s)^2}$$

→ RANDOM MATRiX THEORY

$s = \omega/\Delta$

Efetov:Supersymmetry in disorder and chaos

Random Matrix $\equiv \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}$

$a_{ij}$ random entries

**Spectral Properties**

$$R_2(s) = \delta(s) + 1 - \frac{\sin^2(\pi s)}{(\pi s)^2}$$

$\beta = 2 \quad GUE$

**Level Repulsion**

$$P(s) \sim s^{\beta} e^{-As^2} \qquad s = \frac{E_{i+1} - E_i}{\delta}$$

$\beta = 1 \quad GOE$
$\beta = 2 \quad GUE$
$\beta = 4 \quad GSE$

**Spectral Rigidity**

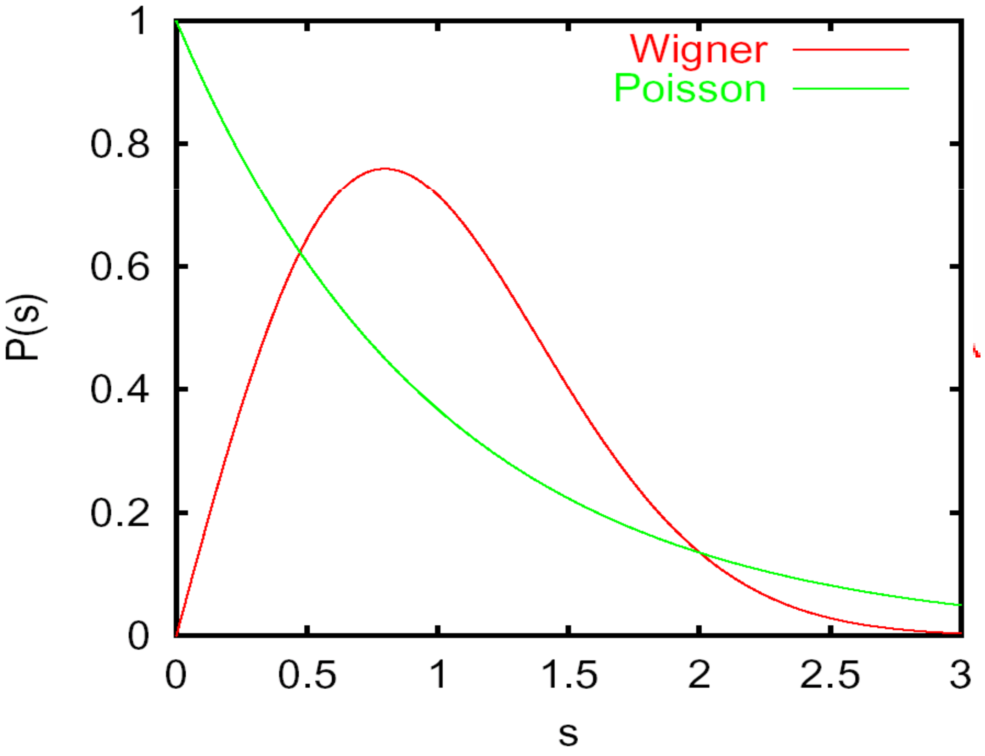
$$\Sigma_2(N) = \langle n(N)^2 \rangle - \langle n(N) \rangle^2 \sim \log N$$

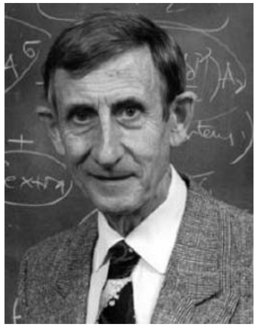
$$N = \frac{E_f - E_i}{\delta} >> 1$$

**UNIVERSAL spectral correlations**

# Uncorrelated spectrum (Poisson)

$$\Sigma_2(N) = N \qquad P(s) = \exp(-s)$$

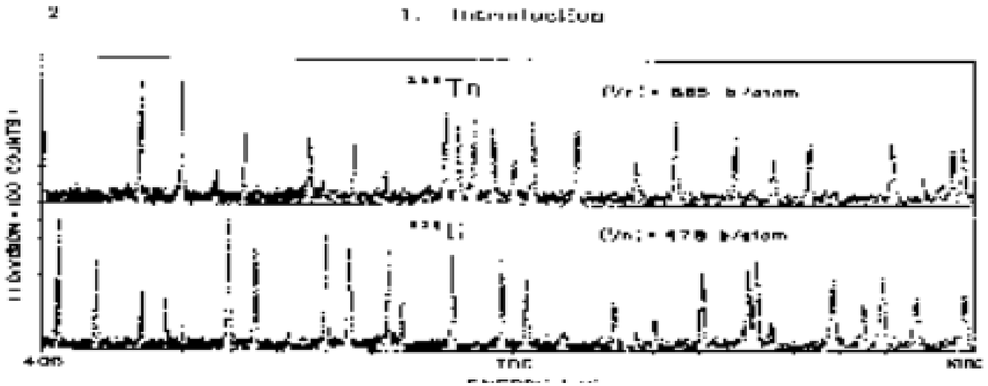


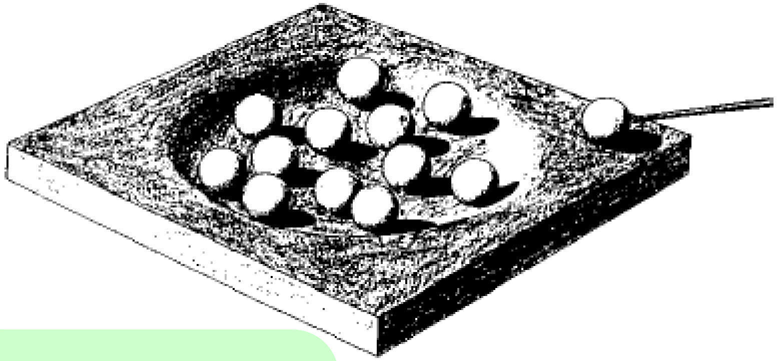



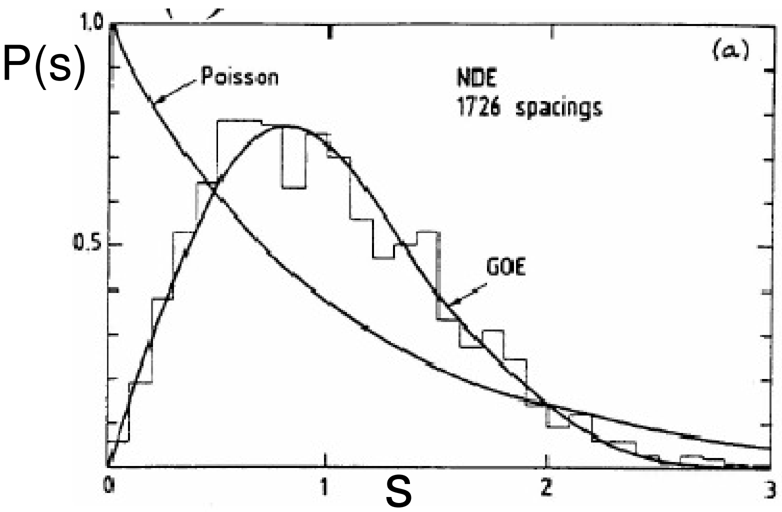
# Universality II

## Nuclear excitations

**Bohr 1936**








$$P(s) = \sum_i \delta\left(s - [E_{i+1} - E_i]/\delta\right)$$

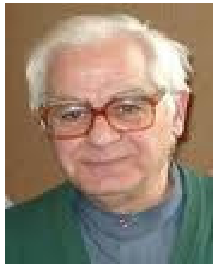
A statistical approach? Maybe Bohr was right

Shell model not feasible

$$P(s) \sim s^{\beta} e^{-As^2}$$

# Universality III

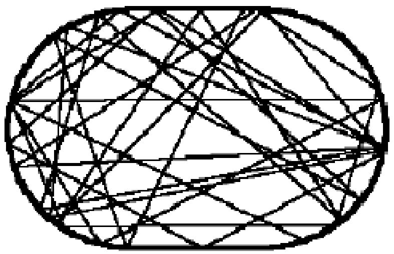
# Quantum chaos

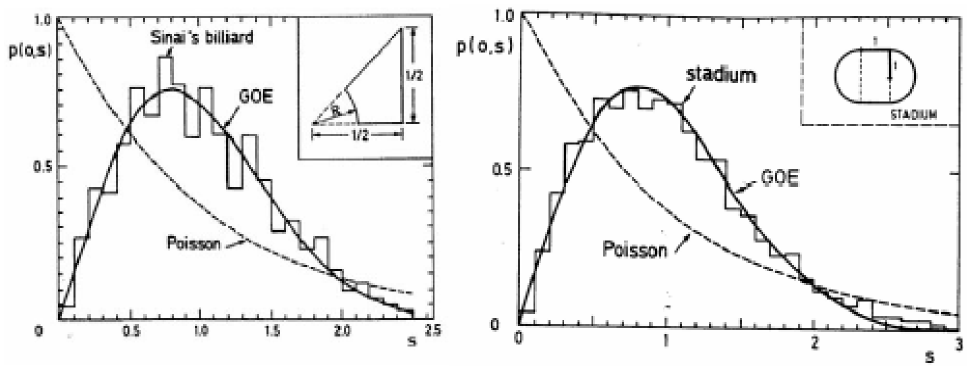


## Bohigas-Giannoni-Schmit conjecture

Phys. Rev. Lett. 52, 1 (1984)

**Classical chaos** ⟷ **Wigner-Dyson**





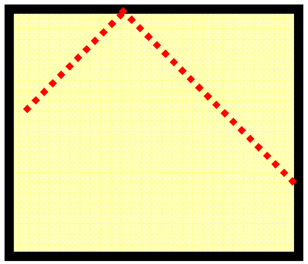

**Energy is the only integral of motion**

**Momentum is not a good quantum number** → **Delocalization**

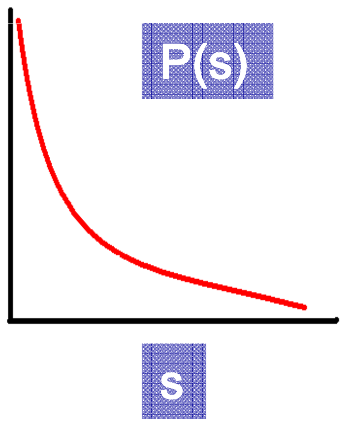
# Gutzwiller-Berry-Tabor conjecture

Integrable classical motion → Poisson statistics (Insulator)

Integrability



Canonical momenta are conserved



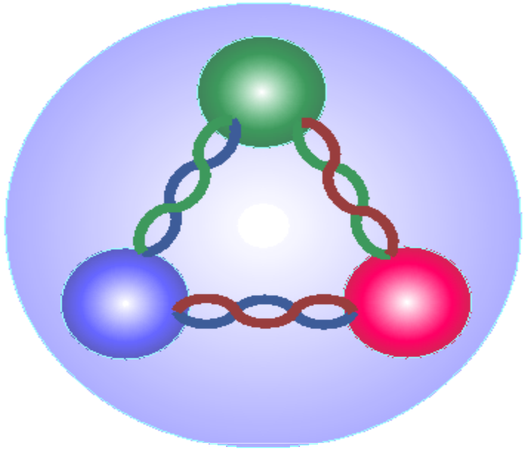

System is localized in momentum space

# Lecture IV

Mesoscopic Physics beyond condensed matter

**QCD vacuum as a disordered medium**

# Inside the Nucleus: What holds the matter together?



Quarks

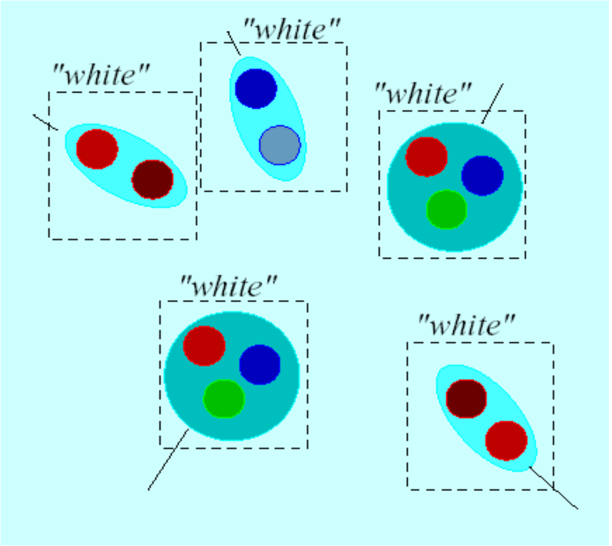
Color charge RED, BLUE, GREEN

Stable matter:  u,d, 3-10MeV

Electric Charge 2/3,1/3

Interact by exchanging gluons

Hadrons are colorless




**Strong color forces govern the interaction among quarks. The relativistic quantum field theory to describe quark interactions is quantum chromodynamics (QCD).**

# A two minute course on non perturbative QCD

## State of the art

| | What? | How? |
|---|---|---|
| $T = 0$ low energy | Chiral Symmetry breaking and Confinement | 1. Lattice QCD<br>2. Instantons…. |
| $T = T_c$ | Chiral and deconfinement transition | Universality (Wilczek and Pisarski) |
| $T > T_c$ | Quark- gluon plasma<br>QCD non perturbative! | AdS-CFT<br>N =4 Yang Mills |

# QCD at T=0, instantons and chiral symmetry breaking

tHooft, Polyakov, Callan, Gross, Shuryak, Diakonov, Petrov,VanBaal

**Instantons: Non perturbative solutions of the Yang Mills equations**

$$D = \partial_\mu \gamma^\mu + g A_\mu^{ins} \gamma^\mu \qquad D\psi_0(r) = 0 \qquad \psi_0(r) \propto 1/r^3$$

**1. Dirac operator has a zero mode**

***2. The smallest eigenvalues of the Dirac operator are controlled by instantons***

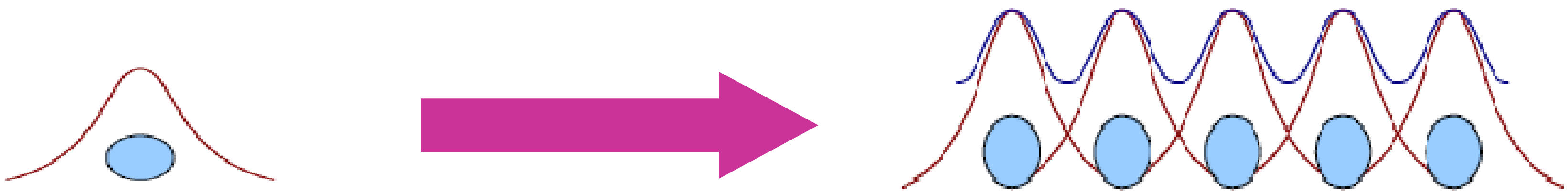
$$\langle \psi\bar{\psi} \rangle = -\frac{1}{V}\langle Tr(D+m)^{-1} \rangle = \int d\lambda \frac{\rho(\lambda)}{m+i\lambda} = -\lim_{m\to m_0} \pi \frac{\langle \rho(m) \rangle}{V}$$

$\langle \psi\bar{\psi} \rangle$ **Order parameter symmetry breaking**

# QCD vacuum as a conductor (T =0)

**Metal:** An electron initially bounded to a single atom gets delocalized due to the overlapping with nearest neighbors

**QCD Vacuum:** Zero modes get delocalized due to the overlapping with the rest of zero modes. (Diakonov and Petrov)



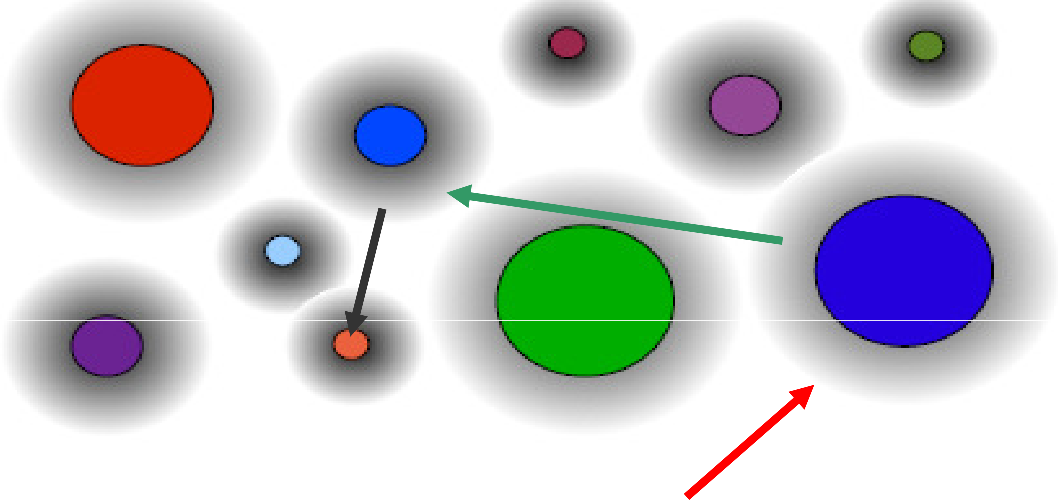
**Differences**

**Disorder: Exponential decay**
**QCD vacuum: Power law decay**

# QCD vacuum as a disordered conductor

Diakonov, Petrov, Verbaarschot, Osborn, Shuryak, Zahed,Janik

**Instanton positions and color orientations vary**



**Ion** ⟷ **Instantons**

**Electron** ⟷ **Quarks**

$T = 0 \quad T_{IA} \sim 1/R^{\alpha}, \ \alpha = 3<4$

$T>0 \quad T_{IA} \sim e^{-R/l(T)}$

**Conductor. RMT applies ?**

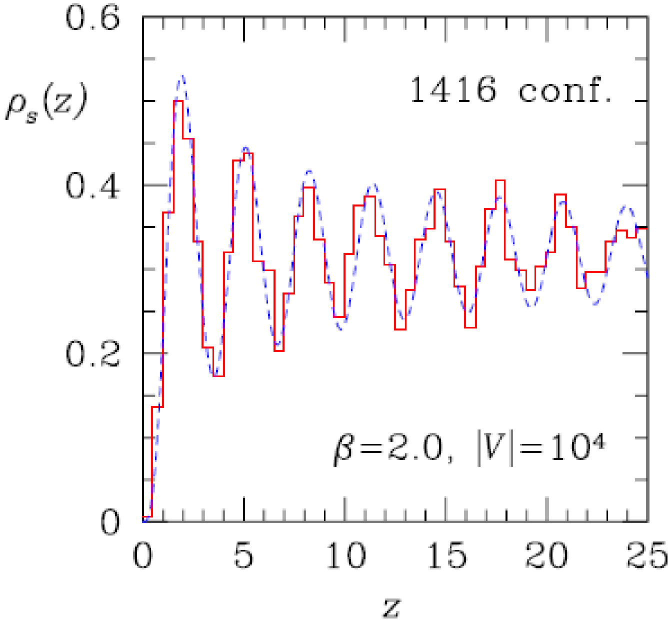
**A transition is possible**

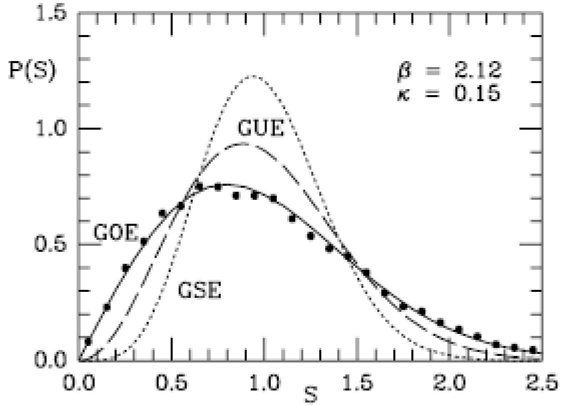
Shuryak,Verbaarschot

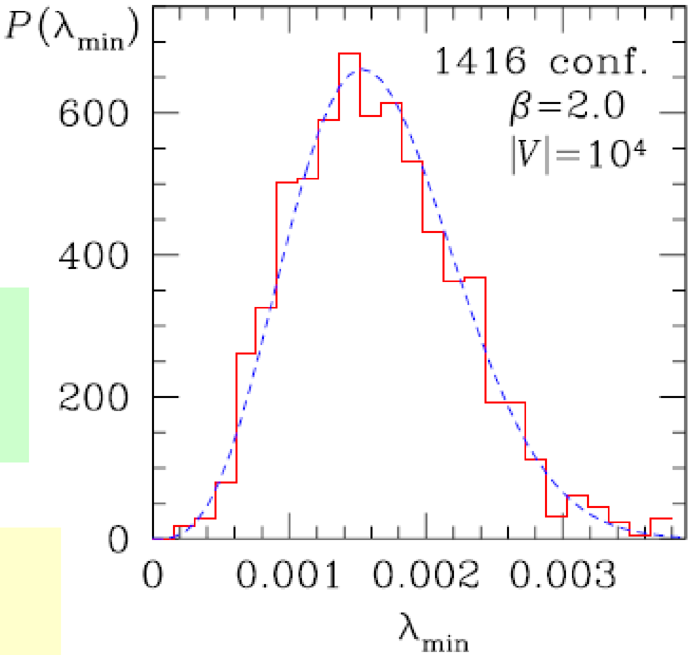
Universality
$E < E_c(L)$

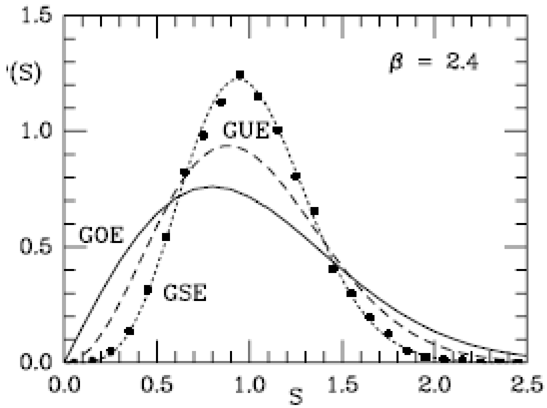
QCD Dirac operator

Chiral random matrix model




Shuryak, Verbaarshot
1993

# Deconfinement and chiral restoration

**Deconfinement:** Confining potential vanishes:

**Chiral Restoration:** Matter becomes light:

**How to explain these transitions?**

**1. Effective, simple, model of QCD close to the phase transition (Wilczek,Pisarski,Yaffe): Universality.**

**2. Classical QCD solutions (t'Hooft): Instantons (chiral), Monopoles and vortices (confinement).**

**QCD Dirac operator**

$$D_\mu^{QCD} = \partial_\mu + gA_\mu$$

Phys.Rev. D75 (2007) 034503

Nucl.Phys. A770 (2006) 141

**with J. Osborn**

$$\gamma^\mu D_\mu^{QCD}\,\psi_n = i\lambda_n \psi_n$$

At the same $T_c$ that the Chiral Phase transition

$$\langle \psi\overline{\psi} \rangle \approx 0$$

$$\left.\begin{matrix}\lambda_n \\ \psi_n\end{matrix}\right\}$$ undergo a *metal - insulator* transition

**A metal-insulator transition in the Dirac operator induces the QCD chiral phase transition**

# Characterization of a metal/insulator

$$H\psi_n = E_n\psi_n$$

1. Eigenvector statistics:

2. Eigenvalue statistics:

$$IPR = L^d \int |\psi_n(r)|^4 d^d r \sim L^{d-D_2}$$

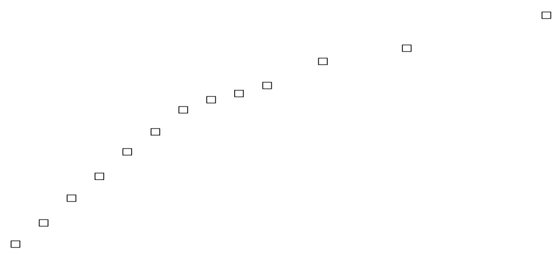
$$P(s) = \sum_i \delta(s - [\lambda_{i+1} - \lambda_i]/\Delta)$$

**Random Matrix**

**Wigner Dyson statistics**

$$\begin{cases} D_2 \sim d \\ P(s) \sim se^{-As^2} \end{cases}$$

**var**



**No correlations**

**Poisson statistics**

$$\begin{cases} D_2 \sim 0 \\ P(s) = e^{-s} \end{cases}$$

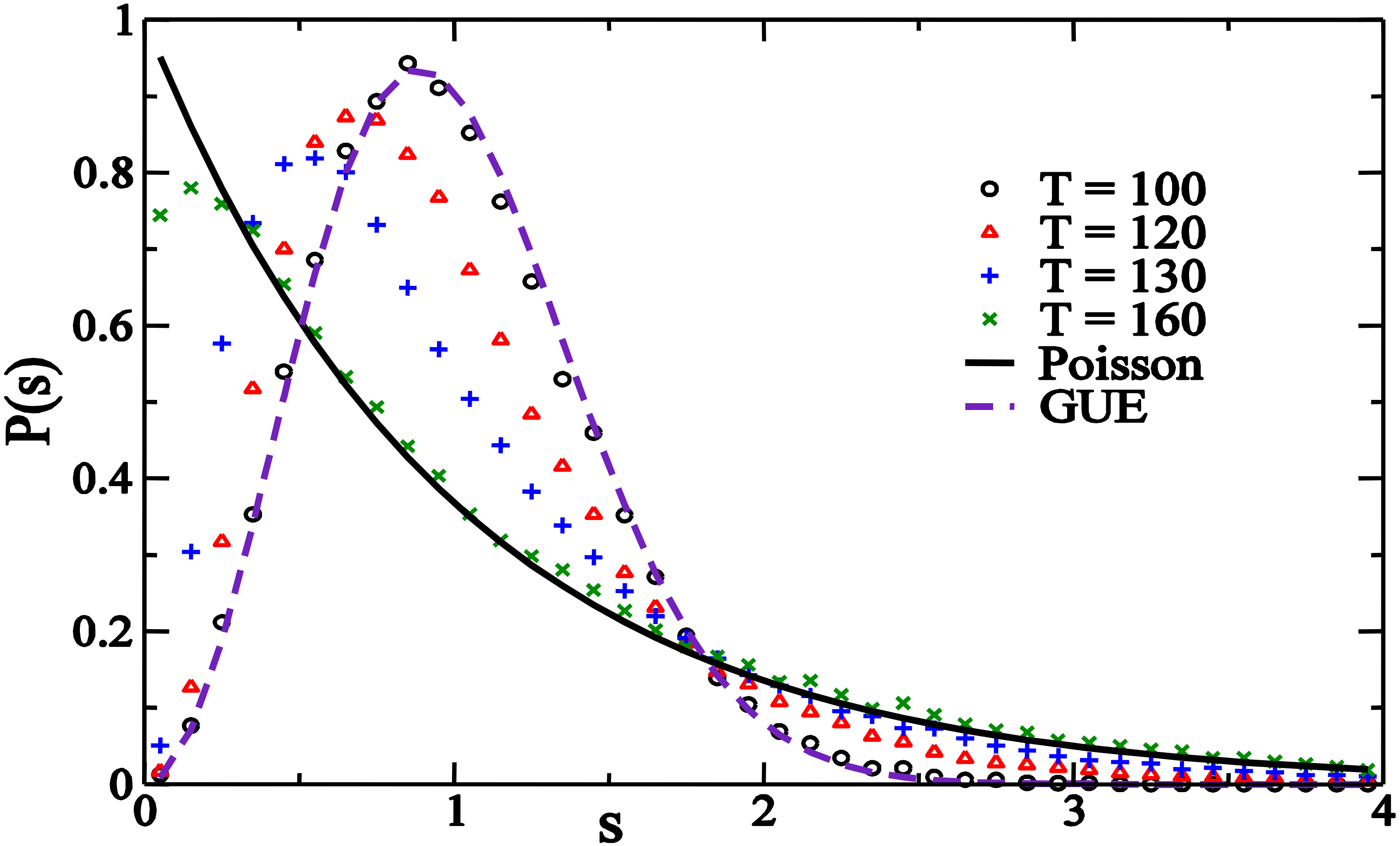
$$\mathrm{var} = \langle s^2 \rangle - \langle s \rangle^2 \quad \langle s^n \rangle = \int s^n P(s) ds$$

**Insulator** $\longleftrightarrow$ **Poisson statistics**

**Disordered metal** $\longleftrightarrow$ **WD statistics**

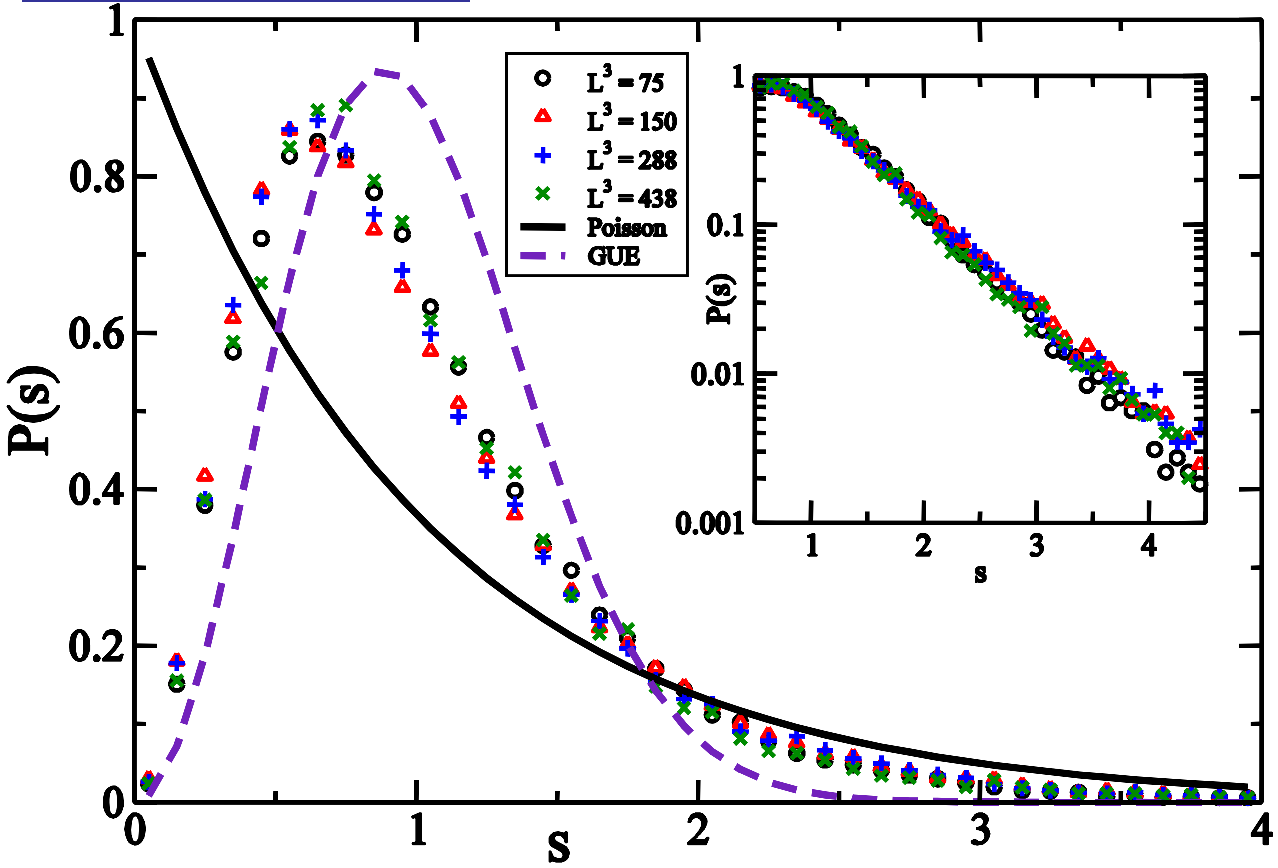
# Metal insulator transition

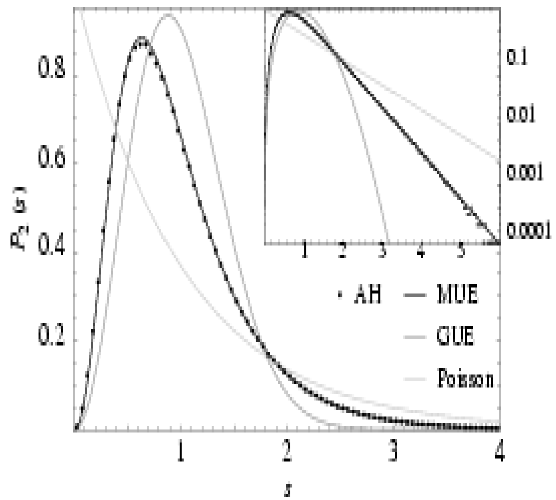
ILM, close to the origin, 2+1 flavors, N = 200



Spectrum is scale invariant

# ILM with 2+1 massless flavors,

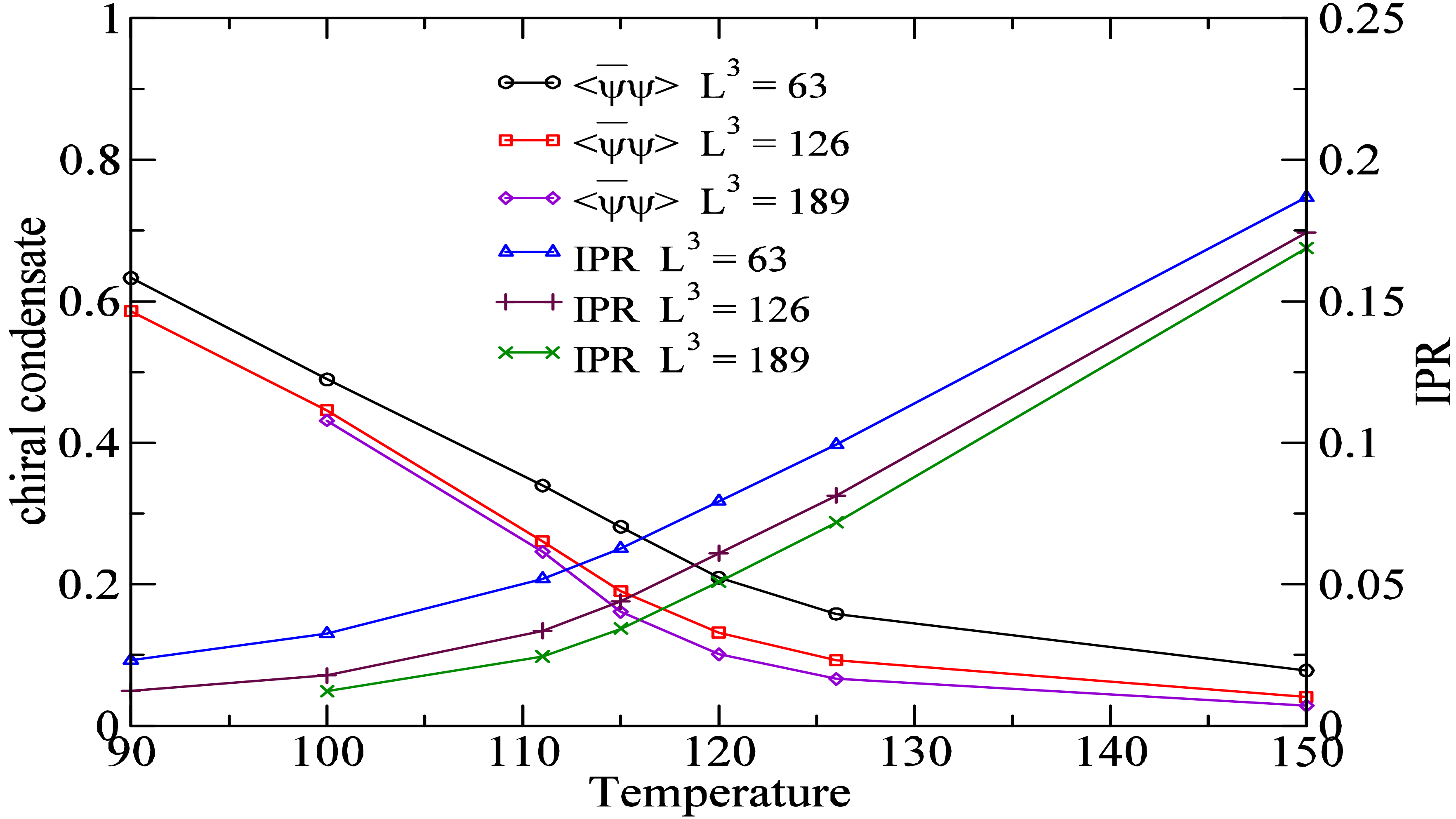


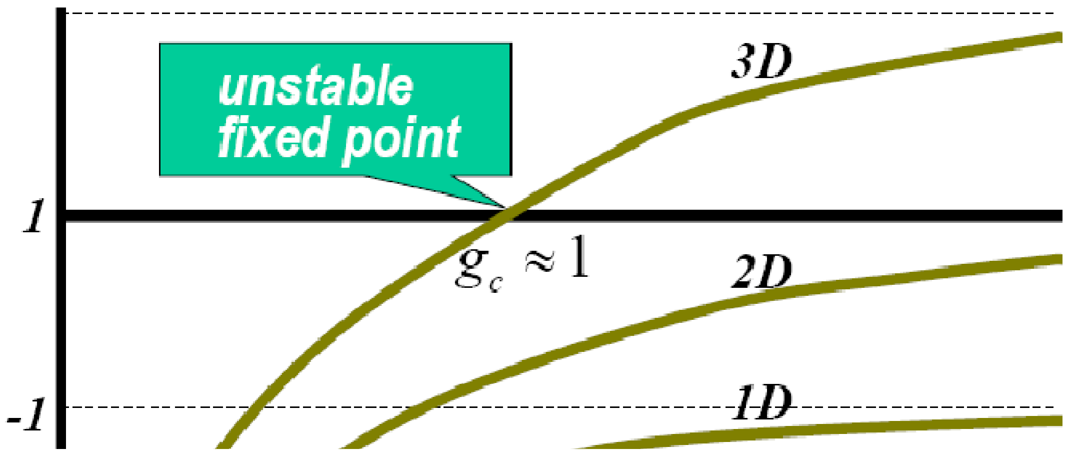

We have observed a metal-insulator transition at T ~ 125 Mev

# Localization versus chiral transition

Instanton liquid model Nf=2, masless




Chiral and localizzation transition occurs at the same temperature



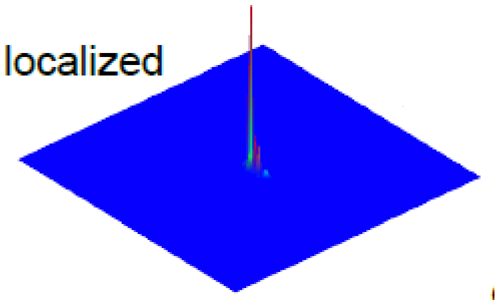

$g = g_c$

# Lecture V

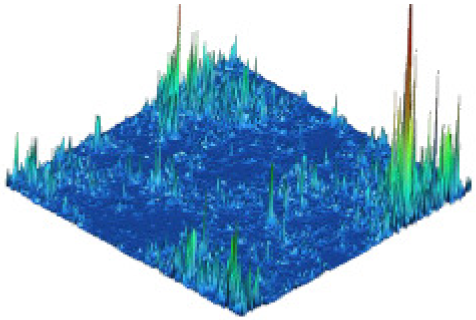
Metal Insulator transitions

New Window of universality

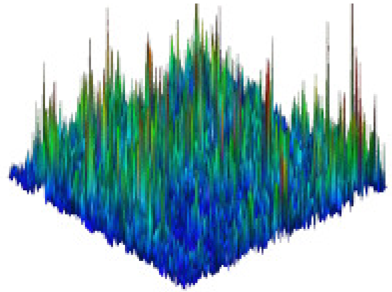
Insulator



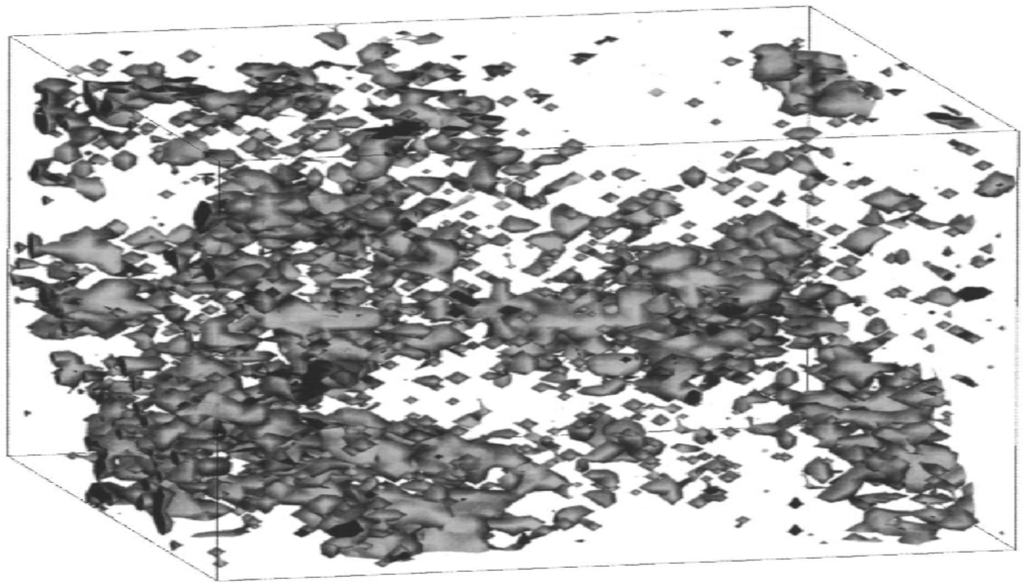

Critical



Metal



Typical Multifractal eigenstate

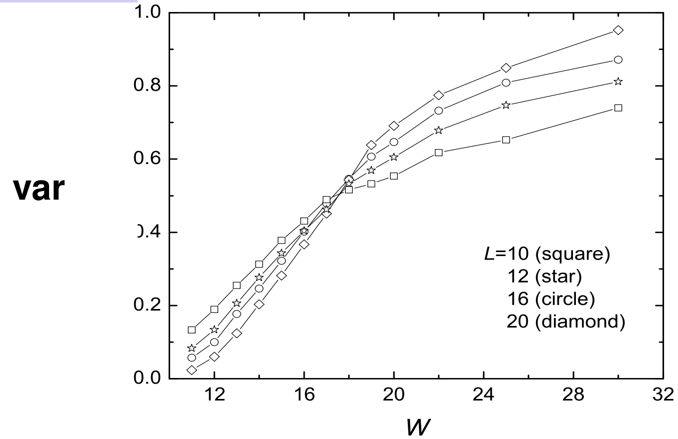


Kramer et al. 1999

# Signatures of a metal-insulator transition

## 1. Scale invariance of the spectral correlations.

Skolovski, Shapiro, Altshuler,90's




## 2.

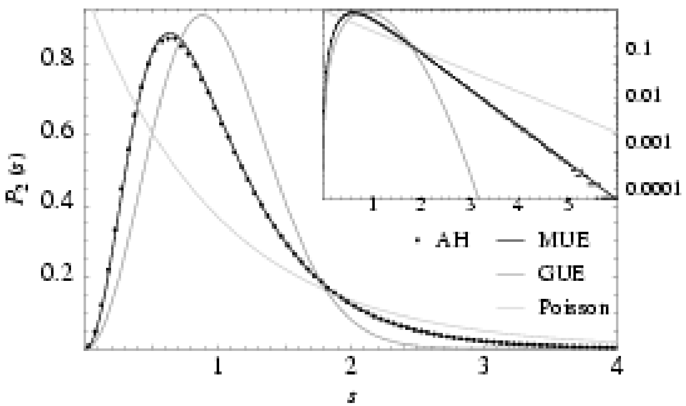
$$P(s) \sim s^{\beta} \qquad s \ll 1$$

$$P(s) \sim e^{-As} \qquad s \gg 1$$

$$\mathrm{var} = \langle s^2 \rangle - \langle s \rangle^2 \qquad \langle s^n \rangle = \int s^n P(s)\,ds$$

## 3. Eigenstates are multifractals

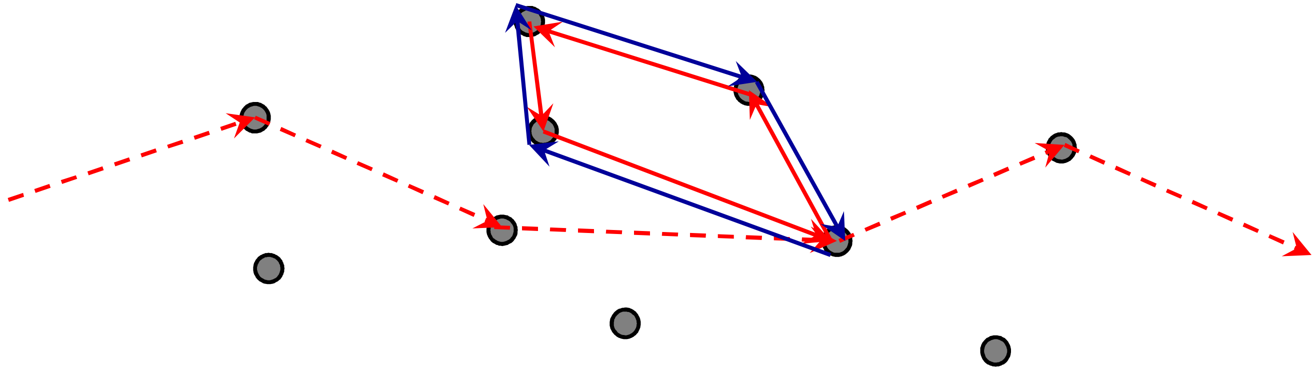
$$\int |\psi_n(r)|^{2q} d^d r \sim L^{-D_q(q-1)}$$




## 4. Difussion is anomalous

$$\langle r^2(t) \propto t^{2/d} \rangle$$

## Self consistent approach to the transition (Wolfle-Volhardt, Imry, Shapiro)



1.Cooperons (Langer-Neal, maximally crossed, responsible for weak localization) and Diffusons (no localization, semiclassical) can be combined.

3. Accurate in d ~2.

$$D(\omega) = D_0 - \frac{k_F^{2-d}}{\pi m} \int_0^{k_0} dk \, \frac{k^{d-1}}{[-i\omega / D(\omega)] + k^2}$$

**No control on the approximation!**

# Predictions of the self consistent theory at the transition

## 1. Critical exponents:

Vollhardt, Wolfle,1982

$$|\psi(r)| \sim e^{-r/\xi} \qquad \xi \propto |E-E_c|^{-\nu}$$

$$\nu = \frac{1}{d-2} \qquad d < 4$$

$$\nu = 1/2 \qquad d > 4$$

**Disagreement with numerical simulations!!**

## 2. Transition for d>2

## 3. Correct for d ~ 2

**Why?**

## Why do self consistent methods fail for d = 3?

1. Always perturbative around the metallic (Vollhardt & Wolfle) or the insulator state (Anderson, Abou Chacra, Thouless) .

A new basis for localization is needed

2. Anomalous diffusion at the transition (predicted by the scaling theory) is not taken into account.

$$D(L) \propto L^{2-d}$$

$$D(q) \propto q^{d-2}$$

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# Semiclassical Theory of the Anderson Transition

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(Received 5 October 2007; published 22 February 2008)

We study analytically the metal-insulator transition in a disordered conductor by combining the self-consistent theory of localization with the one parameter scaling theory. We provide explicit expressions of the critical exponents and the critical disorder as a function of the spatial dimensionality $d$. The critical

Analytical results combining the scaling theory and the self consistent condition.

*Critical exponents, critical disorder, level statistics.*

# Technical details: Critical exponents

$$\tilde{D}(q) = D_0 q^{d-2}$$

$$\omega \to 0 \text{ limit } \xi = \sqrt{-i\omega/\tilde{D}(\omega)}$$

$$\frac{\tilde{D}(\omega)}{D_{clas}} = 1 - \frac{\Delta}{\pi \hbar V D_{clas}} \sum_q \frac{1}{-\frac{i\omega}{\tilde{D}(\omega)\tilde{D}(q)} + q^2}$$

$$\frac{\tilde{D}(\omega)}{D_{clas}} = 1 - \frac{d}{(k_F l)^{d-1}(d-2)\pi} + \frac{d k_F^{2-d}}{\pi k_F l} \int_0^{1/l} dq \frac{|q|^{d-3}}{\frac{1}{D_0 \xi^2} + q^d}$$

The critical exponent ν, can be obtained by solving the above equation for $\xi \propto |E - E_c|^{-\nu}$ with D (ω) = 0.

$$\boxed{\nu = \frac{1}{2} + \frac{1}{d - 2}}$$

# Comparison with numerical results

$$|\psi(r)| \sim e^{-r/\xi} \qquad \xi \propto |E - E_c|^{-\nu}$$

$$\nu = \frac{1}{2} + \frac{1}{d-2}$$

$$\nu_{3T} = 1.5 \qquad \nu_{3N} = 1.52 + 0.06$$

$$\nu_{4T} = 1 \qquad \nu_{4N} = 1.03 \pm 0.07$$

$$\nu_{5T} = 0.83 \qquad \nu_{5N} = 0.84 \pm 0.06$$

$$\nu_{6T} = 0.75 \qquad \nu_{6N} = 0.78 \pm 0.06$$